ANALYSIS OF LOCALIZATION IN A STRUCTURE
BASED ON THE THERMODYNAMICS OF IRREVERSIBLE PROCESS

Junya INOUE[1] and Hideyuki HORII[2]

[1]Dr.Eng., Research Associate, Dept. of Civil Eng., University of Tokyo (7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656 JAPAN)
[2]Member of JSCE, Ph.D., Professor, Dept. of Civil Eng., University of Tokyo (7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656 JAPAN)

This paper presents a method for analysis of strain localization problems which is supported by the thermodynamics of irreversible processes. Stability and bifurcation criteria and a generalized localization analysis method are discussed at the level of structural mechanics. A one dimensional cracking model is examined as a simple example of cracking localization phenomena to show the framework of our method. The method is applied to a practical example of cracking localization in a concrete beam during a bending test. From the results, the importance of the method presented in this paper for a quasi-static strain localization phenomenon in a structure is clarified.

Key Words : *strain localization, thermodynamics, irreversible process, concrete structure*

1. INTRODUCTION

Strain localization behavior, in which homogeneous deformation or damage is replaced by deformation or damage which is concentrated in a relatively small zone in the course of failure of materials, falls into the category of bifurcation phenomena. This type of phenomenon is observed in various materials: structural metals [1], rocks [2], concrete [3], granular materials [4] and so on. For example, **Fig.1** shows the maximum shear strain rate observed in a plane strain biaxial compression test of soft rock [5].

The theoretical description of strain localization is reviewed in an early work [6]. In this study, Hill has presented equations for stationary wave in a solid; for vanishing wave velocity, these are interpreted as a condition for shear band localization. Hill and Hutchinson [7] have studied the strain localization in a rectangular block under plane strain deformation, and showed that the bifurcation into a localized shear band is not possible until the equations governing incremental equilibrium lose ellipticity. Rudnicki and Rice [8]
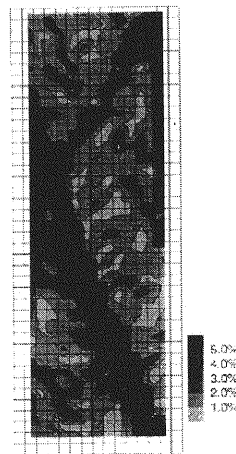
Fig. 1 Maximum shear strain rate in plain-strain compression test on soft rock

have provided a general mathematical theory for analysis of shear band localization. It is shown that both a vertex-like structure of subsequent yield surfaces and non-normality of the plastic strain rate vector to the current yield surface

strongly affect localization; see also reference[9].

To make it possible to follow material behavior into the post-localization range, a number of numerical methods have been provided. For example,

(a) nonlocal models in which either stress or strain is defined as an integral value of the other over a finite material domain (integral limiter) [10].

(b) strain gradient models in which strain is defined in terms of derivatives of order higher than one (gradient limiter) [11],[12].

(c) generalized continuum theories in which an internal material length is originally introduced by considering micro-structures in the material, such as the Cosserat continuum theory.

(d) modeling materials with rate-dependent constitutive relation (rate limiter) [13],[14].

(e) limiting the minimum size of finite elements by considering the fact that constitutive relations are originally obtained by observation of relations between macroscopic quantities in finite size specimens [15],[16].

(f) providing a material model with a descending gradient of stress-strain relation in the post-peak regime as a function of mesh size, such as strain softening materials.

See also references[17],[18]. These theoretical studies clarified which features of the constitutive relation *locally* initiate strain localization; the numerical techniques enabled us to analyze the regime of localization phenomena after initiation of localization, possibly with a help of assumed imperfections either in the boundary conditions or in the material properties.

In addition to the approaches based on constitutive modeling, there are also models which are based on micro-mechanics. Shi and Horii [19] showed that a physical mechanism of localization arises by introducing interaction effects among micro defects during evolution of the micro defects. Okui and Horii [20] investigated a cracking localization phenomenon in a rock mass by taking into account the interaction effects of discrete micro defects by introducing a pseudo-traction. In their study, it was revealed that the non-local constitutive equation can be derived by considering micro-mechanical behavior and, more over,

that a localization mode observed in a rock mass under triaxial loading is one of many possible mechanical equilibrium solutions for the boundary value problem.

Now that it has been demonstrated that localization phenomena can be represented beyond bifurcation by eliminating the loss of ellipticity of constitutive equation in some way, and that a localization mode is one of many possible solutions of a boundary value problem, our principal objective in this paper is to describe a criterion which determines a localization mode in a structural element without introduction of imperfections. In the next section, an elementary boundary value problem is solved to demonstrate the difficulty of finding a post localization path and it is further discussed from thermodynamical point of view in the third section. To clarify the general criterion for localization of deformation in a structure, the thermodynamic theory of irreversible processes [21] is introduced. The method is illustrated by a simple example of one-dimensional cracking localization in the fourth section, and cracking localization phenomena in a concrete beam are analyzed as a practical example.

## 2. DIFFICULTY OF SOLVING A CRACK LOCALIZATION PROBLEM

In this section, the discussion is made for a simple crack opening problem with two crack elements as an example, mainly to describe the general difficulty of solving crack localization problem.

(1) Model Definition and Equilibrium Solution

Let us consider a model with two crack elements which is connected by a single spring element with spring constant k as shown in **Fig.2** where $\bar{u}$ is a prescribed displacement while α_1 and α_2 are the crack opening displacement of the crack elements. For simplicity, it is supposed that the crack elements starts opening when the applied force f reaches the capacity f_c, and that the resistance force changes as a linear function of crack opening displacement α with modulus a,
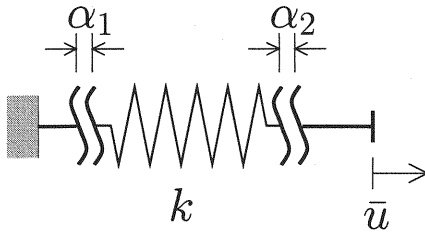
Fig. 2 One dimensional model with springs and crack elements
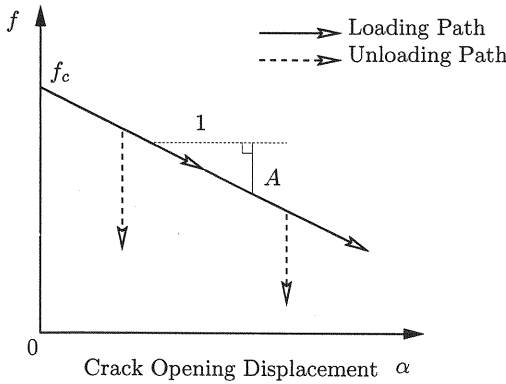
Fig. 3 Tension softening curve representing relationship between crack opening displacement and transmitted force across crack element

