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CIRCULAR RIGID PUNCH ON A SEMI-INFINITE PLANE WITH AN OBLIQUE EDGE CRACK SUBJECTED TO CONCENTRATED FORCES OR POINT DISLOCATIONS

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A circular rigid punch is located on a semi-infinite plane with an oblique edge crack. The punch is acted by an eccentric load to keep the punch vertical, and frictional force is assumed to exist on the contact region. A pair of concentrated forces or point dislocations is assumed to act at arbitrary points in the semi-infinite plane. The analytical solution (Green function) is obtained by a rational mapping function and a complex variable method. To solve the problem, the complex stress functions are divided into two parts; one is the principal part, which corresponds to the fundamental solution of the semi-infinite plane with an oblique edge crack; the other is the holomorphic part of the problem, which can be derived explicitly. The stress intensity factors and resultant moment on the contact region to decide the position of the vertical load on the punch are shown.

Key Words : *circular rigid punch, oblique crack, concentrated force, point dislocation, fundamental solution, Green function, semi-infinite plane*

1. INTRODUCTION

It is well known that the fundamental solutions of an infinite plane and a semi-infinite plane subjected to concentrated force or point dislocation at an arbitrary point play important roles in the analysis of various problems in engineering, especially in the application of Boundary Element Method [1].

When a punch on a semi-infinite plane with a crack is concerned, some difficulties will be met in computation. For example, when the problem is analyzed by numerical method, the modeling of the contact region, the semi-infinite extension of the half plane and the tip of the edge crack will result in much inconvenience [2]. In order to solve the problem effectively, it is necessary to derive the fundamental solution of the problem, and then by making use of the solution, the inherent properties of the problem can be revealed analytically without much computation, and BEM can also be applied efficiently.

A rigid flat or circular punch problem on a semi-infinite plane with an edge crack has been studied in the previous papers [3)-5)] using complex stress functions. The semi-infinite plane with an edge crack is first mapped into a unit circle by a rational mapping function so that the forward derivation can be performed on the mapping plane in an analytical way. The solution of the semi-infinite plane with an edge crack is derived by making use of the regularity of the complex stress functions of the semi-infinite plane. According to the loading and displacement conditions, the punch problem can be transformed into the Riemann-Hilbert problem. To solve the R-H equation, the complex stress functions for the whole problem are divided into two parts, one is the principal part, which is corresponding to the solution of the semi-infinite plane with an edge crack acted by concentrated force or point dislocation; the other is the holomorphic part of the problem. By substituting the first part into the R-H equation, and introducing a Plemelj function, the solution of the second part can be obtained explicitly.

2. THE MAPPING FUNCTION

To analyze the punch problem with an edge crack in the semi-infinite plane subjected to concentrated forces or point dislocations, the first important step is to map the semi-infinite plane with the crack into a unit circle by a rational mapping function.

For the semi-infinite plane with an oblique edge crack (**Fig.1**), the following irrational mapping function can be obtained from Schwarz-Christoffel's formula,

$$z = \omega(\zeta) = b\frac{(1-i)i^{-s}}{2s^{s}(1-s)^{1-s}}\frac{(1+i\zeta)^{s}(1-i\zeta)^{1-s}}{1-\zeta} \quad (1)$$

where b is the crack length, $s = \gamma/180$, γ represents the oblique angle of the crack, and $\zeta = 1$ corresponds to infinity.

To use the above mapping function directly is impossible to obtain an explicit solution of the problem, therefore the following rational mapping function is formed from (1),

$$z = \omega(\zeta) = \frac{E_0}{1-\zeta} + \sum_{k=1}^{N}\frac{E_k}{\zeta_k - \zeta} + E_c \quad (2)$$

where E_0, E_k and $\zeta_k(|\zeta_k| > 1)$ are known constants, E_c is related to the distance from the crack to the origin of the coordinates, and $N = 24$ is used in this paper.

The rational expressions for each irrational term in (1) are formed, and the method of constructing a rational mapping function in a fractional form from an irrational one for a semi-infinite plane with an oblique edge crack has been reported in the previous papers[6], and its high precision has also been proved. The main idea is stated in Appendix A. For an arbitrary point z_0 in the physical plane, the corresponding ζ_0 in the mapping plane can be decided by solving (2) using Newton Method or Muller Iteration Method.

3. SEMI-INFINITE PLANE WITH AN OBLIQUE EDGE CRACK

(1) Case of concentrated force

As shown in **Fig.1**, the semi-infinite plane with an oblique edge crack is assumed to be acted by a pair of concentrated forces q_x, q_y at an arbitrary

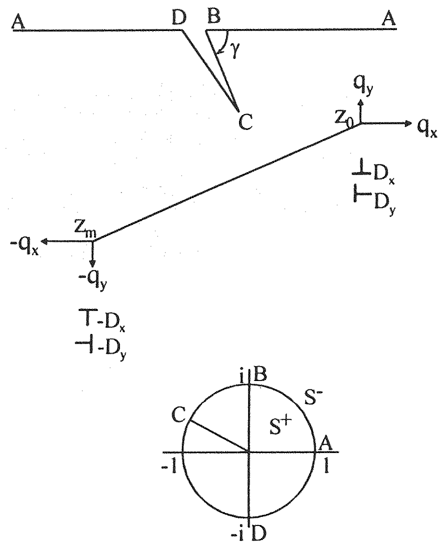

Fig.1 A semi-infinite plane with an oblique edge crack and the unit circle

point z_0, which corresponds to point ζ_0 in the unit circle. It is also supposed that there exists another pair of concentrated forces $-q_x$, $-q_y$ acted at point z_m, which corresponds to point ζ_m in the unit circle. The two pairs of concentrated forces are in self-equilibrium.

It is assumed that the complex stress functions ϕ_q and ψ_q to be obtained is in the following form:

$$\phi_q(\zeta) = \phi_{q1}(\zeta) + \phi_{q2}(\zeta) \quad (3a)$$

$$\psi_q(\zeta) = \psi_{q1}(\zeta) + \psi_{q2}(\zeta) \quad (3b)$$

where $\phi_{q1}(\zeta)$ and $\psi_{q1}(\zeta)$ are the principal parts of the complex potentials $\phi_q(\zeta)$ and $\psi_q(\zeta)$, and represent the complex stress functions of an infinite plane subjected to the concentrated forces, and $\phi_{q2}(\zeta)$ and $\psi_{q2}(\zeta)$ are the holomorphic parts of $\phi_q(\zeta)$ and $\psi_q(\zeta)$, respectively.

