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BEAM EQUATIONS OF MOTION DERIVED FROM DISPLACEMENT FIELD OF LAMB PLATE THEORY

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It has been found that as the elastic surface wave in any medium converges to the Rayleigh surface wave in the limiting case of infinite frequency, the dynamic radius of gyration for a half-infinite medium is definable despite being unbounded. The dynamic radius of gyration for the first mode of phase velocity in a beam in the case of infinite frequency converges to 68% of static values. The same radii converge to zero for modes of phase velocity higher than the second. This result indicates that only the first mode of phase velocity of a flexural wave survives on the surface of a beam medium as a Rayleigh surface wave in the case of infinite frequency.

Key Words *: Lamb plate, Rayleigh-Lamb equation, Timoshenko beams, radius of gyration of area, Rayleigh surface waves, phase velocity curves*

1. INTRODUCTION

Pochhammer's [1)] and Chree's [2)] solutions for the three-dimensional theory of elasticity for beams with solid circular cross-sections, written in the late 19th century, have attracted much attention in regard to problems of wave propagation in infinite beams. As exact elastic theories are complicated and difficult to solve, they are used only in cases in which cross-sections have surfaces with simple outlines, such as circular cylindrical bars. Thus, phase velocity curves were not calculated numerically until the 20th century[3)]. In contrast, the approximate beam theory by Timosheko[4)], which is simple and clear, can be applied to various loading and boundary conditions for finite or infinite beams. However, Timosheko's beam theory is approximate, so only the phase velocity curve of the first mode is reliable.

Our studies have focused on how to correlate Timoshenko's beam theory and Mindlin's plate theory to the three-dimensional theory of elasticity. Specifically, by representing the radius of gyration of a cross-sectional area and the elastic moduli in the equation of motion for a Timoshenko's beam as functions of frequency, we have been able to make the phase velocity curves agree with the results of the Pochhammer-Chree theory in the range of intermediate and high frequency. Similarly, we have been able to make the equation of motion for the Mindlin plate [5)] agree with the results of Lamb plate theory[6)]. However, it is only a numerical agreement of the phase velocity curve, and it is not based on theoretical evaluation.

Pochhammer-Chree's three-dimensional theory and the approximate theory of the Timoshenko beam differ in many ways, so it is

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rather strange that the first mode of each theory conforms with the other. Whereas the three-dimensional theory of elasticity requires a stress free condition at cross-section outlines, the Timoshenko beam theory ignores this condition and instead requires equilibrium of the stress resultants. In the three-dimensional theory of elasticity, trigonometric functions for warping and inplane displacement are adopted. In the Timoshenko beam theory, however, a linear unit warping function and a constant inplane displacement function are adopted regardless of the frequency range.

We have previously described our attempts to bring the results of the Timoshenko beam theory and the Mindlin plate theory close to that of elasticity theory. In this paper, we attempt, in the opposite way, to derive the equation of motion for the Timoshenko beam from elasticity theory. From this study, we can now explain how the radius of gyration of area varies as frequency increases. In concrete, we assume exact displacement fields for the infinite plate used in the Lamb plate theory as displacement fields for a two-dimensional beam. The static basis function of axial warping displacement in these displacement fields is a linear function in the direction of beam height, and becomes gradually more damped near a neutral axis as frequency increases. The static basis function for deflection displacement takes a constant value of 1 in the direction of beam height, and is damped near a neutral axis as frequency increases. The sum of the horizontal and vertical displacements converges to the state of the Rayleigh surface wave at infinite frequency.

By finding the phase velocity curve from this equation of motion based on the exact displacement fields, we investigate whether the curve conforms to that of Lamb plate theory. We then confirm the correctness of the equation of motion. Regardless of the structural system, for example plate or beam, the first mode of the phase velocity for the wave propagated in the medium approaches the Rayleigh surface wave in the high-frequency region. From this fact, even in a semi-infinite medium, we can propose the concept of the dynamic radius of gyration of area for the Rayleigh surface wave. In this paper, we find the closed form solutions and prove that in the case of a Poisson material, they decrease to about 68% of the static radius of gyration of area for the initial case of the beam. Furthermore, we confirm that the radii of gyration of the phase velocity for modes higher than the second become zero. This corresponds to the fact that only Rayleigh waves survive on the surface of the beam or the plate and that the group of phase velocity curves for modes higher than the second, converging to the transverse wave velocity of the medium, finally vanishes at infinite frequency.

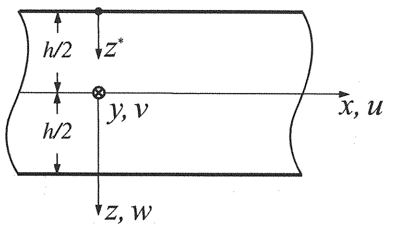

Fig.1 The coordinate system.

2. STRUCTURAL AND COORDINATE SYSTEM

We choose the orthogonal x and y axes in the center of a plate or beam of uniform thickness h. We also choose the z axis perpendicular to the x and y axes in the direction of thickness. The x, y and z axes are in the right-hand coordinate system, and the z axis extends downward from the x axis. At the same time, the z^* axis descends from the upper edge of the plate or beam. Displacement components for the x, y and z directions at point $\mathrm{P}(x, y, z)$ in the plate or beam are u, v and w, respectively.

In tensor notation, x, y and z are denoted by x_1, x_2 and x_3, respectively, and z^* is denoted by x_3^*. Displacements u, v and w are denoted by u_1, u_2 and u_3, respectively (**Fig. 1**).

3. GOVERNING EQUATIONS

(1) Basic conditions

According to the infinitesimal displacement theory, the kinematical relations are expressed as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

where the values of subscripts i and j are between 1 and 3. If the Lamé constants, characteristic values of the material, are represented by λ and μ, the constitutive relations for an elastic medium become

$$\left.\begin{aligned}\sigma_{\alpha\beta} &= 2\mu\varepsilon_{\alpha\beta} + \lambda\delta_{\alpha\beta}\varepsilon_{\gamma\gamma} \\ \sigma_{22} &= \lambda\varepsilon_{\alpha\alpha}\end{aligned}\right\} \tag{2}$$

where the values of subscripts α, β and γ are 1 or 3, and $\delta_{\alpha\beta}$ is Kronecker's delta symbol. Substituting Eq. (1) into Eq. (2) of the constitutive relations for the elastic medium results in the following relation of stress and displacement:

$$\left.\begin{aligned}\sigma_{\alpha\beta} &= \mu(u_{\alpha,\beta} + u_{\beta,\alpha}) + \lambda\delta_{\alpha\beta}u_{\gamma,\gamma} \\ \sigma_{22} &= \lambda u_{\alpha,\alpha}\end{aligned}\right\} \tag{3}$$

Denoting the body force component f_i in direction i, the condition of equilibrium of stresses for an infinitesimal hexagon element is expressed as

$$\sigma_{ij,j} + \rho f_i = \rho u_{i,tt} \tag{4}$$

where ρ denotes the mass density of the medium and t denotes time.

If the stress components for the condition of equilibrium are expressed by displacements, Navier's equation becomes

$$\left.\begin{aligned}(\lambda + \mu)u_{\beta,\beta\alpha} + \mu u_{\alpha,\beta\beta} + \rho f_\alpha &= \rho u_{\alpha,tt} \\ \mu u_{2,\beta\beta} + \rho f_2 &= \rho u_{2,tt}\end{aligned}\right\} \tag{5}$$

(2) Lamé potential

According to Helmholtz's decomposition, we can express the displacement component u_k using the Lamé potentials ϕ and ψ as

$$u_k = \phi_{,k} + e_{klm}\psi_{m,l} \tag{6}$$

In the two-dimensional problem treated here, ϕ and ψ are functions of x_1, x_3 and t only. Therefore, we have

$$\left.\begin{aligned}u_\alpha &= \phi_{,\alpha} + e_{\alpha\beta}\psi_{,\beta} \\ u_2 &= v\end{aligned}\right\} \tag{7}$$

where we define

$$\left.\begin{aligned}\psi(x_\alpha, t) &\equiv \psi_2(x_\alpha, t) \\ v(x_\alpha, t) &\equiv \frac{\partial\psi_3(x_\alpha, t)}{\partial x_1} - \frac{\partial\psi_1(x_\alpha, t)}{\partial x_3}\end{aligned}\right\} \tag{8}$$

and $e_{\alpha\beta}$ is a two-dimensional permutation symbol. In concrete, equation (7) is written

$$u_1(x_\alpha, t) = \frac{\partial\phi}{\partial x_1} + \frac{\partial\psi}{\partial x_3} \tag{9}$$

$$u_2(x_\alpha, t) = v$$

$$u_3(x_\alpha, t) = \frac{\partial\phi}{\partial x_3} - \frac{\partial\psi}{\partial x_1} \tag{10}$$