as shown in Fig.3:

$$f(\alpha) = f_c + a\alpha. \qquad (1)$$

The crack opening displacement α is a monotonically increasing function and its increment is supposed to be non-negative. When a is negative, the crack element shows a softening behavior. As shown in **Fig.3**, this crack element has an opportunity to choose either loading or unloading path everywhere along the loading path as the applied force decreases: one is to follow the solid line, loading path, and another is to follow the dashed line, unloading path. Our problem is to find a stable path of $\{\alpha_1, \alpha_2\}$ for a prescribed displacement $\bar{u}$ at loading point. The given displacement $\bar{u}$ is assumed to be monotonically increasing at constant speed. Before reaching a crack opening criteria, $\bar{u} < f_c/k$, the system behaves as though a single spring element. For simplicity, a loading parameter λ by which the prescribed

displacement $\bar{u}$ is expressed:

$$\bar{u} = \frac{f_c}{k} + \lambda \qquad (2)$$

is introduced. In quasi-static equilibrium, momentum conservation requires that the net force on each crack must vanish; thus accordingly, for $\lambda > 0$

$$f(\alpha_1) = k(\lambda - \alpha_1 - \alpha_2) + f_c \qquad (3)$$
$$f(\alpha_2) = k(\lambda - \alpha_2 - \alpha_2) + f_c \qquad (4)$$

It follows by substituting eq.(1) that

$$\alpha_1 = \alpha_2 = \frac{k}{2k+a}\lambda \qquad (5)$$

which represents a distributed crack opening mode.

(2) Stability of Equilibrium Solution

Let us consider the stability of the solution. A stability of equilibrium state is determined by a work required to make a small perturbation, that is, if the system is stable an external work has to be applied to make small change and the work is positive, otherwise a small perturbation will make energy flow from an internal system to an external system and the work becomes negative.

In this simple case of crack elements the work is defined as

$$\Delta W = \frac{1}{2}a\delta\alpha_2^2 + \frac{1}{2}a\delta\alpha_2^2 + \frac{1}{2}k(\delta\alpha_1 + \delta\alpha_2)^2 \quad (6)$$

Hence, if $a < 0$ the work ΔW is always positive and the system is stable all the time, on the contrary, if $a > 0$ the work can be negative for certain combination of α_i so that the system becomes unstable and a small disturbance will grow naturally. However, from this discussion one can not determine which combination would be the optimum, that is, in the reality what happens when the equilibrium stability condition fails.

(3) Engineering Problem

As shown in the previous sections, a crack opening behavior of crack elements can be solved and the stability of the solution can be examined within the framework of the quasi-statical thermodynamics, and thus the onset of crack localization can be predicted from the discussion. However, what we, engineers, are really interested in

is a behavior of a structure in a post localization *process*, because of the fact that without knowing the behavior at post localization it is almost impossible to predict a stability and a strength of a structure. In order to determine the post localization *process*, the classical stability condition of the thermodynamics of *reversible* process, in which only the stability of a *state* is concerned, is not sufficient. In the following sections, we will discuss the procedure to find a optimum solution for the post localization *process*.

3. APPROACH FROM THERMODYNAMICS

To provide a basis for later discussion, the essential features of the thermodynamics of continuous media are recalled in this section.

(1) Basic assumptions

Throughout this paper, all discussion is based on the following assumptions:
(a) quasi-static rate-independent elastoplasticity process,
(b) small perturbation hypothesis,
(c) principle of local equilibrium (existence of entropy and free energy density), and
(d) uniform temperature.

(2) Thermodynamics of Local Equilibrium

Suppose a mechanical system is in an equilibrium state, in which a body force f and a traction t are acting on the system and the specific entropy, internal energy, and Helmholtz free energy are denoted by s, e, and φ, respectively. The first and second laws of thermodynamics can be stated as:

First law

$$\dot{E} = \dot{W} + \dot{Q} \qquad (7)$$

where

$$\dot{E} = \int_B \rho \dot{e} dV$$

$$\dot{W} = \int_B \rho f \cdot \dot{u} dV + \int_{\delta B} t \cdot \dot{u} dA$$

$$\dot{Q} = -\int_{\delta B} q \cdot n dA$$

Second law

$$\theta \dot{S} \geq \dot{Q} \qquad (8)$$

where θ is a thermodynamic temperature and

$$S = \int_B \rho s dV \qquad (9)$$

A local form of eq.(8) is

$$\rho \dot{s} + \nabla \cdot (q/\theta) \geq 0 \qquad (10)$$

which is known as the Gibbs-Duhem inequality.

For a general case of mechanical behavior, which involves frictional thermoelastic or some other irreversible process, the Gibbs-Duhem inequality is reduced to a form which involves only an internal variable [22] which represents internal rearrangement. For an irreversible process, in which mechanical equilibrium is always satisfied, the work rate $\dot{W}$ is rewritten in the form

$$\dot{W} = \int_B \rho f \cdot \dot{u} dV + \int_{\delta B} t \cdot \dot{u} dA$$
$$= \int_B \sigma_{ij} \dot{u}_{i,j} dA \qquad (11)$$

From the principle of local equilibrium, the free energy density φ can then be defined by

$$\varphi = e - \theta s \qquad (12)$$

$$\rho \theta \dot{s} = \rho \dot{e} - \rho (\dot{\varphi} + s \dot{\theta}) \qquad (13)$$

$$\rho \dot{\varphi} = \rho \frac{\partial \varphi}{\partial u_{i,j}} \dot{u}_{i,j} + \rho \frac{\partial \varphi}{\partial \alpha} \dot{\alpha} + \rho \frac{\partial \varphi}{\partial \theta} \dot{\theta} \qquad (14)$$

where, in analogy with the discussion by Nguyen [23], u and α are a displacement with respect to the reference configuration and an internal variable [22] which represents an irreversible process, respectively. The associated stress and the associated thermodynamic force or affinity are defined by

$$\sigma_{ij} = \rho \frac{\partial \varphi}{\partial u_{i,j}}, \quad A = -\rho \frac{\partial \varphi}{\partial \alpha} \qquad (15)$$

From eqn.(7), (8),(11), (13), and (14), the Gibbs-Duhem inequality becomes

$$A \cdot \dot{\alpha} + \theta q \cdot \nabla(1/\theta) \geq 0. \qquad (16)$$

Because the temperature is assumed to be uniform in our system, the inequality becomes

$$\sigma = A \cdot \dot{\alpha} \geq 0. \qquad (17)$$

where σ is the local entropy production rate. This final inequality means that, for an irreversible mechanical process, the dissipation must always be positive.