It is well known that the expressions of $\phi_{q1}(\zeta)$ and $\psi_{q1}(\zeta)$ can be expressed as

$$\phi_{q1}(\zeta) = \frac{q}{2\pi}\log(\zeta-\zeta_0) - \frac{q}{2\pi}\log(\zeta-\zeta_m) \quad (4a)$$

$$\psi_{q1}(\zeta) = -\frac{\kappa\bar{q}}{2\pi}\log(\zeta-\zeta_0) - \frac{q}{2\pi}\frac{\overline{\omega(\zeta_0)}}{\omega'(\zeta_0)(\zeta-\zeta_0)}$$

$$+\frac{\kappa \bar{q}}{2\pi}\log(\zeta-\zeta_m)+\frac{q}{2\pi}\frac{\overline{\omega(\zeta_m)}}{\omega'(\zeta_m)(\zeta-\zeta_m)} \tag{4b}$$

where $q=-(q_x+iq_y)/(1+\kappa)$, $\kappa=3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress state, respectively, and ν represents the Poisson's ratio of the semi-infinite plane.

Since there exists traction free boundary, another complex stress function $\psi_q(\zeta)$ can be expressed by $\phi_q(\zeta)$ as[7]

$$\psi_q(\zeta)=-\bar{\phi}_q(\frac{1}{\zeta})-\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_q'(\zeta) \tag{5}$$

where

$$\bar{\omega}(\frac{1}{\zeta})=\overline{E_0}-\frac{\overline{E_0}}{1-\zeta}+\sum_{k=1}^{N}\overline{E_k}\zeta_k'-\sum_{k=1}^{N}\frac{\overline{E_k}\zeta_k'^2}{\zeta_k'-\zeta}+\overline{E_c}$$

and $\zeta_k'\equiv 1/\bar{\zeta}_k$.

Substituting (3) into (5) yields

$$\psi_{q2}(\zeta)=-\bar{\phi}_{q2}(\frac{1}{\zeta})-\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_{q2}'(\zeta)$$
$$-\bar{\phi}_{q1}(\frac{1}{\zeta})-\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_{q1}'(\zeta)-\psi_{q1}(\zeta) \tag{6}$$

Since $\psi_{q2}(\zeta)$ $(\zeta\in S^+)$ is regular in the unit circle, the right side of (6) must be also regular. In order to separate the singular parts from the right side of (6), the following derivations are considered:

$$\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_{q1}'(\zeta)=\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\frac{q}{2\pi}\left(\frac{1}{\zeta-\zeta_0}-\frac{1}{\zeta-\zeta_m}\right)$$
$$=\frac{q}{2\pi}\left\{\sum_{k=1}^{N}\frac{\overline{B_k}\zeta_k'^2}{(\zeta_k'-\zeta_0)}\frac{1}{\zeta-\zeta_k'}+\frac{\overline{\omega(1/\bar{\zeta}_0)}}{\omega'(\zeta_0)}\frac{1}{\zeta-\zeta_0}\right\}$$
$$-\frac{q}{2\pi}\left\{\sum_{k=1}^{N}\frac{\overline{B_k}\zeta_k'^2}{(\zeta_k'-\zeta_m)}\frac{1}{\zeta-\zeta_k'}+\frac{\overline{\omega(1/\bar{\zeta}_m)}}{\omega'(\zeta_m)}\frac{1}{\zeta-\zeta_m}\right\}$$
$$+\text{ regular part}\quad(\zeta\in S^+) \tag{7}$$

$$\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_{q2}'(\zeta)=\sum_{k=1}^{N}\frac{A_{qk}\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}+\text{ regular part}\quad(\zeta\in S^+) \tag{8}$$

where $A_{qk}\equiv\phi_{q2}'(\zeta_k')$ and $\overline{B_k}\equiv\overline{E_k}/\omega'(\zeta_k')$.

$\bar{\phi}_{q2}(1/\zeta)$ in (6) must be determined so as to eliminate the irregular parts in the right side of (6). The following expression is then obtained:

$$\bar{\phi}_{q2}\left(\frac{1}{\zeta}\right)=\frac{\kappa\bar{q}}{2\pi}\log(\zeta-\zeta_0)-\frac{\kappa\bar{q}}{2\pi}\log(\zeta-\zeta_m)$$
$$+\frac{q}{2\pi}\frac{\overline{\omega(\zeta_0)}-\overline{\omega(1/\bar{\zeta}_0)}}{\omega'(\zeta_0)}\frac{1}{\zeta-\zeta_0}$$
$$-\frac{q}{2\pi}\frac{\overline{\omega(\zeta_m)}-\overline{\omega(1/\bar{\zeta}_m)}}{\omega'(\zeta_m)}\frac{1}{\zeta-\zeta_m}$$
$$+\frac{q}{2\pi}\sum_{k=1}^{N}\left(\frac{1}{\zeta_k'-\zeta_m}-\frac{1}{\zeta_k'-\zeta_0}\right)\frac{\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}$$
$$-\sum_{k=1}^{N}\frac{A_{qk}\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}\qquad(\zeta\in S^+) \tag{9}$$

From (9), it is easy to deduce that

$$\phi_{q2}(\zeta)=\frac{\kappa q}{2\pi}\log(\zeta-1/\bar{\zeta}_0)-\frac{\kappa q}{2\pi}\log(\zeta-1/\bar{\zeta}_m)$$
$$-\frac{\bar{q}}{2\pi}\frac{\omega(\zeta_0)-\omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}}\frac{(1/\bar{\zeta}_0)^2}{\zeta-1/\bar{\zeta}_0}$$
$$+\frac{\bar{q}}{2\pi}\frac{\omega(\zeta_m)-\omega(1/\bar{\zeta}_m)}{\overline{\omega'(\zeta_m)}}\frac{(1/\bar{\zeta}_m)^2}{\zeta-1/\bar{\zeta}_m}$$
$$-\frac{\bar{q}}{2\pi}\sum_{k=1}^{N}\left(\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_m}-\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_0}\right)\frac{B_k}{\zeta-\zeta_k}$$
$$+\sum_{k=1}^{N}\frac{\overline{A_{qk}}B_k}{\zeta-\zeta_k} \tag{10}$$

and then

$$\phi_{q2}'(\zeta)=\frac{\kappa q}{2\pi}\left\{\frac{1}{\zeta-1/\bar{\zeta}_0}-\frac{1}{\zeta-1/\bar{\zeta}_m}\right\}$$
$$+\frac{\bar{q}}{2\pi}\frac{\omega(\zeta_0)-\omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}}\frac{(1/\bar{\zeta}_0)^2}{(\zeta-1/\bar{\zeta}_0)^2}$$
$$-\frac{\bar{q}}{2\pi}\frac{\omega(\zeta_m)-\omega(1/\bar{\zeta}_m)}{\overline{\omega'(\zeta_m)}}\frac{(1/\bar{\zeta}_m)^2}{(\zeta-1/\bar{\zeta}_m)^2}$$
$$+\frac{\bar{q}}{2\pi}\sum_{k=1}^{N}\left(\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_m}-\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_0}\right)\frac{B_k}{(\zeta-\zeta_k)^2}$$
$$-\sum_{k=1}^{N}\frac{\overline{A_{qk}}B_k}{(\zeta-\zeta_k)^2} \tag{11}$$