Decomposing the body force as in Eq. (7), it becomes

$$\left.\begin{aligned}f_\alpha(x_\beta, t) &= f_{,\alpha} + e_{\alpha\beta}F_{,\beta}, \quad F \equiv F_2 \\ f_2(x_\beta, t) &= F_{3,1} - F_{1,3}\end{aligned}\right\} \tag{11}$$

Substituting the decomposition equation for displacement (7) and that for the body force (11) into the Navier's equation (5), we obtain the following three wave equations:

$$\left.\begin{aligned}\nabla^2\phi + \frac{f}{c_1^2} &= \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \\ \nabla^2 v + \frac{f_2}{c_2^2} &= \frac{1}{c_2^2}\frac{\partial^2 v}{\partial t^2} \\ \nabla^2\psi + \frac{F}{c_2^2} &= \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}\end{aligned}\right\} \tag{12}$$

where $c_\alpha\,(\alpha = 1, 2)$ is defined as

$$c_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\rho}\right)^{1/2} \tag{13}$$

using the Lamé constants. The longitudinal wave velocity c_1 and the transverse wave velocity c_2 of the medium in which the waves are propagated are sometimes denoted by c_L and c_T, respectively.

Substituting the decomposition equation for displacement (7) into Eq. (3), the stress components become

$$\left.\begin{aligned}\sigma_{\alpha\beta} &= \lambda\nabla^2\phi\delta_{\alpha\beta} + 2\mu\phi_{,\alpha\beta} \\ &\quad + \mu(e_{\alpha\beta}\psi_{,\beta\gamma} + e_{\beta\gamma}\psi_{,\alpha\gamma}) \\ \sigma_{22} &= \lambda\nabla^2\phi \\ \sigma_{\alpha 2} &= \mu v_{,\alpha}\end{aligned}\right\} \tag{14}$$

In concrete,

$$\left.\begin{aligned}&\sigma_{11} = \lambda\nabla^2\phi + 2\mu\phi_{,11} + 2\mu\psi_{,13} \\ &\sigma_{22} = \lambda\nabla^2\phi \\ &\sigma_{33} = \lambda\nabla^2\phi + 2\mu\phi_{,33} - 2\mu\psi_{,13} \\ &\sigma_{12} = \mu v_{,1}, \qquad \sigma_{23} = \mu v_{,3} \\ &\sigma_{31} = 2\mu\phi_{,13} + \mu(\psi_{,33} - \psi_{,11})\end{aligned}\right\} \tag{15}$$

where ∇^2 is a two-dimensional Laplace operator:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2} \tag{16}$$

We can consider $\sigma_{22} = 0$ for beams with extremely narrow cross-sections, and such beams are considered to be under the condition of plane stress. In this case, we should transform the Lamé constant λ to $2\mu\lambda/(\lambda + 2\mu)$.

(3) Bending wave for infinite plate

The x_3 axis extends downward in the direction of plate thickness from the center of a plate that extends to infinity in the $\pm(x_1, x_2)-$ directions. Letting the thickness of the plate be h, $x_3 = \pm h/2$ represents a position on the surface of the plate (**Fig. 1**). If there is no body force, the wave equations on the $x_1 - x_3$ plane become

$$\left.\begin{aligned} \nabla^2\phi &= \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \\ \nabla^2\psi &= \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \end{aligned}\right\} \tag{17}$$

According to Eq. (15), the boundary conditions for no stresses on the plate surfaces become

$$\left.\begin{aligned} \sigma_{33}(x_1, \pm h/2) &= \lambda\nabla^2\phi + 2\mu\phi_{,33} - 2\mu\psi_{,13} = 0 \\ \sigma_{31}(x_1, \pm h/2) &= 2\mu\phi_{,13} + \mu(\psi_{,33} - \psi_{,11}) = 0 \end{aligned}\right\} \tag{18}$$

Since the bending wave of the plate has an asymmetric component of displacement, potential functions are given in the forms of

$$\left.\begin{aligned} \phi &= A\sinh k\nu_1 x_3 \exp ik(x_1 - ct) \\ \psi &= B\cosh k\nu_2 x_3 \exp ik(x_1 - ct) \end{aligned}\right\} \tag{19}$$

where A and B are arbitrary complex constants, i is the imaginary unit, and c is phase velocity. We also define the wave number k and the function of phase velocity ν_α as follows:

$$k = \frac{2\pi}{\lambda}, \quad \nu_\alpha = \left(1 - \frac{c^2}{c_\alpha^2}\right)^{1/2} \quad (\alpha = 1, 2) \tag{20}$$

The denominator λ in the equation defines wave k not as the Lamé constant but as the wavelength.

Substituting the form of the potential function (19) into the boundary condition of no stress on the plate surface (18) and representing it in matrices, we obtain the following:

$$\begin{bmatrix} (1+\nu_2^2)\sinh k\nu_1 h/2 & -2i\nu_2 \sinh k\nu_2 h/2 \\ 2i\nu_1 \cosh k\nu_1 h/2 & (1+\nu_2^2)\cosh k\nu_2 h/2 \end{bmatrix} \times \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$

According to the condition in which complex constants A and B have a non-trivial solution, the determinant of the coefficient matrix on the left side of the above equation must be zero. We then obtain the Rayleigh-Lamb frequency equation of a bending wave propagating in the infinite plate;

$$\frac{\tanh k\nu_1 h/2}{\tanh k\nu_2 h/2} = \frac{4\nu_1\nu_2}{(1+\nu_2^2)^2} \tag{22}$$

Furthermore, when Eq. (21) is solved as simultaneous equations for unknowns A and B, the ratio of A and B,

$$\frac{B}{A} = \frac{1+\nu_2^2}{2i\nu_2}\frac{\sinh k\nu_1 h/2}{\sinh k\nu_2 h/2} \tag{23}$$

or

$$\frac{B}{A} = -\frac{2i\nu_1}{1+\nu_2^2}\frac{\cosh k\nu_1 h/2}{\cosh k\nu_2 h/2} \tag{24}$$

is also given.

Substituting the potential function (19) into the equations of displacement (9) and (10) and eliminating the complex constant B from Eq. (23) or (24), we obtain

$$\left.\begin{aligned} u_1 &= -kA\sinh k\nu_1 \frac{h}{2} \cdot \sin k(x_1 - ct) \\ &\quad \times \left(\frac{\sinh k\nu_1 x_3}{\sinh k\nu_1 h/2} - \frac{1+\nu_2^2}{2}\frac{\sinh k\nu_2 x_3}{\sinh k\nu_2 h/2}\right) \\ u_3 &= kA\cosh k\nu_1 \frac{h}{2} \cdot \cos k(x_1 - ct) \\ &\quad \times \left(\nu_1 \frac{\cosh k\nu_1 x_3}{\cosh k\nu_1 h/2} - \frac{2\nu_1}{1+\nu_2^2}\frac{\cosh k\nu_2 x_3}{\cosh k\nu_2 h/2}\right) \end{aligned}\right\} \tag{25}$$

Similarly, substituting the potential function (19) into the stress equation (15) and eliminating the complex constant B from Eq. (23) and (24), we obtain

$$\left.\begin{aligned}
\sigma_{11} &= -\mu k^2 A \sinh k\nu_1 \frac{h}{2} \cdot \cos k(x_1 - ct) \\
&\times\left((1+2\nu_1^2-\nu_2^2)\frac{\sinh k\nu_1 x_3}{\sinh k\nu_1 h/2} - (1+\nu_2^2)\frac{\sinh k\nu_2 x_3}{\sinh k\nu_2 h/2}\right) \\
\sigma_{22} &= \mu k^2 (1 - 2\nu_1^2 + \nu_2^2) A \cos k(x_1 - ct) \\
&\quad \times \sinh k\nu_1 x_3 \\
\sigma_{33} &= \mu k^2 (1 + \nu_2^2) A \sinh k\nu_1 \frac{h}{2} \cdot \cos k(x_1 - ct) \\
&\quad \times\left(\frac{\sinh k\nu_1 x_3}{\sinh k\nu_1 h/2} - \frac{\sinh k\nu_2 x_3}{\sinh k\nu_2 h/2}\right) \\
\sigma_{31} &= -2\mu k^2 \nu_1 A \cosh k\nu_1 \frac{h}{2} \cdot \sin k(x_1 - ct) \\
&\quad \times\left(\frac{\cosh k\nu_1 x_3}{\cosh k\nu_1 h/2} - \frac{\cosh k\nu_2 x_3}{\cosh k\nu_2 h/2}\right)
\end{aligned}\right\} \tag{26}$$

Because the stresses are the exact solution for the two-dimensional theory of elasticity, Hooke's law holds for the shear strain γ_{31} from the equation of the displacement component (25) and shear stress σ_{31} satisfying the equilibrium of stresses, and it is confirmed that there is no inconsistency such as that found in classical beam theory.