(3) Stability of Irreversible Process

Under the assumption of local equilibrium, an irreversible process must satisfy the local equilibrium stability condition and the Lyapounov stability condition[21]. The Lyapounov stability condition is the condition which should be satisfied for a state function of a stable system, the sign of which is fixed to either positive or negative. That is, suppose one can define a state function P which is always positive, the Lyapounov stability condition is

$$P\frac{dP}{dt} < 0. \tag{18}$$

The local equilibrium stability condition is specified by

$$\delta^2 s < 0 \tag{19}$$

or, in global form,

$$\int_B \delta^2 s dV = \delta^2 S < 0 \tag{20}$$

where δ^2 denotes second order differential. By applying the Lyapounov stability condition to the entropy production rate, it follows that

$$\frac{d\sigma^2}{dt} \leq 0 \tag{21}$$

or, because of eqn.(17),

$$\frac{d\sigma}{dt} \leq 0 \tag{22}$$

The global form of the inequality is

$$\frac{DP}{Dt} \leq 0 \tag{23}$$

where

$$P = \int_B \sigma dV \tag{24}$$

If a certain process for a mechanical system is completely characterized by $(u_{i,j}, \boldsymbol{\alpha})$, the local equilibrium stability condition eqn.(19) is explicitly written as

$$\delta^2 s = -\frac{1}{2}\left(\delta\boldsymbol{\alpha}^T\frac{\partial^2\varphi}{\partial\boldsymbol{\alpha}\partial\boldsymbol{\alpha}}\delta\boldsymbol{\alpha} + 2\delta\boldsymbol{\alpha}^T\frac{\partial^2\varphi}{\partial\boldsymbol{\alpha}\partial u_{i,j}}\delta u_{i,j}\right.$$
$$\left. + \delta u_{i,j}^T\frac{\partial^2\varphi}{\partial u_{i,j}\partial u_{i,j}}\delta u_{i,j}\right) < 0 \tag{25}$$

where $\delta u_{i,j}$ and $\delta\boldsymbol{\alpha}$ denote arbitrary perturbations from the current state which is represented by $(u_{i,j}, \boldsymbol{\alpha})$. This inequality must be satisfied for any process, either equilibrium or nonequilibrium. The Lyapounov stability condition eqn.(23) becomes

$$\frac{D}{Dt}\int_B \boldsymbol{A}\cdot\dot{\boldsymbol{\alpha}}dV \leq 0 \tag{26}$$

Because our discussion is limited to a quasi-static mechanical process, it is supposed to be close to an equilibrium state even though the process is irreversible. Hence, the Onsager reciprocity relation[24], which is proved to exist between the affinity $\boldsymbol{A}$ and the internal variable $\boldsymbol{\alpha}$ in a process close to an equilibrium by the statistical thermodynamics, is defined by introducing a phenomenological coefficient $\boldsymbol{L}$ as

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{L}\cdot\boldsymbol{A} \tag{27}$$

where

$$\boldsymbol{L}^T = \boldsymbol{L} \tag{28}$$

By substituting eqn.(27) to eqn.(26),

$$\frac{D}{Dt}\int_B \boldsymbol{A}\cdot\dot{\boldsymbol{\alpha}}dV = \frac{D}{Dt}\int_B L_{ij}A_iA_jdV$$
$$= 2\int_B \dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{\alpha}}dV$$
$$\leq 0 \tag{29}$$

which is locally

$$\dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{\alpha}} \leq 0 \tag{30}$$

where the equality is satisfied only at steady state. If the associated force $\boldsymbol{A}$ is restricted to be in a convex C, which is called elasticity domain, eqn.(30) is equivalent to the inequality:

$$(\boldsymbol{A} - \boldsymbol{A}^*)\cdot\dot{\boldsymbol{\alpha}} \geq 0, \quad \forall\boldsymbol{A}^* \in C \tag{31}$$

which was termed the *maximal-dissipation principle* [25].

From the definition of the associated force $\boldsymbol{A}$, eqn.(15), and the definition of the free energy density eqn.(14), it is concluded that

$$\dot{\boldsymbol{A}} = -\rho\frac{d}{dt}\left(\frac{\partial\varphi}{\partial\boldsymbol{\alpha}}\right)$$
$$= -\rho\left(\frac{\partial^2\varphi}{\partial\boldsymbol{\alpha}\partial u_{i,j}}\dot{u}_{i,j} + \frac{\partial^2\varphi}{\partial\boldsymbol{\alpha}^2}\dot{\boldsymbol{\alpha}}\right) \tag{32}$$

Thus, the local expression of the Lyapounov stability condition for an irreversible process becomes

$$-\dot{\boldsymbol{A}} \cdot \dot{\boldsymbol{\alpha}} = \rho(\dot{\boldsymbol{\alpha}}^T \frac{\partial^2 \varphi}{\partial \boldsymbol{\alpha} \partial u_{i,j}} \dot{u}_{i,j} + \dot{\boldsymbol{\alpha}}^T \frac{\partial^2 \varphi}{\partial \boldsymbol{\alpha}^2} \dot{\boldsymbol{\alpha}}) \geq 0 \quad (33)$$

which is identically satisfied as long as eqn.(25) applies.

4. GENERAL FRAMEWORK

Our objective of the paper is to present a way to determine the evolution of deformation or damage in the course of failure of structures, especially when a homogeneous process becomes unstable. In this section, we will discuss the evolution law of irreversible process at the level of structural mechanics, and strain localization process in a structure is, further, described.