To decide the undetermined values of

$A_{qk} \equiv \phi'_{q2}(\zeta'_k)$, let $\zeta = \zeta'_j (j = 1,2,\ldots,N)$ in (11), it is obtained that

$$A_{qj} + \sum_{k=1}^{N} \frac{B_k}{(\zeta'_j - \zeta_k)^2} \overline{A_{qk}} =$$
$$\frac{\kappa q}{2\pi} \left\{ \frac{1}{\zeta'_j - 1/\bar{\zeta}_0} - \frac{1}{\zeta'_j - 1/\bar{\zeta}_m} \right\}$$
$$+ \frac{\bar{q}}{2\pi} \frac{\omega(\zeta_0) - \omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}} \frac{(1/\bar{\zeta}_0)^2}{(\zeta'_j - 1/\bar{\zeta}_0)^2}$$
$$- \frac{\bar{q}}{2\pi} \frac{\omega(\zeta_m) - \omega(1/\bar{\zeta}_m)}{\overline{\omega'(\zeta_m)}} \frac{(1/\bar{\zeta}_m)^2}{(\zeta'_j - 1/\bar{\zeta}_m)^2}$$
$$+ \frac{\bar{q}}{2\pi} \sum_{k=1}^{N} \left(\frac{1}{\overline{\zeta'_k} - \bar{\zeta}_m} - \frac{1}{\overline{\zeta'_k} - \bar{\zeta}_0} \right) \frac{B_k}{(\zeta'_j - \zeta_k)^2} \quad (12)$$

A_{qj} and $\overline{A_{qj}}$ (j =1,2,...,N) . can then be determined by the above N equations.

When the point force at $z = z_m$ applies at infinity, the solution of the original problem can be obtained by letting $z_m \to \infty$, i.e. $\zeta_m \to 1$.

(2) Case of point dislocations

As shown in **Fig.1**, in this case, the semi-infinite plane is assumed to be subjected to a pair of point dislocations D_x ,D_y at point z_0 in the semi-infinite plane. It is also supposed that there exists another pair of point dislocations $-D_x$,$-D_y$ at another point z_m.

The complex stress functions of the problem to be obtained are assumed in the following forms:

$$\phi_d(\zeta) = \phi_{d1}(\zeta) + \phi_{d2}(\zeta) \quad (13a)$$
$$\psi_d(\zeta) = \psi_{d1}(\zeta) + \psi_{d2}(\zeta) \quad (13b)$$

where $\phi_{d1}(\zeta)$ and $\psi_{d1}(\zeta)$ are the principal parts, which represent the complex stress functions for an infinite plane subjected to the point dislocations, and $\phi_{d2}(\zeta)$ and $\psi_{d2}(\zeta)$ are the holomorphic parts of $\phi_d(\zeta)$ and $\psi_d(\zeta)$, respectively.

$\phi_{d1}(\zeta)$ and $\psi_{d1}(\zeta)$ for point dislocations can be expressed as

$$\phi_{d1}(\zeta) = -\frac{D}{2\pi}\log(\zeta - \zeta_0) + \frac{D}{2\pi}\log(\zeta - \zeta_m) \quad (14a)$$

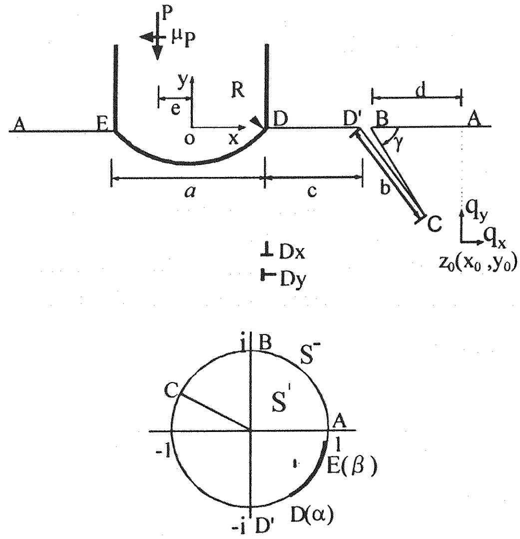

Fig.2 A circular rigid punch on a cracked semi-infinite plane and the unit circle

$$\psi_{d1}(\zeta) = -\frac{\bar{D}}{2\pi}\log(\zeta - \zeta_0) + \frac{D}{2\pi}\frac{\overline{\omega(\zeta_0)}}{\omega'(\zeta_0)(\zeta - \zeta_0)}$$
$$+ \frac{\bar{D}}{2\pi}\log(\zeta - \zeta_m) - \frac{D}{2\pi}\frac{\overline{\omega(\zeta_m)}}{\omega'(\zeta_m)(\zeta - \zeta_m)} \quad (14b)$$

where $D = D_x + iD_y$.

By the same procedures used in the case of concentrated forces, the present solution can be obtained as

$$\phi_d(\zeta) = -\frac{D}{2\pi}\log(\zeta - \zeta_0) + \frac{D}{2\pi}\log(\zeta - \zeta_m)$$
$$+ \frac{D}{2\pi}\log(\zeta - 1/\bar{\zeta}_0) - \frac{D}{2\pi}\log(\zeta - 1/\bar{\zeta}_m)$$
$$+ \frac{\bar{D}}{2\pi} \frac{\omega(\zeta_0) - \omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}} \frac{(1/\bar{\zeta}_0)^2}{\zeta - 1/\bar{\zeta}_0}$$
$$- \frac{\bar{D}}{2\pi} \frac{\omega(\zeta_m) - \omega(1/\bar{\zeta}_m)}{\overline{\omega'(\zeta_m)}} \frac{(1/\bar{\zeta}_m)^2}{\zeta - 1/\bar{\zeta}_m}$$
$$+ \frac{\bar{D}}{2\pi} \sum_{k=1}^{N} \left(\frac{1}{\overline{\zeta'_k} - \bar{\zeta}_m} - \frac{1}{\overline{\zeta'_k} - \bar{\zeta}_0} \right) \frac{B_k}{\zeta - \zeta_k}$$
$$+ \sum_{k=1}^{N} \frac{\overline{A_{dk}} B_k}{\zeta - \zeta_k} \quad (15)$$

where A_{dk} and $\overline{A_{dk}}$ are determined by solving 2N linear simultaneous equations for real and imaginary parts of $A_{dk} \equiv \phi'_{d2}(\zeta'_k)$ (k=1,2,...,N).

4. LOADING AND DISPLACEMENT CONDITIONS OF PUNCH PROBLEM

The punch problem is shown in **Fig.2**, in which the punch is acted by load P with a distance e from the origin of the coordinates. Coulomb's frictional force exists on the contact region. An oblique edge crack with an angle γ $(0<\gamma<180^\circ)$ is located at or away from the right end of the punch. The semi-infinite plane is assumed to be subjected to concentrated forces or point dislocations, respectively.