(4) Limit of zero frequency

We define dimensionless frequency $\overline{\gamma}$ and dimensionless phase velocity $\overline{c}$ as

$$\overline{\gamma} \equiv k\frac{h}{2} = \frac{\pi h}{\lambda}, \qquad \overline{c} = \frac{c}{c_2} \tag{27}$$

In the case of low frequency, i.e., when the wavelength is longer than the plate thickness h, we investigate the phase velocity curve when the frequency approaches zero.

First, we treat phase velocity c as smaller than the transverse wave velocity c_2 of the medium ($0 < c < c_2$). Taking out the first two terms of ascending order of power for the Taylor expansion of two tanh functions in the frequency equation (22) and simplifying the equation, we obtain

$$\frac{c^2}{c_2^2} - \frac{1}{3}k^2h^2\left(1 - \frac{c_2^2}{c_1^2}\right) = 0 \tag{28}$$

Transposing the second term on the left side to the right side and taking the square root of both sides, we obtain

$$\frac{c}{c_2} = \sqrt{\frac{1}{3}}kh\left(1 - \frac{c_2^2}{c_1^2}\right)^{1/2} \tag{29}$$

If we plot dimensionless velocity $\overline{c} = c/c_2$ on the ordinate and dimensionless frequency $\overline{\gamma} \equiv kh/2$ on the abscissa, this equation becomes a straight line, inclined upward from the origin[7].

When $kh \to 0$, the limit of displacement in the direction of the axis x_1 and plate thickness of Eq. (25) is given by

$$\left.\begin{aligned}
u_1 &= -kA \cdot k\nu_1 \frac{h}{2} \cdot \sin k(x_1 - ct) \cdot \left(\frac{1-\nu_2^2}{h}\right) \cdot x_3 \\
u_3 &= \quad kA \cdot 1 \cdot \cos k(x_1 - ct) \cdot \nu_1 \left(\frac{1-\nu_2^2}{1+\nu_2^2}\right) \cdot 1
\end{aligned}\right\} \tag{30}$$

Similarly, we can easily calculate the limit of the stress component of equation (26) when $kh \to 0$.

Next, we treat the phase velocity c as larger than the transverse wave velocity of the medium c_2 ($0 < c_2 < c$). As the dimensionless frequency $\overline{\gamma} \equiv kh/2$ approaches zero, phase velocity c increases rapidly. As a result, the phase velocity function ν_α ($\alpha = 1, 2$) in Eq. (22) becomes complex. If we define

$$p_\alpha = \left(\frac{c^2}{c_\alpha^2} - 1\right)^{1/2} \quad (\alpha = 1, 2) \tag{31}$$

the following relation holds using the imaginary unit i:

$$p_\alpha = i\,\nu_\alpha \quad (\alpha = 1, 2) \tag{32}$$

Then we can rewrite the frequency equation as

$$\frac{\tan kp_1 h/2}{\tan kp_2 h/2} = -\frac{4p_1p_2}{(1 - p_2^2)^2} \tag{33}$$

This equation shows that phase velocity c becomes infinite as the dimensionless frequency $\overline{\gamma} \equiv kh/2$ approaches zero. Then, at the limit of $\overline{\gamma} \to 0$, the denominator on the left side coverges to $\tan kp_2 h/2 \to 0$, namely,

$$kp_2\frac{h}{2} \longrightarrow \frac{\pi}{2} \tag{34}$$

Although a part of what is described here has already been written in Mindlin[7] and Eringen *et al.*[6], we note them in order to clarify the limited form (equation (30)) of the displacement function in the direction of the axis x_1 and plate thickness.

(5) Limit of infinite frequency

We consider the case of high frequency, namely, the case in which the wavelength is much shorter than the plate thickness h. In this case, there are two situations depending on the range of phase velocity. If phase velocity c is smaller than the transverse wave velocity of the medium c_2, the left side of the Rayleigh-Lamb frequency equation (22) becomes

$$\frac{\tanh k\nu_1 h/2}{\tanh k\nu_2 h/2} \longrightarrow 1 \tag{35}$$

when $\overline{\gamma} \equiv kh/2 \to \infty$. Then the equation changes to

$$1 = \frac{4\nu_1\nu_2}{(1+\nu_2^2)^2} \tag{36}$$

This equation is independent of frequency. That is, it becomes the equation for Rayleigh wave velocity with no dispersion;

$$\left(2-\frac{c^2}{c_2^2}\right)^2 - 4\left(1-\frac{c^2}{c_1^2}\right)^{1/2}\left(1-\frac{c^2}{c_2^2}\right)^{1/2} = 0 \tag{37}$$

The right side of the displacement equation (25) and stress equation (26) are composed of the quotient forms of hyperbolic functions. By extracting only these quotient forms, changing the hyperbolic functions to exponential functions and simplifying them, we obtain a transformed equation, as follows:

$$\left.\begin{aligned}
\frac{\sinh k\nu_\alpha x_3}{\sinh k\nu_\alpha h/2} &= \frac{e^{k\nu_\alpha x_3} - e^{-k\nu_\alpha x_3}}{e^{k\nu_\alpha h/2} - e^{-k\nu_\alpha h/2}} \\
&= -e^{-k\nu_\alpha(\frac{h}{2}+x_3)}\frac{1-e^{2k\nu_\alpha x_3}}{1-e^{-2k\nu_\alpha h/2}} \\
&= -e^{-k\nu_\alpha x_3^*}\frac{1-e^{-2k\nu_\alpha(h/2-x_3^*)}}{1-e^{-2k\nu_\alpha h/2}} \\
\frac{\cosh k\nu_\alpha x_3}{\cosh k\nu_\alpha h/2} &= \frac{e^{k\nu_\alpha x_3} + e^{-k\nu_\alpha x_3}}{e^{k\nu_\alpha h/2} + e^{-k\nu_\alpha h/2}} \\
&= e^{-k\nu_\alpha(\frac{h}{2}+x_3)}\frac{1+e^{2k\nu_\alpha x_3}}{1+e^{-2k\nu_\alpha h/2}} \\
&= e^{-k\nu_\alpha x_3^*}\frac{1+e^{-2k\nu_\alpha(h/2-x_3^*)}}{1+e^{-2k\nu_\alpha h/2}}
\end{aligned}\right\} \tag{38}$$

where the x_3^* axis extends downward in the direction of plate thickness and the origin of the axis is on the upper surface of the plate. According to **Fig. 1**, the following relation holds:

$$x_3^* = \frac{h}{2} + x_3 \tag{39}$$

By making the frequency infinite, these quotient forms become the following:

$$\left.\begin{aligned}
\frac{\sinh k\nu_\alpha x_3}{\sinh k\nu_\alpha h/2} &\longrightarrow -e^{-k\nu_\alpha x_3^*}\ (\overline{\gamma} \equiv kh/2 \to \infty) \\
\frac{\cosh k\nu_\alpha x_3}{\cosh k\nu_\alpha h/2} &\longrightarrow \ e^{-k\nu_\alpha x_3^*}\ (\overline{\gamma} \equiv kh/2 \to \infty)
\end{aligned}\right\} \tag{40}$$

At infinite frequency, the displacement equation (25) and stress equation (26) for the Rayleigh-Lamb plate can be rewritten as follows:

$$\left.\begin{aligned}
u_1 &= -C\sin k(x_1 - ct) \\
&\quad \times\left(e^{-k\nu_1 x_3^*} - \frac{1+\nu_2^2}{2}e^{-k\nu_2 x_3^*}\right) \\
u_3 &= C\cos k(x_1 - ct) \\
&\quad \times\left(\nu_1 e^{-k\nu_1 x_3^*} - \frac{2\nu_1}{1+\nu_2^2}e^{-k\nu_2 x_3^*}\right)
\end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned}
\sigma_{11} &= \mu kC\cos k(x_1 - ct) \\
&\quad \times\left(-(1+2\nu_1^2-\nu_2^2)e^{-k\nu_1 x_3^*} + (1+\nu_2^2)e^{-k\nu_2 x_3^*}\right) \\
\sigma_{22} &= -(1-2\nu_1^2+\nu_2^2)\mu kC\cos k(x_1-ct)\, e^{-k\nu_1 x_3^*} \\
\sigma_{33} &= (1+\nu_2^2)\mu kC\cos k(x_1 - ct) \\
&\quad \times\left(e^{-k\nu_1 x_3^*} - e^{-k\nu_2 x_3^*}\right) \\
\sigma_{31} &= 2\nu_1\mu kC\sin k(x_1 - ct) \\
&\quad \times\left(e^{-k\nu_1 x_3^*} - e^{-k\nu_2 x_3^*}\right)
\end{aligned}\right\} \tag{42}$$