(1) Evolution of Irreversible Process

The concept of deformation of the structure is generalized to represent the process in which the displacement $\boldsymbol{u}(t)$ and the internal variable $\boldsymbol{\alpha}(t)$ are controlled, so that both the condition of local equilibrium and the Lyapunov stability condition are satisfied at each instance of time t for a prescribed boundary displacement $\boldsymbol{\lambda}(t)$. The total free energy Φ and and the entropy production rate P for the system are defined by

$$\Phi = \int_B \rho\varphi dV - \int_B \rho\boldsymbol{f} \cdot \boldsymbol{u} dV - \int_{\delta B} \boldsymbol{t} \cdot \boldsymbol{u} dA \quad (34)$$
$$P = \int_B \sigma dV$$
$$= \boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}} \quad (35)$$

where

$$\boldsymbol{A} = -\frac{\partial \Phi}{\partial \boldsymbol{\alpha}} \quad (36)$$

which represents a governing equation for the plastic deformation.

Following the discussion of Nguyen [23], if the total free energy is a positive definite functional of the displacement $\boldsymbol{u}$, so that the system is geometrically stable, then the mechanical equilibrium condition provides a solution for $\boldsymbol{u}$ in the form of a functional of the internal variable $\boldsymbol{\alpha}$ and the imposed boundary displacement $\boldsymbol{\lambda}$, represented by

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \quad (37)$$

It follows that the total free energy $\Phi(\boldsymbol{u}, \boldsymbol{\alpha}, \boldsymbol{\lambda})$ can be re-interpreted as a functional of internal variable and imposed boundary loading, that is,

$$\Phi^*(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = \Phi(\boldsymbol{u}(\boldsymbol{\alpha}, \boldsymbol{\lambda}), \boldsymbol{\alpha}, \boldsymbol{\lambda}) \quad (38)$$

Thus, the condition of stability of instantaneous equilibrium state requires that

$$\delta^2 S = -\frac{1}{2} \left(\delta\dot{\boldsymbol{\alpha}} \frac{\partial^2 \Phi^*}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}} \delta\dot{\boldsymbol{\alpha}} + 2\delta\dot{\boldsymbol{\alpha}} \frac{\partial^2 \Phi^*}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\lambda}} \dot{\boldsymbol{\lambda}} \right) \Delta t^2$$
$$< 0 \quad (39)$$

where Δt is an arbitrary infinitesimally small time increment, and $\dot{\boldsymbol{\alpha}}$ and $\dot{\boldsymbol{\lambda}}$ are the time rates of change of the internal variable $\boldsymbol{\alpha}$ and the loading parameter $\boldsymbol{\lambda}$, respectively. Thus, if the functional

$$I(q) = q \frac{\partial^2 \Phi^*}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}} q + 2q \frac{\partial^2 \Phi^*}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\lambda}} \dot{\boldsymbol{\lambda}} \quad (40)$$

is introduced, the result (39) implies that $I(q)$ is minimum when $q = \dot{\boldsymbol{\alpha}}$, the actual rate of change of the internal variable.

The Lyapunov stability condition for the process is obtained by assuming the validity of the Onsager reciprocity condition which, in this case, implies that

$$\frac{D}{Dt} \int_B \sigma dV = 2\dot{\boldsymbol{A}} \cdot \dot{\boldsymbol{\alpha}}$$
$$= -2 \frac{d}{dt} \left(\frac{\partial \Phi^*}{\partial \boldsymbol{\alpha}} \right) \cdot \dot{\boldsymbol{\alpha}}$$
$$= -2 \left(\frac{\partial^2 \Phi^*}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}} \dot{\boldsymbol{\alpha}} + \frac{\partial^2 \Phi^*}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\lambda}} \dot{\boldsymbol{\lambda}} \right) \cdot \dot{\boldsymbol{\alpha}}$$
$$\leq 0 \quad (41)$$

which is satisfied as long as eqn.(39) is satisfied. This simple outcome is perhaps the reason why the study of the thermodynamics of irreversible processes is focused mainly on nonlinear phenomenon.

Hence, it is concluded that, if a structure is geometrically stable, an irreversible deformation of the structure should be the process which minimize the functional $I(\dot{\boldsymbol{\alpha}})$, and that such a process is always stable as far as the Onsager reciprocity relation can be applied.

(2) Bifurcation: strain localization in structure

In the present paper, a strain localization is considered as the process in which a homogeneous

deformation becomes unstable and a localized deformation is selected as a stable path.

During a certain process, if the functional $I(\dot\alpha)$ is positive definite for any $\dot\alpha$, then the solution which makes $I(\dot\alpha)$ minimum is unique, and hence the process of the system is simply defined by

$$\dot\alpha = -\left(\frac{\partial^2\Phi^*}{\partial\alpha\partial\alpha}\right)^{-1}\frac{\partial^2\Phi^*}{\partial\alpha\partial\lambda}\dot\lambda \qquad (42)$$

which is so-called fundamental solution of the system, and is considered as a homogeneous defromation. However, $I(\dot\alpha)$ can lose the positive definiteness under a certain condition, and when the functional $I(\dot\alpha)$ is not positive definite, the uniqueness of the solution which makes $I(\dot\alpha)$ minimum is no longer guaranteed, and a bifurcation is supposed to take place. This state is considered as an onset of a strain localization. After the fundamental solution of a system becomes unstable, the process will folllow one of the possible paths which make $I(\dot\alpha)$ locally minimum. Of course, if the minima of I does not exist for the admissible $\dot\alpha$ of our assumption, then the system will lose stability in a dynamic manner, which is beyond the scope of our approach, otherwise such a bifurcation path is always stable as far as the Onsager reciprocity relation can be applied.

Note: Within the assumptions adopted by the present paper, the solution might be equivalent to what can be derived by the *simplified abstract formulation* [23], which adapts the maximum dissipation principle as the fundamental concept, because of the fact that the Onsager reciprocity leads one of the stability conditions of the non-equilibrium thermodynamics to the same principle, as shown in eqn.(31). More over, since our discusion has been limited to the process which is very close to the global equilibrium, the final criteria for the localization and the stability of the bifurcation path has the identical form with the theory of stability of inelastic structure [26], which is based on the classical thermodynamics of equilibrium. However, further discussion of stability of more general irreversible process, such as rate-dependent plasticity, might be done only from the view point discribed in this paper, but not from the *simplified abstract formulation* nor from the classical thermodynamics of equilibrium.