The loading and displacement conditions of the problem can be presented as follows:

$$p_x = p_y = 0 \qquad \text{on} \quad L = L_1 + L_2 \tag{16a}$$

$$p_x = \mu p_y, \ \int p_y \, ds = P \quad \text{on} \quad M \tag{16b}$$

$$V = x^2 / 2R \qquad \text{on} \quad M \tag{16c}$$

The displacement in (16c) is given by

$$V = -\sqrt{R^2 - x^2} = -R\sqrt{1-(x/R)^2} \approx x^2/(2R) \tag{16d}$$

owing to the fact that V is very small compared with R. The conditions related to the concentrated forces or point dislocations are expressed as

$$Q(x,y) = (q_x + iq_y)\delta(z,z_0) - (q_x + iq_y)\delta(z,z_m) \tag{17a}$$

$$G(x,y) = (D_x + iD_y)\delta(z,z_0) - (D_x + iD_y)\delta(z,z_m) \tag{17b}$$

where $L_1 = ABCD'D$, $L_2 = EA$, $M = DE$ in **Fig.2**; μ represents the Coulomb's frictional coefficient on M; p_x and p_y represents the components of traction in x and y directions on the surface of the semi-infinite plane; $Q(x,y)$ and $G(x,y)$ represent the forces and dislocations in the semi-infinite plane, respectively; $\delta(z,z_0)=1$ when $z=z_0$ and 0 when $z \neq z_0$, so does $\delta(z,z_m)$. R represents the radius of curvature of the punch.

5. FUNDAMENTAL SOLUTIONS OF THE PUNCH PROBLEM

According to the above loading and displacement conditions, the problem can be transformed into the Riemann-Hilbert problem as follows:[3),4)]

$$\phi^+(\sigma) - \phi^-(\sigma) = f_L \qquad \text{on} \quad L = L_1 + L_2 \tag{18a}$$

$$\phi^+(\sigma) + \frac{1}{g}\phi^-(\sigma) = f_M \quad \text{on} \quad M \tag{18b}$$

where $\phi^+(\sigma)$ denotes the value of $\phi(\sigma)$ on the unit circle approaching from inside region S^+ and $\phi^-(\sigma)$ from the outside region S^- (see **Fig.1**), and

$$f_L = i\int (p_x + ip_y)ds \tag{19a}$$

$$f_M = \frac{4(1-i\mu)GiV + (1+i\mu)(1+\kappa)S(\sigma)}{(\kappa+1) - i\mu(\kappa-1)} \tag{19b}$$

$$S(\zeta) = \phi(\zeta) + \frac{1-i\mu}{1+i\mu}\overline{\phi\left(\frac{1}{\overline{\zeta}}\right)} \tag{19c}$$

$$\frac{1}{g} = \frac{(\kappa+1) + i\mu(\kappa-1)}{(\kappa+1) - i\mu(\kappa-1)} \tag{19d}$$

$S(\zeta)$ is a function to be determined so as to satisfy (19c), and G is the shear modulus of the semi-infinite plane. R is included in the expression of $V = [\omega(\sigma)]^2/(2R)$ in (19b). The solution of punch problem can be easily obtained from the present paper by letting q and D zero.

(1) The semi-infinite plane acted by concentrated forces

The complex stress functions to be obtained are represented by two terms:

$$\phi(\zeta) = \phi_1(\zeta) + \phi_2(\zeta) \tag{20a}$$

$$\psi(\zeta) = \psi_1(\zeta) + \psi_2(\zeta) \tag{20b}$$

where $\phi_1(\zeta)$ and $\psi_1(\zeta)$ correspond to $\phi_q(\zeta)$ and $\psi_q(\zeta)$ of the semi-infinite plane acted by the two pairs of concentrated forces, as presented by (3), (4), (10) and (5); $\phi_2(\zeta)$ and $\psi_2(\zeta)$ are the holomorphic parts of $\phi(\zeta)$ and $\psi(\zeta)$.

Substituting (20a) into (18), it is obtained that

$$\phi_2^+(\sigma) - \phi_2^-(\sigma) = f_{L2}(\sigma) \tag{21a}$$

$$\phi_2^+(\sigma) + \frac{1}{g}\phi_2^-(\sigma) = f_{M2}(\sigma) + C\left[\overline{\phi_q(\sigma)} - \phi_q(\sigma)\right] = f_{M2}(\sigma) + \frac{C}{2\pi}\left\{(\bar{q}-\kappa q)f_1 + (\kappa\bar{q}-q)f_2 + qg_1 + \bar{q}g_2 + 2\pi g_3\right\} \tag{21b}$$

where

$$f_{L2}(\sigma)=\begin{cases}0 & on\ L_1\\ P(1-i\mu) & on\ L_2\end{cases} \quad (22a)$$

$$f_{M2}(\sigma)=\frac{4(1-i\mu)GiV+(1+i\mu)(1+\kappa)S(\sigma)}{(\kappa+1)-i\mu(\kappa-1)} \quad (22b)$$

$$S(\zeta)=\phi_2(\zeta)+\frac{1-i\mu}{1+i\mu}\overline{\phi_2\left(\frac{1}{\bar{\zeta}}\right)} \quad (22c)$$

$$C=\frac{(1-i\mu)(\kappa+1)}{(\kappa+1)-i\mu(\kappa-1)} \quad (22d)$$

$$f_1=\log(\sigma-1/\bar{\zeta}_0)-\log(\sigma-1/\bar{\zeta}_m) \quad (22e)$$

$$f_2=\log(\sigma-\zeta_0)-\log(\sigma-\zeta_m) \quad (22f)$$

$$g_1=\frac{\overline{\omega(\zeta_0)}-\overline{\omega(1/\bar{\zeta}_0)}}{\omega'(\zeta_0)}\frac{1}{\sigma-\zeta_0}-\frac{\overline{\omega(\zeta_m)}-\overline{\omega(1/\bar{\zeta}_m)}}{\omega'(\zeta_m)}\frac{1}{\sigma-\zeta_m}-\sum_{k=1}^{N}\frac{\overline{B_k}\zeta_k'^2}{(\sigma-\zeta_k')}\left(\frac{1}{\zeta_k'-\zeta_0}-\frac{1}{\zeta_k'-\zeta_m}\right) \quad (22g)$$

$$g_2=\frac{\omega(\zeta_0)-\omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}}\frac{(1/\bar{\zeta}_0)^2}{\sigma-1/\bar{\zeta}_0}-\frac{\omega(\zeta_m)-\omega(1/\bar{\zeta}_m)}{\overline{\omega'(\zeta_m)}}\frac{(1/\bar{\zeta}_m)^2}{\sigma-1/\bar{\zeta}_m}-\sum_{k=1}^{N}\frac{B_k}{(\sigma-\zeta_k)}\left(\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_0}-\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_m}\right) \quad (22h)$$

$$g_3=-\sum_{k=1}^{N}\frac{A_{qk}\overline{B_k}\zeta_k'^2}{\sigma-\zeta_k'}-\sum_{k=1}^{N}\frac{\overline{A_{qk}}B_k}{\sigma-\zeta_k} \quad (22i)$$