Representing the convergent value of the phase velocity at infinite frequency as c_R, with $\overline{\nu}_1 = (1 - c_R^2/c_1^2)^{1/2}$, we rewrite the equation as follows:

$$-C = \lim_{\overline{\gamma}\to\infty} kA\sinh k\overline{\nu}_1\frac{h}{2} = \lim_{\overline{\gamma}\to\infty} kA\cosh k\overline{\nu}_1\frac{h}{2} \tag{43}$$

In the case of a Poisson's ratio of $\nu = 1/4$, we have that $c_R = 0.9194c_2$. Then the equations above become the following:

$$\left.\begin{aligned}
u_1 &= -C\sin k(x_1 - c_R t) \\
&\quad \times\left(e^{-0.8475kx_3^*} - 0.5773e^{-0.3933kx_3^*}\right) \\
u_3 &= C\cos k(x_1 - c_R t) \\
&\quad \times\left(-0.8475e^{-0.8475kx_3^*} + 1.4679e^{-0.3933kx_3^*}\right) \\
\sigma_{11} &= \mu kC\cos k(x_1 - c_R t) \\
&\quad \times\left(-2.2817e^{-0.8475kx_3^*} + 1.1547e^{-0.3933kx_3^*}\right) \\
\sigma_{22} &= 0.2817\mu kC\cos k(x_1 - c_R t)\, e^{-0.8475kx_3^*} \\
\sigma_{33} &= 1.1547\mu kC\cos k(x_1 - c_R t) \\
&\quad \times\left(e^{-0.8475kx_3^*} - e^{-0.3933kx_3^*}\right) \\
\sigma_{31} &= 1.6950\mu kC\sin k(x_1 - c_R t) \\
&\quad \times\left(e^{-0.8475kx_3^*} - e^{-0.3933kx_3^*}\right)
\end{aligned}\right\} \tag{44}$$

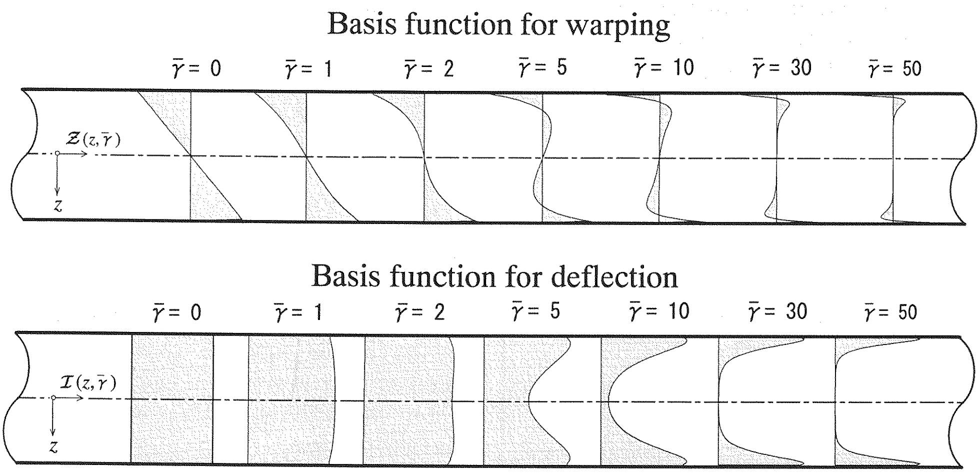

Fig. 2 Distribution geometry of basis functions.

These are the displacement and stress equations of a Rayleigh surface wave[6] for a Poisson material ($\nu = 1/4$).

When the phase velocity c is larger than the transverse wave velocity c_2, the Rayleigh-Lamb frequency equation (22) is expressed by

$$\frac{\tanh k\nu_1 h/2}{\tan kp_2 h/2} = -\frac{4\nu_1 p_2}{(1-p_2^2)^2} \tag{45}$$

when the dimensionless frequency $\bar{\gamma} \equiv kh/2 \to \infty$. The phase velocity c, included in ν_1 and p_2 in the equation, approaches the transverse wave velocity of the medium c_2 from the upper part of the phase velocity curves.

(6) Setting basis functions

Using the results of the exact solution of elasticity theory, we reconstruct a beam theory. Hereafter, the coordinate system is represented by $x-y-z$. In the static beam theory, the basis function of warping u in the direction of the x ($\equiv x_1$) axis is z ($\equiv x_3$), and the basis function of deflection displacement w is represented by 1. According to equation (25), the displacement function is given by

$$\left.\begin{aligned} u(x,z,\bar{\gamma}) &= -\mathcal{Z}(z,\bar{\gamma})\psi(x,\bar{\gamma}) \\ w(x,z,\bar{\gamma}) &= \mathcal{I}(z,\bar{\gamma})W(x,\bar{\gamma}) \end{aligned}\right\} \tag{46}$$

such that the basis functions of the dynamic beam corresponds to those of the static beam when motion stops ($\bar{\gamma}=0$). Henceforth, $\psi(x,\bar{\gamma})$ represents the rotation of the beam cross-section.

These equations are not in the form of completely *separated variables.* Both unknown functions with independent variables of axial coordinate x and basis functions with independent variables of inplane coordinate z are also functions of frequency $\bar{\gamma}$. Therefore, the basis functions can be normalized by multiplying an arbitrary dummy function with only one independent variable; frequency. With reference to equation (30), we determine the basis functions as follows:

$$\left.\begin{aligned} \mathcal{Z}(z,\bar{\gamma}) &= \frac{h}{2}\frac{2}{1-\nu_2^2}\left(\frac{\sinh k\nu_1 z}{\sinh k\nu_1 h/2} - \frac{1+\nu_2^2}{2}\frac{\sinh k\nu_2 z}{\sinh k\nu_2 h/2}\right) \\ \mathcal{I}(z,\bar{\gamma}) &= \frac{1+\nu_2^2}{1-\nu_2^2}\left(-\frac{\cosh k\nu_1 z}{\cosh k\nu_1 h/2} + \frac{2}{1+\nu_2^2}\frac{\cosh k\nu_2 z}{\cosh k\nu_2 h/2}\right) \end{aligned}\right\} \tag{47}$$

A limit of zero frequency ($\bar{\gamma} \equiv kh/2 \to 0$) means a static case. In this case, we have $\mathcal{Z}(z,\bar{\gamma}) \to z$ and $\mathcal{I}(z,\bar{\gamma}) \to 1$. Thus, we can confirm that this determination is correct. Distribution patterns of basis functions for some values of frequency $\bar{\gamma}$ are given in **Fig. 2**. As dimensionless frequency $\bar{\gamma}$ gets larger, displacements near a neutral axis disappear and the beam surface approaches the Rayleigh wave state.

(7) Strain and stress of beams

Substituting the assumptive equation for displacement (46) into the kinematical relations (1) and the constitutive equations (3), we obtain

$$\left.\begin{aligned} \varepsilon_{xx}(x,z,\overline{\gamma}) &= -\mathcal{Z}(z,\overline{\gamma})\psi'(x,\overline{\gamma}) \\ \varepsilon_{zz}(x,z,\overline{\gamma}) &= \dot{\mathcal{I}}(z,\overline{\gamma})W(x,\overline{\gamma}) \\ \gamma_{zx}(x,z,\overline{\gamma}) &= \mathcal{I}(z,\overline{\gamma})\frac{dW}{dx} - \dot{\mathcal{Z}}(z,\overline{\gamma})\psi(x,\overline{\gamma}) \end{aligned}\right\} \quad (48)$$

$$\left.\begin{aligned} \sigma_{xx}(x,z,\overline{\gamma}) &= -\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\mathcal{Z}(z,\overline{\gamma})\psi'(x,\overline{\gamma}) \\ &\quad + \frac{\nu E}{(1+\nu)(1-2\nu)}\dot{\mathcal{I}}(z,\overline{\gamma})W(x,\overline{\gamma}) \\ \sigma_{zz}(x,z,\overline{\gamma}) &= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\dot{\mathcal{I}}(z,\overline{\gamma})W(x,\overline{\gamma}) \\ &\quad - \frac{\nu E}{(1+\nu)(1-2\nu)}\mathcal{Z}(z,\overline{\gamma})\psi'(x,\overline{\gamma}) \\ \tau_{zx}(x,z,\overline{\gamma}) &= G\left(\mathcal{I}(z,\overline{\gamma})\frac{dW}{dx} - \dot{\mathcal{Z}}(z,\overline{\gamma})\psi(x,\overline{\gamma})\right) \end{aligned}\right\} \quad (49)$$

as components of strain and stress for beams, where $(\ )'$ and $(\dot{\ })$ mean differentiation with respect to x and z, respectively.