5. ONE DIMENSIONAL EXAMPLE WITH CRACK ELEMENTS

In this section cracking localization phenomena in one dimensional model is examined with the method described in the last section. The same two-crack problem as the one which is described in section 2 is solved.

(1) Cracking Localization Analysis

For the two-crack model, the plastic potential [27] , which represents the reversible energy stored by modification of the internal structure, is given by

$$\Phi^p(\alpha) = \frac{1}{2}a\alpha^2 \qquad (43)$$

Then, the free energy of the structure is given by

$$\Phi(\alpha_1,\alpha_2) = \frac{1}{2}a\alpha_2^2 + \frac{1}{2}a\alpha_2^2 + \frac{1}{2}k(\lambda - \alpha_1 - \alpha_2)^2 \quad (44)$$

where α_1 and α_2 correspond to the crack opening displacement of each crack. The functional I, which is defined by eq.(40), is given by

$$I(\dot\alpha) = (k+a)(\dot\alpha_1^2 + \dot\alpha_2^2) + 2k\dot\alpha_1\dot\alpha_2$$
$$-2k(\dot\alpha_1 + \dot\alpha_2)\dot\lambda \qquad (45)$$

The fundamental solution of the system is

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial I}{\partial\dot\alpha_1} = (k+a)\dot\alpha_1 + k\dot\alpha_2 - k\dot\lambda = 0 \\[2mm] \dfrac{1}{2}\dfrac{\partial I}{\partial\dot\alpha_2} = k\dot\alpha_1 + (k+a)\dot\alpha_2 - k\dot\lambda = 0 \end{cases} \quad (46)$$

$$\rightarrow \dot\alpha_1 = \dot\alpha_2 = \frac{k}{2k+a}\dot\lambda \quad (47)$$

Since the eigenvalue of the second derivative of the functional I is $2k + a$ and a, the fundamental solution is stable only if $a > 0$, otherwise the system may have two different solutions $(-k < a \leq 0)$ or be completely unstable $(a \leq -k)$. When $-k < a < 0$, the minima of I is either $(\dot\alpha_1, \dot\alpha_2) = (k/(k+A)\dot\lambda, 0)$ or $(\dot\alpha_1, \dot\alpha_2) = (0, k/(k+A)\dot\lambda)$, that is, either left or right crack will be activated. These solutions correspond to the localized solution of the system.

6. PRACTICAL EXAMPLE: CRACKING LOCALIZATION IN CONCRETE BEAM

In this section, more general cracking localization phenomena in concrete structures is examined.
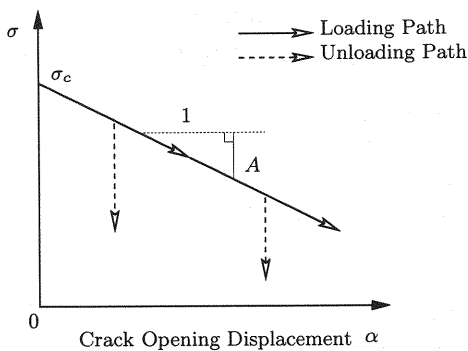
Fig. 4 Tension softening relationship assumed between crack opening displacement and transmitted stress

Fig. 5 Shape functions corresponding to crack opening displacements

(1) Simple model of concrete

For simplicity, nonlinear behavior of material is ignored, so that a crack opening displacement is the only source for nonlinearity of a structure behavior, in this problem. Cracks are initiated when a maximum tensile stress exceeds the tensile strength σ_c, and the crack opening displacement is controlled by a certain tensile-softening relation. In this paper, a simplified Dugdale-Barenblatt type model is adopted. As shown in **Fig.4** by a solid line, a tensile stress transmitted across a single crack is assumed to be a linear function of crack opening displacement α, in which the softening modulus is a. A crack closure in elastic unloading is ignored for simplicity, that is, when a crack is under unloading condition, the transmitted stress across the crack decreases without any change of crack opening displacement, as shown in **Fig.4** by a dotted line.

(2) Finite Element Formulation

In the FEM analysis, cracks are assumed to have enough length to cut one element into two. The crack initiation criteria and the tension-softening relation are examined with a stress state at the nearest gauss point from the crack.

The displacement field caused by the crack opening displacements are interpolated by a shape functions, as shown in **Fig.5**, which are suggested by Wan [28] and Dvorkin[29] to reduce a mesh dependencies in the shear localization analysis of soil foundation. The explicit forms of the
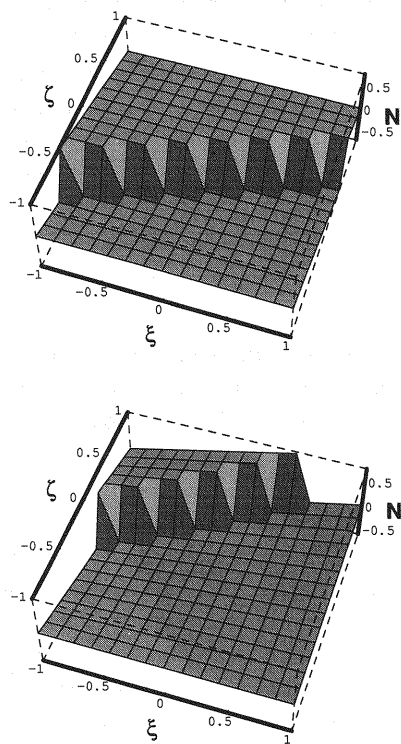
shape functions in **Fig.5** are respectively

$$N^c(x) = N_1(x) + N_2(x) - N_3(x) - N_4(x) \quad (48)$$

and

$$N^c(x) = -N_4(x) \quad (49)$$

where $N_i(x)$ are the isoparametric quadrilateral shape functions for 4-node element, the number of whose nodes is counted counter-clockwise from the node $(-1, -1)$ in the local coodinate system (ξ, ζ). In the shape functions, the displacement jump is assumed to be constant along the crack. When the maximum tensile stress reaches the tensile strength, a crack is embedded in the direction perpendicular to the maximum tensile stress as shown in **Fig.6**. The tensile softening relationship shown in **Fig.4** is assumed between normal stress and normal opening displacement across a crack. For simplicity shear deformation along the crack is constrained to zero.
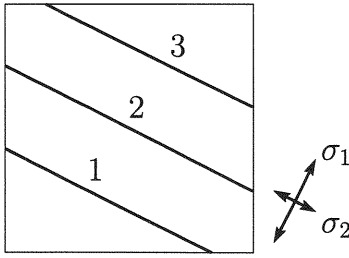
Fig. 6 Cracked element used in cracking model of concrete