By introducing the Plemelj function $\chi(\zeta)=(\zeta-\alpha)^m(\zeta-\beta)^{1-m}$, $m=0.5-i\ln g/2\pi$, the solution of (21) can be expressed as[3),7)]

$$\phi_2(\zeta)=H_1(\zeta)+H_2(\zeta)+H_3(\zeta)+\frac{1+i\mu}{2}J(\zeta)+Q(\zeta)\chi(\zeta) \quad (23)$$

where

$$H_1(\zeta)=P(1-i\mu)\frac{\chi(\zeta)}{2\pi i}\int_\beta^1\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} \quad (24a)$$

$$\begin{aligned}H_2(\zeta)&=\frac{Gi(1-i\mu)}{R(\kappa+1)}\frac{\chi(\zeta)}{2\pi i}\oint_M\frac{\{\omega(\sigma)\}^2}{\chi(\sigma)(\sigma-\zeta)}d\sigma\\
&=\frac{Gi(1-i\mu)}{R(\kappa+1)}\Bigg[\{\omega(\zeta)\}^2-\frac{\chi(\zeta)}{\chi(1)}\times\\
&\left\{\frac{E_0^2}{1-\zeta}\left[\frac{m}{1-\alpha}+\frac{1-m}{1-\beta}\right]+\frac{2E_0E_c}{1-\zeta}+\frac{E_0^2}{(1-\zeta)^2}\right\}\\
&-\sum_{k=1}^{N}\frac{2E_0E_k}{\zeta_k-1}\left\{\frac{\chi(\zeta)}{\chi(1)(1-\zeta)}-\frac{\chi(\zeta)}{\chi(\zeta_k)(\zeta_k-\zeta)}\right\}\\
&-\sum_{k\neq l}^{N}\frac{E_kE_l}{\zeta_k-\zeta_l}\left\{\frac{\chi(\zeta)}{\chi(\zeta_l)(\zeta_l-\zeta)}-\frac{\chi(\zeta)}{\chi(\zeta_k)(\zeta_k-\zeta)}\right\}\\
&-\sum_{k=l}^{N}\frac{\chi(\zeta)}{\chi(\zeta_k)}\left\{\frac{E_k^2}{\zeta_k-\zeta}\left[\frac{m}{\zeta_k-\alpha}+\frac{1-m}{\zeta_k-\beta}\right]\right.\\
&\left.+\frac{2E_kE_c}{\zeta_k-\zeta}+\frac{E_k^2}{(\zeta_k-\zeta)^2}\right\}\Bigg]\end{aligned} \quad (24b)$$

$$H_3(\zeta)=\frac{C\chi(\zeta)}{4\pi^2 i}\int_M\frac{(\bar{q}-\kappa q)f_1+(\kappa\bar{q}-q)f_2+qg_1+\bar{q}g_2+2\pi g_3}{\chi^+(\sigma)(\sigma-\zeta)}d\sigma \quad (24c)$$

$$J(\zeta)=\frac{\chi(\zeta)}{2\pi i}\oint_M\frac{R(\sigma)}{\chi(\sigma)(\sigma-\zeta)}d\sigma \quad (24d)$$

and $Q(\zeta)$ is a function to be determined. $H_1(\zeta)$ is related to the load on the punch. Though it is in integral form, its first derivative can be expressed in the form without integration[3)]; $H_2(\zeta)$ is related to the vertical displacement on the contact region induced by the radius of curvature of the punch. Owing to the use of the rational mapping function, the integration of $H_2(\zeta)$ has been carried out; $H_3(\zeta)$ is related to the concentrated forces in the semi-infinite plane. The final expression of $H_3(\zeta)$ can be obtained as

$$H_3(\zeta)=\frac{1-i\mu}{4\pi}\left[(\bar{q}-\kappa q)F_1+(\kappa\bar{q}-q)F_2+qG_1+\bar{q}G_2+2\pi G_3\right] \quad (25)$$

where

$$F_1=\log(\zeta-1/\bar{\zeta}_0)-\log(\zeta-1/\bar{\zeta}_m)$$

$$+\chi(\zeta)\int_{1/\bar{\zeta}_0}^{1/\bar{\zeta}_m}\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} \quad (26a)$$

$$F_2=\log(\zeta-\zeta_0)-\log(\zeta-\zeta_m)$$
$$+\chi(\zeta)\int_{\zeta_0}^{\zeta_m}\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} \quad (26b)$$

$$G_1=\frac{\overline{\omega(\zeta_0)}-\overline{\omega(1/\bar{\zeta}_0)}}{\overline{\omega'(\zeta_0)}}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_0)}\right]\frac{1}{\zeta-\zeta_0}$$
$$-\frac{\overline{\omega(\zeta_m)}-\overline{\omega(1/\bar{\zeta}_m)}}{\overline{\omega'(\zeta_m)}}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_m)}\right]\frac{1}{\zeta-\zeta_m}$$
$$-\sum_{k=1}^{N}\left(\frac{1}{\zeta_k'-\zeta_0}-\frac{1}{\zeta_k'-\zeta_m}\right)\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}$$
(26c)

$$G_2=\frac{\omega(\zeta_0)-\omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}}\left[1-\frac{\chi(\zeta)}{\chi(1/\bar{\zeta}_0)}\right]\frac{(1/\bar{\zeta}_0)^2}{\zeta-1/\bar{\zeta}_0}$$
$$-\frac{\omega(\zeta_m)-\omega(1/\bar{\zeta}_m)}{\overline{\omega'(\zeta_m)}}\left[1-\frac{\chi(\zeta)}{\chi(1/\bar{\zeta}_m)}\right]\frac{(1/\bar{\zeta}_m)^2}{\zeta-1/\bar{\zeta}_m}$$
$$-\sum_{k=1}^{N}\left(\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_0}-\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_m}\right)\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{B_k}{\zeta-\zeta_k}$$
(26d)

$$G_3=-\sum_{k=1}^{N}\frac{A_{qk}\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k')}\right]$$
$$-\sum_{k=1}^{N}\frac{\overline{A_{qk}}B_k}{\zeta-\zeta_k}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k)}\right] \quad (26e)$$

The method of deriving F_1 and F_2 is stated in Appendix B. Though the last terms of F_1 and F_2 are in integral forms, their first derivatives can be expressed in the form without integration[3].