(8) Equation of motion and stress resultants

Using the displacement, strain, and stress described above, we derive an equation of motion and define stress resultants. Hamilton's principle is expressed by

$$\delta \int_{t_0}^{t_1} (T - U - V)\,dt = 0 \qquad (50)$$

using a symbol of variation δ, where t_0 and t_1 represent the beginning and end of time t. Kinetic energy T, strain energy U and the potential external force V of a system in the time interval are given by

$$T = \frac{1}{2}\int_V \rho\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\} dV \qquad (51)$$

$$U = \frac{1}{2}\int_V \{\sigma_{xx}\varepsilon_{xx} + \sigma_{zz}\varepsilon_{zz} + \tau_{zx}\gamma_{zx}\}\, dV \qquad (52)$$

$$V = -\int_V (p_x u + p_z w)\, dV - \left(\sum P_x u + \sum P_z w\right) \qquad (53)$$

Substituting the displacement equation (46) and the strain equation (48) into Hamilton's principle, we obtain the relations between stress resultants, distrubuted load, and inertial forces;

$$\frac{\partial Q}{\partial x} - q_{se} + \int_A p_z \mathcal{I} dA - \rho A^*(\overline{\gamma})\frac{\partial^2 W}{\partial t^2} = 0 \qquad (54)$$

$$\frac{\partial M}{\partial x} - Q_{se} + \int_A p_x \mathcal{Z} dA + \rho I^*(\overline{\gamma})\frac{\partial^2 \psi}{\partial t^2} = 0 \qquad (55)$$

Let I and A of the beam cross-section be an ordinary static moment of inertia and a cross-sectional area, respectively. Then $I^*(\overline{\gamma})$ and $A^*(\overline{\gamma})$ in the equation above become a dynamic moment of inertia and a cross-sectional area, respectively, taking into account frequency changes.

The boundary conditions become the following:

$$\left[\delta W\left(Q - \sum P_z \mathcal{I}\right)\right]_0^l = 0 \qquad (56)$$

$$\left[\delta \psi\left(M - \sum P_x \mathcal{Z}\right)\right]_0^l = 0 \qquad (57)$$

The stress resultants in the equation above are defined as follows:

$$M(x,\overline{\gamma}) = \int_A \sigma_{xx}(x,z,\overline{\gamma})\,\mathcal{Z}(z,\overline{\gamma})\,dA \qquad (58)$$

$$q_{se}(x,\overline{\gamma}) = \int_A \sigma_{zz}(x,z,\overline{\gamma})\,\dot{\mathcal{I}}(z,\overline{\gamma})\,dA \qquad (59)$$

$$Q(x,\overline{\gamma}) = \int_A \tau_{zx}(x,z,\overline{\gamma})\,\mathcal{I}(z,\overline{\gamma})\,dA \qquad (60)$$

$$Q_{se}(x,\overline{\gamma}) = \int_A \tau_{zx}(x,z,\overline{\gamma})\,\dot{\mathcal{Z}}(z,\overline{\gamma})\,dA \qquad (61)$$

Substituting stress equation (49) into the definition of the stress resultants (58)~(61), we obtain

$$M(x,\overline{\gamma}) = -\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}I^*(\overline{\gamma})\psi'(x,\overline{\gamma}) + \frac{\nu E}{(1+\nu)(1-2\nu)}B^*(\overline{\gamma})W(x,\overline{\gamma}) \qquad (62)$$

$$q_{se}(x,\overline{\gamma}) = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}H^*(\overline{\gamma})W(x,\overline{\gamma}) - \frac{\nu E}{(1+\nu)(1-2\nu)}B^*(\overline{\gamma})\psi'(x,\overline{\gamma}) \qquad (63)$$

$$Q(x,\overline{\gamma}) = GA^*\left(k_0(\overline{\gamma})\frac{\partial W}{\partial x} - k_1(\overline{\gamma})\psi(x,\overline{\gamma})\right) \qquad (64)$$

$$Q_{se}(x,\overline{\gamma}) = GA^*\left(k_1(\overline{\gamma})\frac{\partial W}{\partial x} - k_2(\overline{\gamma})\psi(x,\overline{\gamma})\right) \qquad (65)$$

where the quantities for beam cross-section $I^*(\overline{\gamma})$, $A^*(\overline{\gamma})$, $B^*(\overline{\gamma})$ and $H^*(\overline{\gamma})$ and three kinds of shear correction factors $k_0(\overline{\gamma})$, $k_1(\overline{\gamma})$ and $k_2(\overline{\gamma})$, which are used in the above equations, are defined as

$$\left.\begin{aligned} I^*(\overline{\gamma}) &= \int_A \mathcal{Z}(z,\overline{\gamma})\mathcal{Z}(z,\overline{\gamma})dA \\ A^*(\overline{\gamma}) &= \int_A \mathcal{I}(z,\overline{\gamma})\mathcal{I}(z,\overline{\gamma})dA \\ B^*(\overline{\gamma}) &= \int_A \dot{\mathcal{I}}(z,\overline{\gamma})\mathcal{Z}(z,\overline{\gamma})dA \\ H^*(\overline{\gamma}) &= \int_A \dot{\mathcal{I}}(z,\overline{\gamma})\dot{\mathcal{I}}(z,\overline{\gamma})dA \end{aligned}\right\} \quad (66)$$

$$\left.\begin{aligned} k_0(\overline{\gamma}) &= \frac{1}{A^*(\overline{\gamma})}\int_A \mathcal{I}(z,\overline{\gamma})\mathcal{I}(z,\overline{\gamma})dA \equiv 1 \\ k_1(\overline{\gamma}) &= \frac{1}{A^*(\overline{\gamma})}\int_A \mathcal{I}(z,\overline{\gamma})\dot{\mathcal{Z}}(z,\overline{\gamma})dA \\ k_2(\overline{\gamma}) &= \frac{1}{A^*(\overline{\gamma})}\int_A \dot{\mathcal{Z}}(z,\overline{\gamma})\dot{\mathcal{Z}}(z,\overline{\gamma})dA \end{aligned}\right\} \quad (67)$$

If there is a differentiation relation between the basis function for warping $\mathcal{Z}(z,\overline{\gamma})$ and the basis function for deflection $\mathcal{I}(z,\overline{\gamma})$ similar to the case of static beams, the three shear correction factors $k_0(\overline{\gamma})$, $k_1(\overline{\gamma})$, $k_2(\overline{\gamma})$ all become 1. However, considering the change of frequency, a differentiation relation does not exist except in the static case ($\overline{\gamma}=0$), as shown in Eq. (47);

$$\dot{\mathcal{Z}}(z,\overline{\gamma}) \neq \mathcal{I}(z,\overline{\gamma}) \quad (68)$$

(9) Equation of motion for displacement

Substituting the equations for stress resultants (62)~(65) into the equations of motion (54) and (55), we obtain the equations of motion for displacement as follows:

$$\left[GA^*\left(k_0\frac{\partial W}{\partial x} - k_1\psi\right)\right]' - \left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}H^*W - \frac{\nu E}{(1+\nu)(1-2\nu)}B^*\psi'\right] + \int_A p_z\mathcal{I}dA - \rho A^*(\overline{\gamma})\frac{\partial^2 W}{\partial t^2} = 0 \quad (69)$$

$$\left[\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}I^*\psi' - \frac{\nu E}{(1+\nu)(1-2\nu)}B^*W\right]' + \left[GA^*\left(k_1\frac{\partial W}{\partial x} - k_2\psi\right)\right] - \int_A p_x\mathcal{Z}dA - \rho I^*(\overline{\gamma})\frac{\partial^2\psi}{\partial t^2} = 0 \quad (70)$$

Here, we write the equations of motion for the Timoshenko beam[8] for comparison;

$$\left[GAk\left(\frac{\partial w}{\partial x} - \psi\right)\right]' + \int_A p_z dA - \rho A\frac{\partial^2 w}{\partial t^2} = 0 \quad (71)$$

$$[EI\psi']' + \left[GAk\left(\frac{\partial w}{\partial x} - \psi\right)\right] - \int_A p_x z dA - \rho I\frac{\partial^2\psi}{\partial t^2} = 0 \quad (72)$$

where the cross-sectional quantities are defined as

$$I = \int_A z \cdot z\, dA \quad (73)$$

$$A = \int_A 1 \cdot 1\, dA \quad (74)$$

$$R = \int_A S(z) \cdot S(z) dA \quad (75)$$

$$k = \frac{1}{A} \cdot I \cdot R^{-1} \cdot I \quad (76)$$

$S(z)$ in equation (75) for shearing resistance R is a unit shearing stress function that is obtained by integrating unit warping function z, where k is the shear correction factor[9].