The displacement field $u^e(x)$ is interpolated as

$$u_i^e(x) = N_{ij}(x)u_j + N_{ij}^c(x)\alpha_j, \qquad (50)$$

in which u and α are the nodal displacement and crack opening displacement of each crack, respectively, and N and N^c are the shape functions for nodal displacement u and crack opening displacement α, respectively. Strain field is given as

$$\epsilon_{ij}(x) = \frac{1}{2}\{u_{i,j}^e(x) + u_{j,i}^e(x)\}$$
$$= B_{ijk}(x)u_k + B_{ijk}^c(x)\alpha_k, \qquad (51)$$

where

$$B_{ijk} = \frac{1}{2}\left(\frac{\partial N_{ik}}{\partial x_j} + \frac{\partial N_{jk}}{\partial x_i}\right) \qquad (52)$$

$$B_{ijk}^c = \frac{1}{2}\left(\frac{\partial N_{ik}^c}{\partial x_j} + \frac{\partial N_{jk}^c}{\partial x_i}\right) \qquad (53)$$

for a small displacement.

Since the increment of crack opening displacement is assumed to be non-negative, they represent purely irreversible variables. On the other hand, all degrees of freedom corresponding to the nodal displacement contribute as components of the reversible variables.

(3) Crack Localization Analysis

By integrating over the system, the total free energy $\Phi(u, \alpha, \lambda)$ can be written in the vector form by

$$\Phi = \frac{1}{2}u \cdot K^{uu} \cdot u + \frac{1}{2}\alpha \cdot K^{\alpha\alpha} \cdot \alpha + \frac{1}{2}\lambda \cdot K^{\lambda\lambda} \cdot \lambda$$
$$+ u \cdot K^{u\alpha} \cdot \alpha + u \cdot K^{u\lambda} \cdot \lambda + \alpha \cdot K^{\alpha\lambda} \cdot \lambda$$
$$- F \cdot u \qquad (54)$$

where u, α, λ, and F are respectively nodal displacement, crack opening displacement, prescribed displacement, and nodal force in the global configuration. The explicit form of Ks are found in Appendix A. Following the discussion in section 2, if the structure is geometrically stable, that is, K^{uu} is positive definite,

$$u(\alpha, \lambda) = -(K^{uu})^{-1}\{K^{u\alpha} \cdot \alpha + K^{u\lambda} \cdot \lambda - F\} \quad (55)$$

then the free energy is rewritten by

$$\Phi^*(\alpha, \lambda) = \Phi(u(\alpha, \lambda), \alpha, \lambda)$$
$$= \frac{1}{2}\alpha \cdot K^{*\alpha\alpha} \cdot \alpha + \alpha \cdot K^{*\alpha\lambda} \cdot \lambda$$
$$+ \frac{1}{2}\lambda \cdot K^{*\lambda\lambda} \cdot \lambda$$
$$- F^{*\alpha} \cdot \alpha - F^{*\lambda} \cdot \lambda \qquad (56)$$

see Appendix B. Hence, the functional I is given by

$$I(\dot{\alpha}, \dot{\lambda}) = \dot{\alpha} \cdot K^{*\alpha\alpha} \cdot \dot{\alpha} + 2\dot{\alpha} \cdot K^{*\alpha\lambda} \cdot \dot{\lambda}$$
$$+ \dot{\lambda} \cdot K^{*\lambda\lambda} \cdot \dot{\lambda} \qquad (57)$$

and the fundamental solution to the crack opening displacement becomes

$$\dot{\alpha} = -(K^{*\alpha\alpha})^{-1} K^{*\alpha\lambda} \cdot \dot{\lambda} \qquad (58)$$

The stability of the fundamental solution is examined by checking the eigenvalue of the Hessian matrix $K^{*\alpha\alpha}$. In the present FEM analysis, the eigenvalues of the Hessian matrix $K^{*\alpha\alpha}$ are evaluated at every time step. When at least one of the eigenvalues becomes negative, the fundamental solution is judged to be unstable, then the program tries to find a point which makes the functional $I(\dot{\alpha})$ minimum. In this study, modified Downhill-Simplex [30] method, in which all the αs are suppressed to be positive in each incremental modification of a simplex, is introduced to solve this optimization problem.

(4) Four-point bending test

The numerical simulation of crack localization phenomena is carried out for a rectangular beam without any notch as shown in **Fig.7**, in which $L_1 : L_2 : L_3 : D = 10 : 5 : 10 : 3$. Plane strain conditions are assumed. The beam is supported by hinge at the left bottom corner and roller at the right bottom conner. The beam is supposed to be loaded downward simultaneously at two points near the center of the beam from the
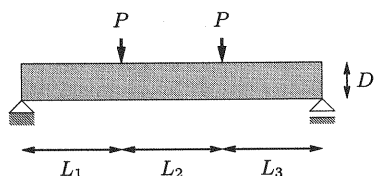
Fig. 7 Configuration of four-point bending test in FEM analysis

Fig. 8 Mesh used in simulation of four-point bending test

top so that approximately constant moment distribution can be observed between the two loading points. The loading is done by displacement control. The finite element discretization is based on bilinear displacement rectangular elements as shown in **Fig.8**.