Since there exists traction free boundary for the problem, another stress function $\psi(\zeta)$ can be expressed by $\phi(\zeta)$ as[7]

$$\psi(\zeta)=-\overline{\phi(1/\bar{\zeta})}-\frac{\overline{\omega(1/\bar{\zeta})}}{\omega'(\zeta)}\phi'(\zeta) \quad (27)$$

Substituting (20) into (27), it is obtained that

$$\psi_2(\zeta)=-\overline{\phi_2(1/\bar{\zeta})}-\frac{\overline{\omega(1/\bar{\zeta})}}{\omega'(\zeta)}\phi_2'(\zeta)$$
$$-\psi_1(\zeta)-\overline{\phi_1(1/\bar{\zeta})}-\frac{\overline{\omega(1/\bar{\zeta})}}{\omega'(\zeta)}\phi_1'(\zeta) \quad (28)$$

Since there also exists free boundary for the semi-infinite plane acted by concentrated forces, $\psi_1(\zeta)$ can be expressed by $\phi_1(\zeta)$ as presented in (5). By substituting (5) into (28), it is obtained that

$$\psi_2(\zeta)=-\overline{\phi_2(1/\bar{\zeta})}-\frac{\overline{\omega(1/\bar{\zeta})}}{\omega'(\zeta)}\phi_2'(\zeta) \quad (29)$$

It is noticed that there exists irregular term in the right side of (29) as

$$\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_2'(\zeta)=\sum_{k=1}^{N}\frac{A_k\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}+\text{regular part}$$
$$(\zeta\in S^+) \quad (30)$$

where $A_k\equiv\phi_2'(\zeta_k')$.

Since $\psi_2(\zeta)$ must be regular in S^+, $\overline{\phi_2(1/\bar{\zeta})}$ must cancel the irregular term of (30), i.e.,

$$\overline{\phi_2(1/\bar{\zeta})}=-\sum_{k=1}^{N}\frac{A_k\overline{B_k}\zeta_2'^2}{\zeta-\zeta_k'}+\text{regular part}$$
$$(\zeta\in S^+) \quad (31)$$

From (23) it is obtained that

$$\overline{\phi_2(1/\bar{\zeta})}=\overline{H_1(1/\bar{\zeta})}+\overline{H_2(1/\bar{\zeta})}+\overline{H_3(1/\bar{\zeta})}$$
$$+\frac{1-i\mu}{2}\overline{J(1/\bar{\zeta})}+\overline{Q(1/\bar{\zeta})\chi(1/\bar{\zeta})} \quad (32)$$

On the other hand, it can be proved that

$$\overline{F_1(1/\bar{\zeta})}=-F_2(\zeta) \quad (33a)$$
$$\overline{G_1(1/\bar{\zeta})}=-G_2(\zeta) \quad (33b)$$
$$\overline{G_3(1/\bar{\zeta})}=-G_3(\zeta) \quad (33c)$$

Making use of (33), it is easy to prove that

$$\overline{H_3(1/\bar{\zeta})}=-\frac{1+i\mu}{1-i\mu}H_3(\zeta) \quad (34)$$

The following equations can also be derived[3]

$$\overline{H_1(1/\bar{\zeta})}=-\frac{1+i\mu}{1-i\mu}H_1(\zeta) \quad (35a)$$
$$\overline{H_2(1/\bar{\zeta})}=-\frac{1+i\mu}{1-i\mu}H_2(\zeta) \quad (35b)$$
$$\overline{J(1/\bar{\zeta})}=\frac{1+i\mu}{1-i\mu}J(\zeta) \quad (35c)$$

(34) and (35) mean that the first four terms of the right side of (32) are all regular in S^+.

Therefore it must be that

$$\overline{Q(1/\bar{\zeta})\chi(1/\bar{\zeta})} = -\sum_{k=1}^{N} \frac{A_k B_k \zeta_k'^2}{\zeta - \zeta_k'} + \text{regular part} \qquad (\zeta \in S^+) \quad (36)$$

Finally

$$Q(\zeta)\chi(\zeta) = -\sum_{k=1}^{N} \frac{\chi(\zeta)\overline{A_k B_k}}{\chi(\zeta_k)(\zeta_k - \zeta)} \qquad (37)$$

Making use of (34) and (35), $J(\zeta)$ can be determined so as to satisfy (22c, 24d) as[3]

$$J(\zeta) = -\sum_{k=1}^{N}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{\overline{A_k B_k}}{\zeta_k - \zeta} + \frac{1 - i\mu}{1 + i\mu}\sum_{k=1}^{N}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{A_k \overline{B_k} \zeta_k'^2}{\zeta_k' - \zeta} + \text{const} \quad (38)$$

Therefore each term of $\phi_2(\zeta)$ expressed by (23) has been determined, i.e., $H_1(\zeta)$ is expressed by (24a), $H_2(\zeta)$ by (24b), $H_3(\zeta)$ by (25), $J(\zeta)$ by (38) and $Q(\zeta)\chi(\zeta)$ by (37), respectively.

A_k and $\overline{A_k}$ are determined by solving 2N linear simultaneous equations for real and imaginary parts of $A_k = \phi_2'(\zeta_k')$ (k=1,2,3,...,N). In this paper, $N = 24$ and thus 48 linear simultaneous equations must be solved.

(2) The semi-infinite plane subjected to point dislocations

The complex stress functions in the present case are also divided into two parts as expressed in (20), where $\phi_1(\zeta)$ and $\psi_1(\zeta)$ correspond to $\phi_d(\zeta)$ and $\psi_d(\zeta)$, which represent the solution of the semi-infinite plane subjected to two pairs of point dislocations presented by (15). In the same procedure described in the case of concentrated force, the solution of the present case can be obtained as

$$\begin{aligned}\phi(\zeta) &= \phi_d(\zeta) + \phi_2(\zeta)\\ &= \phi_d(\zeta) + H_1(\zeta) + H_2(\zeta) + H_4(\zeta)\\ &\quad + \frac{1 + i\mu}{2} J(\zeta) + Q(\zeta)\chi(\zeta) \qquad (39)\end{aligned}$$

where $\phi_d(\zeta)$ is expressed by (15), $H_1(\zeta)$ by (24a), $H_2(\zeta)$ by (24b), $J(\zeta)$ by (38) and $Q(\zeta)\chi(\zeta)$ by (37), respectively.