After comparing the equations of motion (69) and (70) obtained above with Timosheko's equations (71) and (72), the equations in our theory are characterized as follows: (1) The second term $[\cdots]$ of equation (69), consisting of two partial terms, exists because we consider the normal stress σ_{zz} that the engineering beam theory ignores. (2) There are three shear correction factors that depend on the frequency; $k_0(\overline{\gamma})$, $k_1(\overline{\gamma})$ and $k_2(\overline{\gamma})$. (3) The first term $[\cdots]'$ of Eq. (70) is composed of two partial terms, and the expressions of the material constants become rather complicated because we consider the Poisson's ratio exactly, which the engineering beam theory ignores.

For beams of a constant cross-section, equations (69) and (70) can be rewritten as

$$\left(GA^*k_0\frac{\partial^2}{\partial x^2} - \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}H^* - \rho A^*\frac{\partial^2}{\partial t^2}\right)W + \left(-GA^*k_1\frac{\partial}{\partial x} + \frac{\nu E}{(1+\nu)(1-2\nu)}B^*\frac{\partial}{\partial x}\right)\psi = -\int_A p_z\mathcal{I}dA \quad (77)$$

$$\left(GA^*k_1\frac{\partial}{\partial x} - \frac{\nu E}{(1+\nu)(1-2\nu)}B^*\frac{\partial}{\partial x}\right)W + \left(-GA^*k_2 + \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}I^*\frac{\partial^2}{\partial x^2} - \rho I^*\frac{\partial^2}{\partial t^2}\right)\psi = \int_A p_x\mathcal{Z}dA \quad (78)$$

We divide the first equation by GA^*k_1, differentiate the second equation once by x, and divide the differentiated equation by EI^*. Furthermore, we define the longitudinal wave velocity c_0, the transverse wave velocity c_{Qi} for the beam material, and the radius of gyration

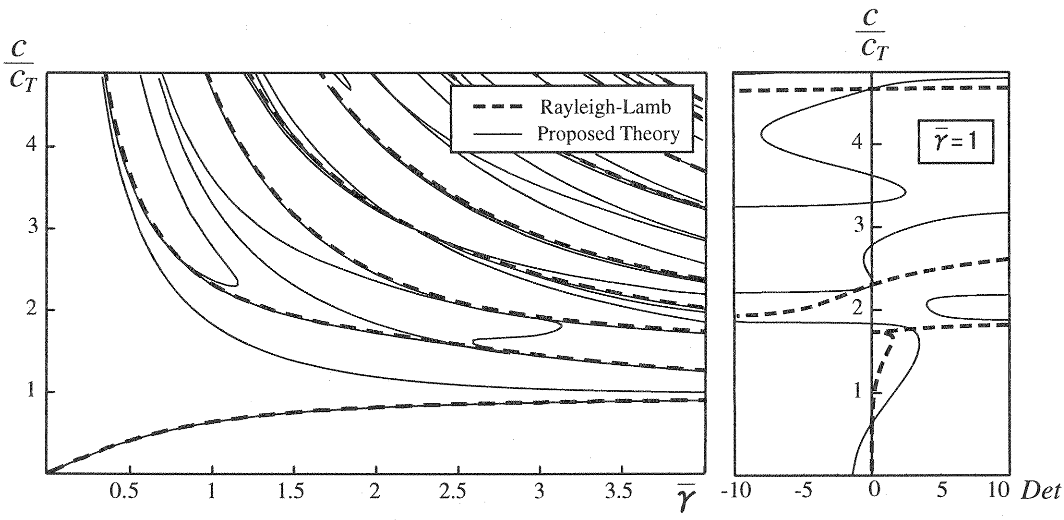

Fig. 3 Phase velocity curve for a beam with rectangular cross-section.

of area for a beam $r^*(\bar{\gamma})$, consider its frequency as follows, and rewrite the equation;

$$\left.\begin{aligned} c_0^2 &= \frac{E}{\rho},\quad c_{Qi}^2(\bar{\gamma}) = \frac{Gk_i(\bar{\gamma})}{\rho} \quad (i=0,\,1,\,2) \\ r^{*2}(\bar{\gamma}) &= \frac{I^*(\bar{\gamma})}{A^*(\bar{\gamma})} \end{aligned}\right\} \tag{79}$$

Then we obtain

$$\left(\frac{k_0}{k_1}\frac{\partial^2}{\partial x^2} - \frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{c_0^2}{c_{Q1}^2}\frac{H^*}{A^*} - \frac{1}{c_{Q1}^2}\frac{\partial^2}{\partial t^2}\right)W + \left(-\frac{\partial}{\partial x} + \frac{\nu}{(1+\nu)(1-2\nu)}\frac{c_0^2}{c_{Q1}^2}\frac{B^*}{A^*}\frac{\partial}{\partial x}\right)\psi = -\frac{\int_A p_z \mathcal{I} dA}{GA^* k_1(\bar{\gamma})} \tag{80}$$

$$\left(\frac{1}{r^{*2}}\frac{c_{Q1}^2}{c_0^2}\frac{\partial^2}{\partial x^2} - \frac{\nu}{(1+\nu)(1-2\nu)}\frac{1}{r^{*2}}\frac{B^*}{A^*}\frac{\partial^2}{\partial x^2}\right)W + \left(-\frac{1}{r^{*2}}\frac{c_{Q2}^2}{c_0^2} + \frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial\psi}{\partial x} = \frac{\int_A p_x' \mathcal{Z} dA}{EI^*(\bar{\gamma})} \tag{81}$$

Three transverse wave velocities are used in the above equations. The first, c_{Q0}, is equal to the transverse wave velocity $c_2(\equiv c_T)$ for a medium as a three-dimensional elastic body. c_{Q1} and c_{Q2} are the transverse wave velocities depending on the shape of a member's cross-section. In equation (13), we defined the longitudinal wave velocity c_1 and the transverse wave velocity c_2 for a medium that makes up the beam member.

We find phase velocity curves for an infinite beam without transverse and longitudinal distributed loads. The deflection displacement $W(x,\bar{\gamma})$ and bending rotation for beam cross-section $\psi(x,\bar{\gamma})$ are assumed to be

$$\left.\begin{aligned} W(x,\bar{\gamma}) &= a\sin\frac{2\pi}{\lambda}(x-ct) \\ \psi(x,\bar{\gamma}) &= b\cos\frac{2\pi}{\lambda}(x-ct) \end{aligned}\right\} \tag{82}$$

where λ is the wavelength, c is the phase velocity, and a and b are the amplitudes of the displacement components. Setting the load term on the right-hand side of the equations of motion (77) and (78) to zero, substituting the assumed equation (82) into the equation of motion, and enclosing the product of the amplitude and trigonometrical function, we obtain the characteristic equation

$$\begin{vmatrix} \frac{k_0}{k_1} - \frac{c^2}{c_{Q1}^2} + \left(\frac{\lambda}{2\pi}\right)^2 \frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{c_0^2}{c_{Q1}^2}\frac{H^*}{A^*} & \left(\frac{\lambda}{2\pi}\right)^2\left(1 - \frac{\nu}{(1+\nu)(1-2\nu)}\frac{c_0^2}{c_{Q1}^2}\frac{B^*}{A^*}\right) \\ \frac{1}{r^{*2}}\frac{c_{Q1}^2}{c_0^2}\left(1 - \frac{\nu}{(1+\nu)(1-2\nu)}\frac{c_0^2}{c_{Q1}^2}\frac{B^*}{A^*}\right) & \frac{1-\nu}{(1+\nu)(1-2\nu)} - \frac{c^2}{c_0^2} + \left(\frac{\lambda}{2\pi}\right)^2\frac{1}{r^{*2}}\frac{c_{Q2}^2}{c_0^2} \end{vmatrix} = 0 \tag{83}$$

as the condition for having a non-trivial solution. This equation formally conforms to the phase velocity equation for Timosheko's beam[8] if we rewrite $k_0/k_1 \to 1$, $c_{Q\alpha} \to c_Q$, $B^* \to 0$, $H^* \to 0$, $r^*(\overline{\gamma}) \to r$ and $(1-\nu)/(1+\nu)(1-2\nu) \to 1$.