Two types of numerical computations were conducted: 1) with localization judgment and 2) without localization judgment. **Fig.9** and **Fig.10** show the history of crack initiation and propagation in the analysis with and without localization judgment. The lines with arrows inside each element represent the orientation and the amount of crack opening displacements (incremental) in a certain increment. It is observed that, in the simulation with localization judgment, distributed crack openings gradually concentrate into several crack paths and, in the final stage of the calculation, crack opening displacements are observed in only two crack paths. While in the computation without localization analysis, the result just follows the fundamental solutions and crack opening displacements remains distributed. What is more remarkable aspect of cracking pattern clarified in the numerical simulation is that, even in the loading step in which whole structure shows hardening, cracking localization occurs, and that all the cracks which are selected to be stable in a certain increment will not always be activated after the increment, that is, the cracking pattern often changes in the course of loading.
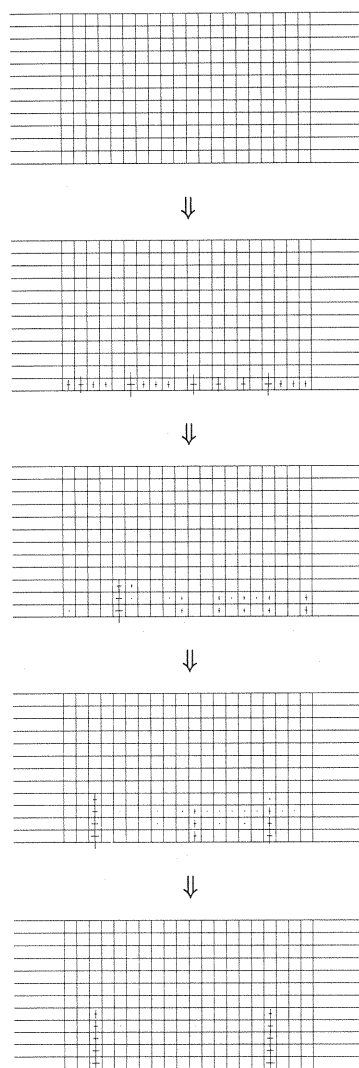
Fig. 9 History of crack propagation in analysis with localization check

7. CONCLUSION

In the present paper, an analytical and numerical method for strain localization in a structural level is presented. The evolution law of irreversible process has been given from the thermodynamics of irreversible process, within the concept of the Onsager's theory. Localization problem for rate-independent elastoplasticity is shown to be a minimization problem of second order differential of total entropy, which is defined in the
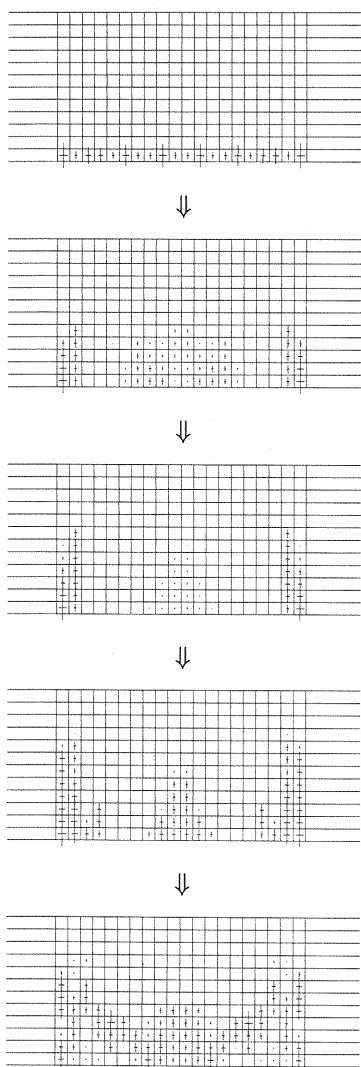
Fig. 10 History of crack propagation in analysis without localization check

velocity field $\dot{\alpha}$.

In the example of four point bending test of concrete, the results of calculation, such as loading history, are not compared quantitatively with the experimental data, since this study is aimed at establishing a analysis method for a strain localization phenomena. However, the remarkable crack localization pattern changes which will not be derived from a calculation without localization analysis happen to be found by applying a simple cracking model of concrete. This result shows that stability checking analysis as shown in this paper is necessary for a simulation of the structure which has a opportunity to have an unstable behavior.

From the practical point of view, the nonlocal theories, which are very effective to solve localization problem, is enough for *reproducing* a behavior of structures. However, these conventional techniques cannot answer to a more general problem in which there is no empirical information about the localization mode. In order to solve such kind of problem, a stability checking and optimization analysis that is suggested in the present paper is necessary.

Our discussion is limited to the rate-independent elastoplastic material with a displacement control boundary condition and uniform temperature. However, the basic idea of the thermodynamics of irreversible process is not limited to such conditions. The extension to a more general problem will be our future work.

Finally, since our discussion is limited by such assumptions, the final criteria of localization and the stability condition of bifurcation path has an identical form with the theory described by Bažant [26], and possibly the resulting solution is equivalent to the analysis outlined by Nguyen [23]. However, for a more general case of irreversible process in which, for instance, temperature- and rate-dependent behavior is considered the non-equilibrium approach might be the only possible approach.

APPENDIX A

K^{uu}, $K^{\lambda\lambda}$, $K^{u\alpha}$, $K^{u\lambda}$, and $K^{\alpha\lambda}$ are

$$K^{ij} = \int_\Omega B^i : D : B^j dV \qquad (59)$$

where

$$B^u = B^\lambda = B \qquad (60)$$
$$B^\alpha = B^c \qquad (61)$$

within an element which contains coresponding nodes, u, α and λ, and Ω is a domain in which concrete are filled, while $K^{\alpha\alpha}$ is

$$K^{\alpha\alpha} = \int_\Omega B^c : D : B^c dV + \int_\Gamma N^c \cdot A \cdot N^c dS \qquad (62)$$

where Γ is a crack surface and $\boldsymbol{A}$ is a hardning parameter. The external nodal force $\boldsymbol{F}$ is

$$\boldsymbol{F} = \int_{\delta\Omega} \boldsymbol{N} \cdot \boldsymbol{f} dS \qquad (63)$$