$H_4(\zeta)$ in this case is given by

$$H_4(\zeta) = \frac{1 - i\mu}{4\pi}\left[-(\overline{D} + D)F_1 + (\overline{D} + D)F_2 + DG_4 + \overline{D}G_5 + 2\pi G_3\right] \qquad (40)$$

where F_1, F_2 and G_3 are expressed by (26) and

$$\begin{aligned}G_4 &= \frac{\overline{\omega(1/\bar{\zeta}_0)} - \omega(\zeta_0)}{\omega'(\zeta_0)}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_0)}\right]\frac{1}{\zeta - \zeta_0} - \frac{\overline{\omega(1/\bar{\zeta}_m)} - \omega(\zeta_m)}{\omega'(\zeta_m)}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_m)}\right]\frac{1}{\zeta - \zeta_m}\\ &\quad + \sum_{k=1}^{N}\left(\frac{1}{\zeta_k' - \zeta_0} - \frac{1}{\zeta_k' - \zeta_m}\right)\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{\overline{B_k}\zeta_k'^2}{\zeta - \zeta_k'} \quad (41)\end{aligned}$$

$$\begin{aligned}G_5 &= \frac{\omega(1/\bar{\zeta}_0) - \overline{\omega(\zeta_0)}}{\overline{\omega'(\zeta_0)}}\left[1 - \frac{\chi(\zeta)}{\chi(1/\bar{\zeta}_0)}\right]\frac{(1/\bar{\zeta}_0)^2}{\zeta - 1/\bar{\zeta}_0} - \frac{\omega(1/\bar{\zeta}_m) - \overline{\omega(\zeta_m)}}{\overline{\omega'(\zeta_m)}}\left[1 - \frac{\chi(\zeta)}{\chi(1/\bar{\zeta}_m)}\right]\frac{(1/\bar{\zeta}_m)^2}{\zeta - 1/\bar{\zeta}_m}\\ &\quad + \sum_{k=1}^{N}\left(\frac{1}{\overline{\zeta_k'} - \bar{\zeta}_0} - \frac{1}{\overline{\zeta_k'} - \bar{\zeta}_m}\right)\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{B_k}{\zeta - \zeta_k} \quad (42)\end{aligned}$$

6. STRESS INTENSITY FACTORS

The stress intensity factors of the crack are calculated by[6]

$$\begin{aligned}&K_I - iK_{II}\\ &= 2\sqrt{2\pi}\lim_{\sigma \to \sigma_0}\left\{\sqrt{[\omega(\sigma) - \omega(\sigma_0)]e^{-i\delta}}\,\phi'(\sigma)/\omega'(\sigma)\right\}\\ &= 2\sqrt{\pi}e^{-i\delta/2}\phi'(\sigma_0)/\sqrt{\omega''(\sigma_0)} \qquad (43)\end{aligned}$$

where $\sigma_0 = (1 - 2s + i)/(1 - 2s - i)$ is ζ on the unit circle corresponding to the tip of the crack (point C on the unit circle), and $\delta = -i\gamma\pi/180$ represents the angle between the x-axis and the crack.

The non-dimensional stress intensity factors are defined as

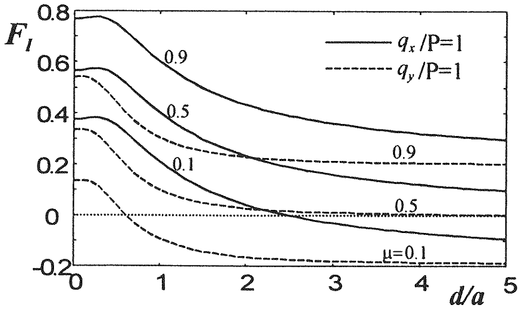

Fig.3 F_I with different μ and d, $y_0 = 0$ and $\gamma = 90^\circ$

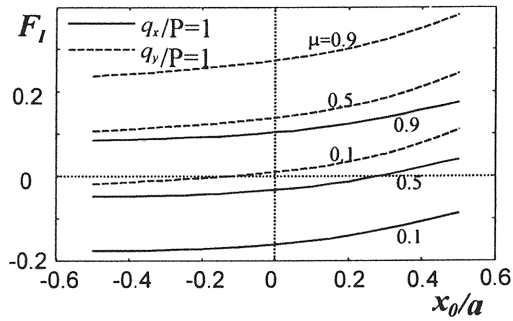

Fig.5 F_I with different μ and x_0, $y_0 = -a$ and $\gamma = 60^\circ$

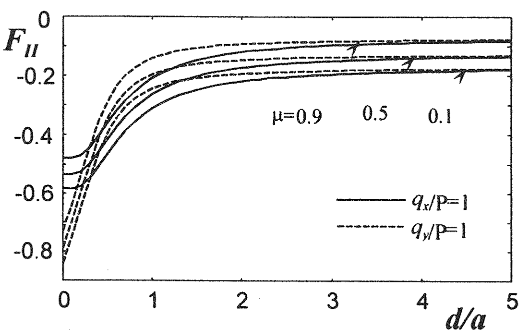

Fig.4 F_{II} with different μ and d, $y_0 = 0$ and $\gamma = 90^\circ$

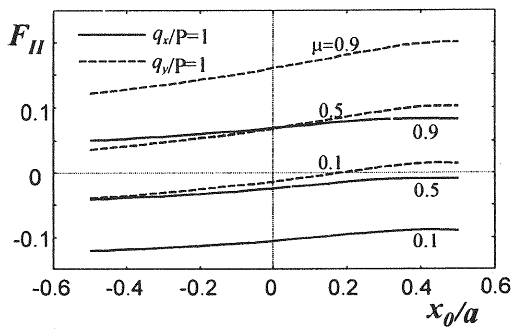

Fig.6 F_{II} with different μ and x_0, $y_0 = -a$ and $\gamma = 60^\circ$

$$F_I + iF_{II} = \frac{\sqrt{a}(K_I + iK_{II})}{P\sqrt{\pi}} \tag{44}$$

In the following examples, $b/a = 0.5$, $c/a = 0$, κ=2 and $Ga^2/(PR) = 1$ are selected for calculation**Figs.3** and **4** show F_I and F_{II} with different μ and d, where $d = x_0 - a/2 - c$ and $y_0 = 0$, which represents that q_x or q_y acts on the surface of the semi-infinite plane. The incline angle of the crack is typically taken as $\gamma = 90^\circ$. When the concentrated force acts on the boundary, the solution can also be obtained by using (19c). F_I usually decreases with the increase of d, and tends to a stable value (for the case of punch only) for each μ. The influence of q_x on F_I is greater than that of q_y on F_I owing to the fact that the edge crack is in vertical direction; F_{II} increases with the increase of d and also tends to a stable value for each μ. Both F_I and F_{II} increase with the increase of μ.