From this equation for phase velocity c, we find the phase velocity curves for the infinite beam with a solid rectangular cross-section. They are shown in **Fig. 3** along with the Lamb plate theory. The curves for the Rayleigh-Lamb theory (broken line) and our theory (solid line) fit perfectly even in the higher modes. However, the cause of the extra lines in the curves for our theory must be considered. For example, the relation between the dimensionless phase velocity c/c_T at the dimensionless frequency $\overline{\gamma} = 1.0$ and the value of the determinant Det on the left-hand side of the characteristic equation (83) is plotted in the right-hand portion of **Fig. 3**. The c/c_T axis intercepts are plotted on the left of the figure for clarity. While the broken lines for the Rayleigh-Lamb theory cross the c/c_T axis only once, the solid lines for our theory cross it more than twice. Although one of the intercepts matches perfectly with that of the Rayleigh-Lamb equation, the others are extra roots. Compared with the Rayleigh-Lamb equation, which is represented by a simple summation of two tanh functions, the characteristic equation (83) of our theory has a complicated structure requiring the division and multiplication of functions $A^*(\overline{\gamma})$, $B^*(\overline{\gamma})$ and $H^*(\overline{\gamma})$, which contain phase velocity c. Thus, some sort of rationalization effect generates extra roots.

(10) Radius of gyration of area for Rayleigh surface wave

Whether the medium is a plate or a beam, we confirmed in section **(5)** that waves propagated in media converge to the Rayleigh surface wave at the limit of high frequency ($\overline{\gamma} \to \infty$). Using this fact, we can determine the closed solutions of the radius of gyration of area r_R for the Rayleigh surface wave propagated even in semi-infinite media.

Substituting the basis function (47) into the right-hand side of equation (66) of the dynamic moment of inertia $I^*(\overline{\gamma})$ and sectional area $A^*(\overline{\gamma})$, and integrating from $-h/2$ to $+h/2$ in the direction of beam height, we finally obtain

$$
\begin{aligned}
I^*(\overline{\gamma}) = \left(\frac{h}{1-\nu_2^2}\right)^2 \Bigg\{ & \frac{-k\nu_1 h + \sinh k\nu_1 h}{2k\nu_1 \sinh^2 k\nu_1 h/2} + \frac{(1+\nu_2^2)^2}{4}\frac{-k\nu_2 h + \sinh k\nu_2 h}{2k\nu_2 \sinh^2 k\nu_2 h/2} \\
& - \frac{\nu_1(1+\nu_2^2)}{k(\nu_1^2-\nu_2^2)}\frac{\sinh k(\nu_1+\nu_2)h/2 - \sinh k(\nu_1-\nu_2)h/2}{\sinh k\nu_1 h/2 \cdot \sinh k\nu_2 h/2} \\
& + \frac{\nu_2(1+\nu_2^2)}{k(\nu_1^2-\nu_2^2)}\frac{\sinh k(\nu_1+\nu_2)h/2 + \sinh k(\nu_1-\nu_2)h/2}{\sinh k\nu_1 h/2 \cdot \sinh k\nu_2 h/2} \Bigg\}
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
A^*(\overline{\gamma}) = \left(\frac{1+\nu_2^2}{1-\nu_2^2}\right)^2 \Bigg\{ & \frac{k\nu_1 h + \sinh k\nu_1 h}{2k\nu_1 \cosh^2 k\nu_1 h/2} + \frac{4}{(1+\nu_2^2)^2}\frac{k\nu_2 h + \sinh k\nu_2 h}{2k\nu_2 \cosh^2 k\nu_2 h/2} \\
& - \frac{4\nu_1}{k(1+\nu_2^2)(\nu_1^2-\nu_2^2)}\frac{\sinh k(\nu_1+\nu_2)h/2 + \sinh k(\nu_1-\nu_2)h/2}{\cosh k\nu_1 h/2 \cdot \cosh k\nu_2 h/2} \\
& + \frac{4\nu_2}{k(1+\nu_2^2)(\nu_1^2-\nu_2^2)}\frac{\sinh k(\nu_1+\nu_2)h/2 - \sinh k(\nu_1-\nu_2)h/2}{\cosh k\nu_1 h/2 \cdot \cosh k\nu_2 h/2} \Bigg\}
\end{aligned} \tag{85}
$$

for a solid rectangular cross-section with unit width in the case of the first mode. We make the results dimensionless using the moment of inertia $I \equiv 1 \cdot h^3/12$ and the cross-sectional area $A \equiv 1 \cdot h$. The results are shown in **Fig. 4**(a) as functions of frequency $\overline{\gamma}$. The medium is a Poisson material ($\nu = 1/4$). Both curves start from 1 and a gradient of 0 (horizontal line) and decrease slowly as the frequency $\overline{\gamma}$ increases. At the limit of infinite frequency, both the dynamic moment of inertia $I^*(\overline{\gamma})$ and cross-sectional area $A^*(\overline{\gamma})$ gradually approach zero. Thus, the limit of the dynamic radius of gyration of area $r^*(\overline{\gamma})$ defined in equation (79.c) becomes $0/0$, however the results of the calculation converge to the following finite value:

$$
\lim_{\overline{\gamma}\to\infty} \frac{r^{*2}(\overline{\gamma})}{r^2} = 3\,\frac{4\overline{\nu}_2(\overline{\nu}_1+\overline{\nu}_2) - 8\overline{\nu}_1\overline{\nu}_2(1+\overline{\nu}_2^2) + \overline{\nu}_1(\overline{\nu}_1+\overline{\nu}_2)(1+\overline{\nu}_2^2)^2}{4\overline{\nu}_1(\overline{\nu}_1+\overline{\nu}_2) - 8\overline{\nu}_1\overline{\nu}_2(1+\overline{\nu}_2^2) + \overline{\nu}_2(\overline{\nu}_1+\overline{\nu}_2)(1+\overline{\nu}_2^2)^2} \tag{86}
$$

where we define

$$
\overline{\nu}_\alpha = \left(1 - \frac{c_R^2}{c_\alpha^2}\right)^{1/2} \quad (\alpha = 1,\ 2) \tag{87}
$$

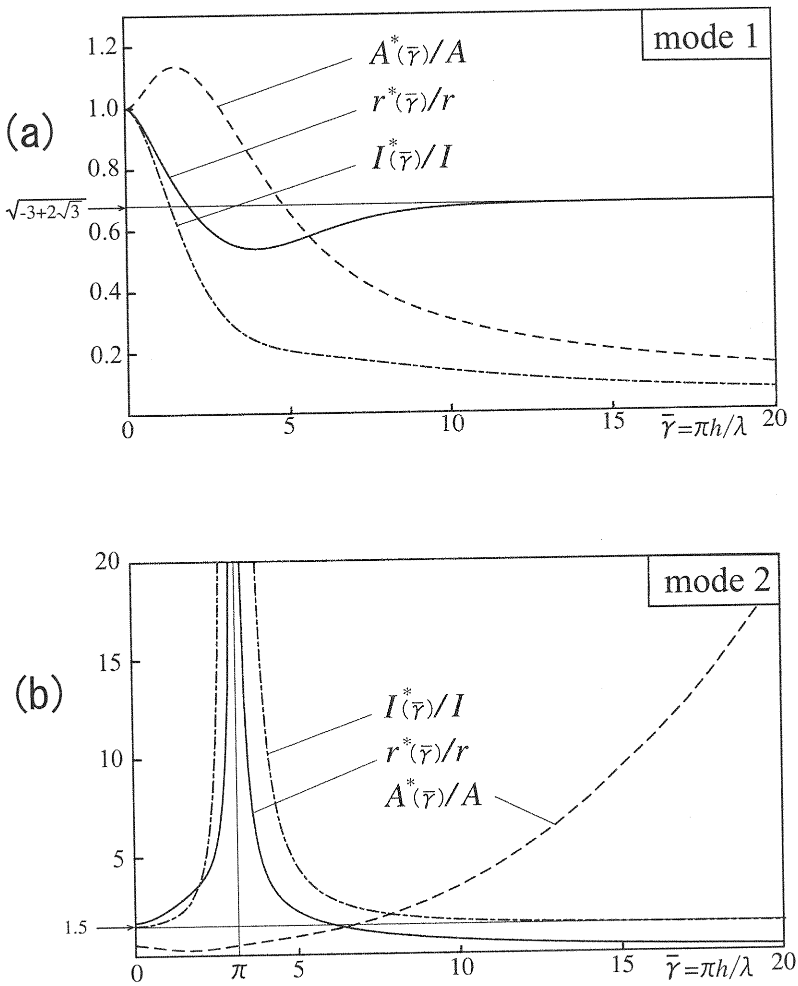

Fig. 4 Relation between frequency and $A^*(\overline{\gamma})$, $I^*(\overline{\gamma})$ and $r^*(\overline{\gamma})$ for $\nu=1/4$.