APPENDIX B

$\boldsymbol{K}^{*\alpha\alpha}$, $\boldsymbol{K}^{*\lambda\lambda}$, and $\boldsymbol{K}^{*\alpha\lambda}$ are

$$\boldsymbol{K}^{*\alpha\alpha} = \boldsymbol{K}^{\alpha\alpha} - \boldsymbol{K}^{\alpha u} \cdot (\boldsymbol{K}^{uu})^{-1} \cdot \boldsymbol{K}^{u\alpha} \qquad (64)$$

$$\boldsymbol{K}^{*\lambda\lambda} = \boldsymbol{K}^{\lambda\lambda} - \boldsymbol{K}^{\lambda u} \cdot (\boldsymbol{K}^{uu})^{-1} \cdot \boldsymbol{K}^{u\lambda} \qquad (65)$$

$$\boldsymbol{K}^{*\alpha\lambda} = \boldsymbol{K}^{\alpha\lambda} - \boldsymbol{K}^{\alpha u} \cdot (\boldsymbol{K}^{uu})^{-1} \cdot \boldsymbol{K}^{u\lambda} \qquad (66)$$

and $\boldsymbol{F}^{*\alpha}$ and $\boldsymbol{F}^{*\lambda}$ are

$$\boldsymbol{F}^{*\alpha} = \boldsymbol{K}^{\alpha u} \cdot (\boldsymbol{K}^{uu})^{-1} \cdot \boldsymbol{F} \qquad (67)$$

$$\boldsymbol{F}^{*\lambda} = \boldsymbol{K}^{\lambda u} \cdot (\boldsymbol{K}^{uu})^{-1} \cdot \boldsymbol{F} \qquad (68)$$

REFERENCES

1) Nadai,A.: *Theory of Flow and Fracture of Solids*. McGraw-Hill, 1950.

2) Friedman,M. and Logan,J.M.: Luders' bands in experimentally deformed sandstone and limestone. *Geological Society of America Bulletin*, 84:1465–1476, 1973.

3) Torrenti,J.M.: Some remarks upon concrete softening. *Materiaux et Constructions*, 19:391–394, 1986.

4) Mandel,G., de Jong,L.N.J. and Maltha,A.: Shear zone in granular material. *Rock Mechanics*, 9:95–144, 1977.

5) Horii,H., Takamatsu,K., Inoue,J. and Sasaki,N.: Measurement of displacement field by "matching method" and observation of strain localization in soft rock. In J.D. Frost and S. McNeil, editors, *Proceedings of Second International Conference on Imaging Technologies*, pages 10–19. ASCE, May 1998.

6) Hill,R.: Acceleration waves in solids. *J. Mech. Phis. Solids*, 10:713–726, 1962.

7) Hill,R. and Hutchinson,J.W.: Bifurcation phenomena in the plane tension test. *J. Mech. Phis. Solids*, 23:239–264, 1975.

8) Rudnicki,J.W. and Rice,J.R.: Conditions for the localization of deformation in pressuresensitive dilatant materials. *J. Mech. Phis. Solids*, 23:371–394, 1975.

9) Needleman,A.: Non-normality and bifurcation in plane strain tension and compression. *J. Mech. Phis. Solids*, 27:231–254, 1979.

10) Eringen,A.C. and Edelen,D.G.B.: On nonlocal elasticity. *Int. J. Engng. Sci.*, 10:233–248, 1972.

11) Aifantis,E.C.: On the microstructural origin of certain inelastic models. *J. Eng. Mater. Technol.*, 106:326–330, 1984.

12) Oka,F., Yashima,A. and Aifantis,E.C.: Instability of gradient dependent viscoplastic model for clay saturated with water and fem analysis. *Appl. Mech. Rev.*, 45(3):140–148, 1992.

13) Needleman,A.: Material rate dependence and mesh sensitivity in localization problems. *Comp. Meth. Appl. Mech. Engng.*, 67:69–85, 1988.

14) Leroy,Y.M. and Chapuis,O.: Localization in strain-rate-dependent solids. *Comp. Meth. Appl. Mech. Engng.*, 90:969–986, 1991.

15) Bažant,Z.P.: Crack band model for fracture of geomatrials. In *Int. Conf. Numer. Meth. Geo.*, volume 3, pages 1137–1152, Edmonton, 1982.

16) Belytschko,T. and Fish,J.: Spectral superposition on finite elements for shear banding problems. In *Proc. 5th Int. Sympo. Numer. Meth. Engng.*, volume 1, pages 19–29, 1989.

17) de Brost,R. and Mühlhous,H.-B.: *Continuum models for discontinuous media*, pages 601–618. E. & F. N. Spon, London, 1991.

18) Tomita,Y.: Simulation of plastic instabilities in solid mechanics. *App. Mech. Rev.*, 47(6):171–205, 1994.

19) Shi,Z.H. and Horii,H.: Microslip model of strain localization in sand deformation. *Mech. Mater.*, 8:89–102, 1989.

20) Okui,Y. and Horii,H.: *A Micromechanics-based Continuum Theory for Microcracking Localization of Rock under Compression*, chapter 2. John Wiley & Sons Ltd., 1995.

21) Glansdorff,P. and Prigogine,I.: *Thermodynamic theory of structure, stability and fluctuations*. Jhon Wiley & Sons, Ltd., London-New York-Sydney-Tronto, 1971.

22) Rice,J.R.: Inelastic constitutive relations for solids: An internal variable theory and its application to metal plasticity. *J. Mech. Phys. Solids*, 19:433–455, 1971.

23) Nguyen,Q.S.: Bifurcation and post-bifurcation analysis in plasticity and brittle fracture. *J. Mech. Phis. Solids*, 35:303–324, 1987.

24) Onsager,L.: Reciprocal relations in irreversible process. *Phys. Rev.*, 37:405–426, 1931.

25) Hill,R.: A variational principle of maximum plastic work in classical plasticity. *Q.J. Mech. Appl. Math.*, 1:18–28, 1948.

26) Bažant,Z.P.: *Stability of Structures*. Oxford University Press, 1991.

27) Nguyen,Q.S.: *Contribution à la théorie de l'élasto-plasticité avec écrouissage*. PhD thesis, Université Pierre-et-Marie Curie, Paris, 1973.

28) Wan,R.G.: *The numerical modeling of shear bands in geological materials*. PhD thesis, University of Alberta, 1990.

29) Dvorkin,E.N., Cuitino,A.M. and Gioia,G.: Finite elements with displacement interpolated embedded localization lines insensitive to mesh size and distortions. *International Journal for Numerical Methods in Engineering*, 30:541–564, 1990.

30) Nelder,J.A. and Mead,R.: Downhill simplex method. *Computer Journal*, 7:308–313, 1965.