Figs.5 and **6** show F_I and F_{II} with different μ and x_0, and $y_0 = -a$. The angle of the crack is taken as $\gamma = 60^\circ$. F_I and F_{II} usually increase with the increase of x_0 and the influence of q_x on F_I and F_{II} is smaller than that of q_y on them

7. RESULTANT MOMENT

The resultant moment on the contact region about the origin of the x-y coordinates is necessary to decide the position of the load P on the punch (see **Fig.2**), and is calculated by [7]

$$R_m = -\mathrm{Re}\left[\int_\alpha^\beta \omega(\sigma)\overline{\phi'}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma^2} + \int_\alpha^\beta \overline{\omega}\left(\frac{1}{\sigma}\right)\phi'(\sigma)d\sigma\right] \tag{45}$$

The position of the load P is decided by

$$Pe = R_m \tag{46}$$

The non-dimensional resultant moment which also means the length e/a is defined by

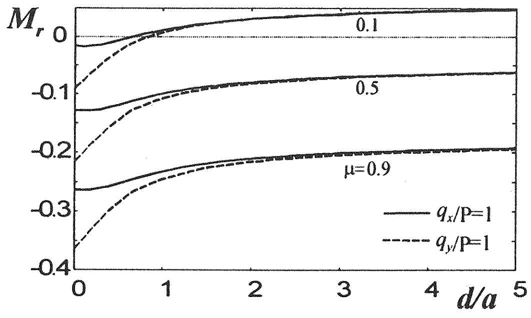

Fig.7 M_r with different μ and d, $y_0 = 0$ and $\gamma = 90^\circ$

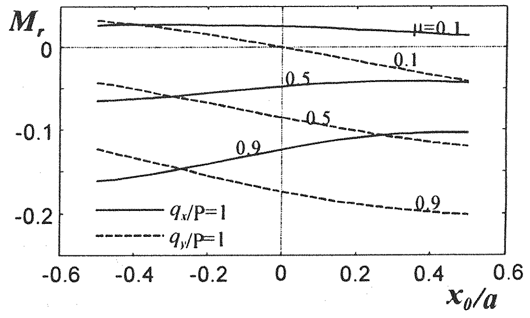

Fig.8 M_r with different μ and x_0, $y_0 = -a$ and $\gamma = 60^\circ$

$$M_r = \frac{R_m}{Pa} = \frac{e}{a} \tag{47}$$

Fig.7 shows M_r with q_x and q_y act on the surface of the semi-infinite plane with $\gamma = 90^\circ$. M_r increases with the increase of d, and tends to a stable value for each μ, which corresponds to M_r for the punch problem without q_x and q_y in the semi-infinite plane. The influence of q_y on M_r is greater than q_x when d is relatively small but tends to the same value with the increase of d since the influences of q_x and q_y become smaller and smaller with the increase of d for each μ. **Fig.8** shows M_r with q_x and q_y that act in the semi-infinite plane with $y_0 = -a$ and $\gamma = 60^\circ$. When q_y is applied, M_r decreases with the increase of x_0 for each μ; while q_x is applied, M_r usually increases with the increase of x_0 for large μ but decreases for small μ. For all cases, the larger the value of μ becomes, the smaller the value of M_r becomes. It is noted that the positive value of M_r represents anti-clockwise moment on the contact region, which corresponds to the case that the load P is on the left side of the y-axis. In the same process, the results for the punch on the semi-infinite plane subjected to point dislocation can be obtained.

8. CONCLUSIONS

The solution of circular rigid punch on a cracked semi-infinite plane subjected to concentrated force or point dislocation was derived. Since one part of the complex stress functions of the punch problem is selected as the solution (see(20)) of the corresponding cracked semi-infinite plane subjected to the concentrated force or point dislocation, the influence of the concentrated force or point dislocation and the boundary conditions of the semi-infinite plane can be reflected completely. The derivation of the solutions (10) and (15) for a semi-infinite plane and (20) for a punch problem does not need tedious integration because they have been derived from the condition of the regularity of $\psi(\zeta)$. The concentrated forces or point dislocations are located at arbitrary points in the semi-infinite plane. If V in (16c) is changed, the punch problem of other shapes can be solved. If the radius of curvature of the punch tends to infinity, the fundamental solution of flat-ended punch problem can be obtained. If the coefficients E_k (k=1,2,3,...,N) in (2) are taken to be zero, the solution of the punch on a semi-infinite plane without crack can be obtained. The first derivatives of (24a) and (26a,b) can be expressed in the form without integration, therefore the expression to decide the A_k and stress components does not include any integral terms so that the numerical integration is not needed for the calculation of stress components, stress intensity factors as well as resultant moment on the contact region. Since the punch is assumed to be vertical on the semi-infinite plane, the resultant moment on the contact region is needed to balance the moment produced by the eccentric load on the punch. $Ga^2/(PR)$ is a non-dimensional parameter which provides a relation among the length a of the contact region, the load P on the punch, the radius R of the punch and the material constant G of the semi-infinite plane. The solutions in the present paper can be progressively used to analyze more complicated problems related to punch problems with internal crack or hole, which is very efficient compared to common computational methods, such as FEM, BEM, etc..

Not only the analytical property of the punch and edge crack can be reflected efficiently without numerical modeling around the contact region, but the crack and the semi-infinite plane, the stress intensity factors of the crack and resultant moment on the contact region can also be obtained directly.

APPENDIX A

The two irrational terms in (1) can be approximated by the following rational functions

$$(1+i\zeta)^{s}=1+\sum_{j=1}^{12}\left(A_j+\frac{-A_j}{1+i\alpha_j\zeta}\right)$$

$$(1-i\zeta)^{1-s}=1+\sum_{k=1}^{12}\left(B_k+\frac{-B_k}{1-i\beta_k\zeta}\right)$$

The coefficients A_j, α_j (j=1,2,...,12) and B_k, β_k (k=1,2,...,12) can be determined by solving a nonlinear algebraic equation. The method of solving the equation was described in reference 6).

Substituting the above expressions into (1), expression (2) can then be obtained.

APPENDIX B

The first integral in (24c) can be expressed by

$$\int_M \frac{f_1(\sigma)}{\chi^+(\sigma)(\sigma-\zeta)}d\sigma = C_1\oint_M \frac{f_1(\sigma)}{\chi(\sigma)(\sigma-\zeta)}d\sigma$$

$$=2\pi i C_1\cdot\frac{1}{2\pi i}\oint_M \frac{\log\left(s-1/\overline{\zeta_0}\right)-\log\left(\sigma-1/\overline{\zeta_m}\right)}{\chi(\sigma)(\sigma-\zeta)}d\sigma$$

$$=2\pi i C_1\left\{\frac{\log(\zeta-1/\overline{\zeta_0})-\log(\zeta-1/\overline{\zeta_m})}{\chi(\zeta)}+\int_{1/\overline{\zeta_m}}^{1/\overline{\zeta_0}}\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)}\right\}$$

where $\chi^-(\sigma)=-\chi^+(\sigma)/g$ on M is used, and $C_1=\left[(\kappa+1)-i\mu(\kappa-1)\right]/\left[2(\kappa+1)\right]$.

Therefore

$$F_1=\log(\zeta-1/\overline{\zeta_0})-\log(\zeta-1/\overline{\zeta_m})+\chi(\zeta)\int_{1/\overline{\zeta_0}}^{1/\overline{\zeta_m}}\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)}$$

In the same procedure, F_2 can be derived.

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