according to equation (20.b). In the case of a Poisson material with $\nu=1/4$, we have

$$\frac{c_1^2}{c_2^2}=3,\quad \frac{c_R^2}{c_2^2}=2-\frac{2}{\sqrt{3}},\quad \frac{c_R^2}{c_1^2}=\frac{1}{3}\left(2-\frac{2}{\sqrt{3}}\right)$$

Substituting the above into equations (86) and (87), we obtain

$$\overline{\nu}_1=\frac{1}{3}\sqrt{3+2\sqrt{3}}\simeq 0.847\,487$$

$$\overline{\nu}_2=\sqrt{-1+\frac{2}{\sqrt{3}}}\simeq 0.393\,320$$

$$\lim_{\overline{\gamma}\to\infty}\frac{r^{*2}(\overline{\gamma})}{r^2}=-3+2\sqrt{3}\simeq 0.464\,102$$

Namely, the dynamic radius of gyration of area decreases as frequency increases, and finally decreases to about 68% of the static value according to the following equation:

$$\lim_{\overline{\gamma}\to\infty}\frac{r^*(\overline{\gamma})}{r}=\sqrt{-3+2\sqrt{3}}\simeq 0.681\,250$$

The shape of the curves we find here suggests that three functions $A^*(\overline{\gamma})$, $I^*(\overline{\gamma})$ and $r^*(\overline{\gamma})$ assumed in reference 5) and 10) for the new Mindlin plate theory[5] and for the arbitrarily higher order plate theory[10], respectively, are valid.

In the case of modes higher than the second, we can calculate the values of the cross-section in the same way. We show only the case of the second mode in **Fig. 4**(b) as an example. The dynamic moment of inertia $I^*(\overline{\gamma})$ converges to

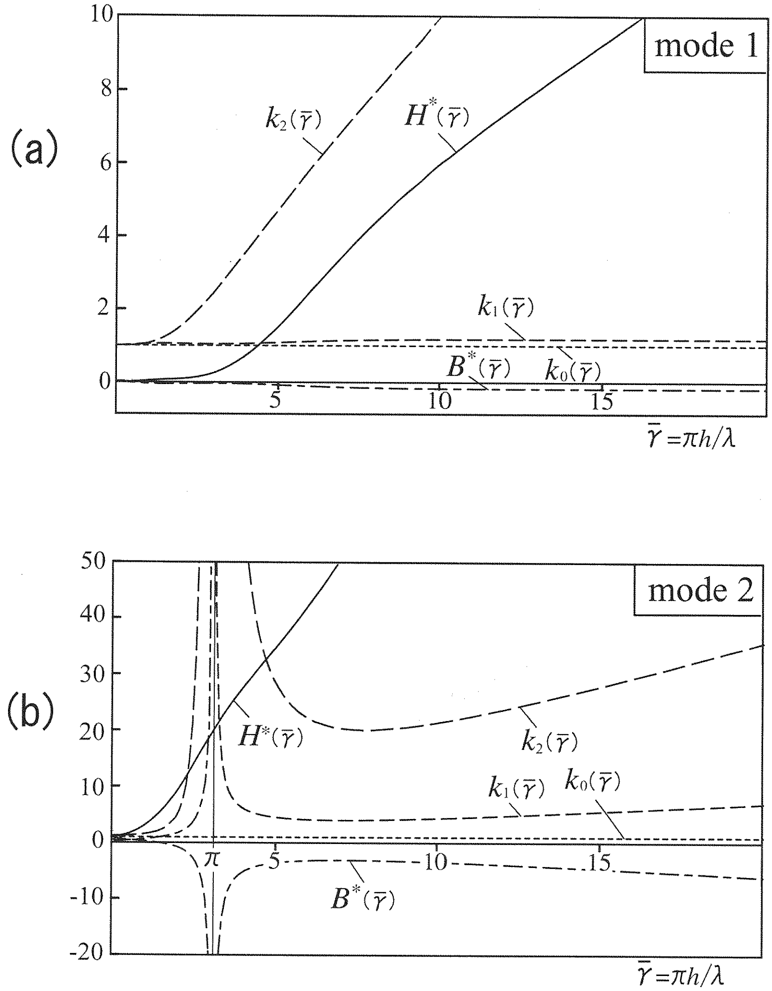

Fig. 5 Relation between frequency and $B^*(\overline{\gamma})$, $H^*(\overline{\gamma})$, $k_0(\overline{\gamma})$, $k_1(\overline{\gamma})$ and $k_2(\overline{\gamma})$ for $\nu = 1/4$.

a constant value at the limit of infinite frequency, and the dynamic cross-sectional area $A^*(\overline{\gamma})$ diverges to infinity at the limit of infinite frequency. Therefore, the dynamic radius of gyration of area $r^*(\overline{\gamma})$ converges to zero at the limit of infinite frequency. This behavior is common in all the higher modes. This fact indicates that only the first mode survives as a Rayleigh surface wave and all the modes higher than the second disappear at the limit of infinite frequency, proving that the phenomenon actually exists.

As the denominator of the basis function $\mathcal{Z}(z, \overline{\gamma})$ for axial warping of the beam becomes zero at $\overline{\gamma} = \pi$, the dynamic moment of inertia $I^*(\overline{\gamma})$ and the dynamic radius of gyration of area $r^*(\overline{\gamma})$ become infinite at that point. However, the axial displacement $u(x, z, \overline{\gamma})$ itself changes slowly and does not exhibit singularity because the longitudinal function $\psi(x, \overline{\gamma})$ becomes zero at that point.

For reference, the dependence of the other functions on frequency is given in **Fig. 5** for the cross-sectional values $B^*(\overline{\gamma})$ and $H^*(\overline{\gamma})$, and the three shear correction factors $k_0(\overline{\gamma})$, $k_1(\overline{\gamma})$ and $k_2(\overline{\gamma})$ in the first and second modes.

4. CONCLUSIONS

We have derived a beam equation of motion using the displacement field as a basis function, with consideration of the Lamb-plate frequency, giving the exact solution of elastic theory. We have confirmed the correctness of the equation by comparing the phase velocity curves for the Lamb plate theory with those for our theory. At the same time, we find that superfluous solutions are produced as well as the true solution in the process of converting the equation to the beam equation.

We have obtained a dynamic moment of inertia and a dynamic cross-sectional area, which are the coefficients of the beam equation of motion, calculated the dynamic radius of gyration of area from the dynamic moment of inertia and cross-sectional area, and investigated the dependence of the first three items on frequency. The results are as follows:

(1) In the first mode, the dynamic moment of inertia and the cross-sectional area gradually approach zero at the limit of infinite frequency. However, the dynamic radius of gyration of area, which is the square of the ratio of the dynamic moment of inertia to cross-sectional area, converges to a constant that is slightly less than 70% of the static value. The phase velocity converges to the propagation velocity of the Rayleigh surface wave at infinite frequency.

(2) In modes higher than the second, the dynamic moment of inertia converges to a constant other than 0, whereas the dynamic cross-sectional area diverges to positive infinity at infinite frequency. As a result, the dynamic radius of gyration of area in this case converges to zero, and the modes higher than the second disappear.

(3) From these results, whether considering a beam, a plate, or a semi-infinite elastic body, we can confirm the existence of the physical phenomenon in which motion is governed only by Rayleigh surface waves in the range of high frequency.

Whereas we used a solid rectangular cross-section of the Lamb-plate type in this paper, the generality holds for a beam cross-section because regardless of the shape of the cross-section, the basis function for static warping of the beam is $\mathcal{Z}(z,\overline{\gamma})=z$, and the basis function for in-plane deflection is $\mathcal{I}(z,\overline{\gamma})=1$. Another reason is that in the high-frequency range, regardless of the shape of the cross-section, these basis functions become the states of the Rayleigh surface wave. In the intermediate frequency region, the distinct characteristics for each beam cross-section must appear, however in a compact cross-section, the phase velocity of intermediate frequency becomes a monotonic decreasing curve that is only negligibly influenced by frequency changes.

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