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DEGENERATION OF BIFURCATION HIERARCHY OF A RECTANGULAR PLATE DUE TO BOUNDARY CONDITIONS†

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A simply-supported rectangular plate subject to a pure bending undergoes successive hierarchial bifurcation. The bifurcation structure of this plate arises from the "hidden (circular) symmetry" of the periodic nature of its deflection. We will arrive at this structure by means of the concept of irreducible representations in the group-theoretic bifurcation theory. The boundary conditions are revealed to significantly alter the bifurcation structure.

Key Words *: Bifurcation, group theory, hidden symmetry, irreducible representations.*

1. INTRODUCTION

Most of bifurcation can be ascribed with the "symmetry-breaking bifurcation" due to the (partial) loss of the symmetry of a system. For the Euler buckling, for example, from the trivial solution with the reflection symmetry with respect to the member axis, a bifurcated one without it emerges. For the buckling of a cylindrical shell, a bifurcated solution without axisymmetry branches from the axisymmetric trivial solution. It is customary to employ groups which are made up of reflections and rotations in describing the symmetry[1)]. The symmetry of governing equations is described by the group-equivariance condition, which shows the objectivity of these equations. The mathematical framework of a system equivariant to a group can be known a *priori* by means of the group-theoretic bifurcation theory[2),3),4)]. Block-diagonalization method, which can decompose the governing equations into a series of independent equations, is established as a method to exploit symmetry[5),6),7)].

In the field of structural engineering, the group-representation theory has come to be employed to describe bifurcation behavior of structures [8),9),10),11)]. Extensive research has been conducted on axisymmetric systems (equivariant to a group D_∞ for circular symmetry or to a group D_n for regular $n$gonal symmetry) to arrive at a bifurcation diagram[9)], and to present a block-diagonalization method for the tangent stiffness matrix for discrete systems[8),10),11)]. It is advantageous in numerical analysis to grasp the mechanism of bifurcation by means of the bifurcation diagram, and is numerically efficient and stable to put the tangent stiffness matrix into a block-diagonal form.

The symmetry of structures can be categorized into "natural symmetry" and "hidden one." The bifurcation mechanism of the former, which expresses the geometrical symmetry (such as the circumferential symmetry of axisymmetric shells, can be obtained by the group-theoretic bifurcation theory. The latter, such as the bifurcation of a beam on a foundation, is ascribed with the periodic nature of solutions, and its mechanism suffers from the "degeneration" due to the boundary conditions. The secondary bifurcation of a beam[12)] serves its example.

In this paper, we focus on a simply-supported rectangular plate in **Fig.1** as an example of the "hidden symmetry." Although this plate only has the reflection with respect to the y-axis in view of the geometric symmetry, in fact, is equivariant to D_∞ due to the hidden periodic symmetry in the x-direction. The mechanism of the degeneration of the hierarchical structure of the bifurcation of this plate is investigated by the group representation theory. This system is shown to

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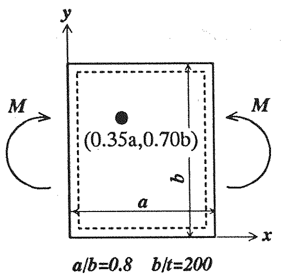


Fig. 1 Simply-supporter rectangular plate under pure bending

have a different bifurcation diagram than a D_∞-equivariant system due to "geometrical symmetry." The tangential stiffness matrix of the plate is shown to be block-diagonalized by expanding the displacement by the Fourier series and appropriately permutating their order. A formula for this permutation is presented. The bifurcation analysis of the plate is carried out to assess the validity of the present theory.

2. BIFURCATION THEORY OF SYSTEMS WITH SYMMETRY

In this section a method for describing the symmetry of a system with geometric symmetry is introduced[3),4),10),11)].

(1) Group-equivariance of equilibrium equations

Denote by

$$F(f, u) = 0 \tag{1}$$

a set of N-dimensional equilibrium equations. Here f stands for a loading parameter and $\boldsymbol{u}$ for a displacement vector, respectively. In the vicinity of an equilibrium point $(f, \boldsymbol{u})$, we rewrite Eq. (1) into an incremental form

$$\boldsymbol{F} = J\mathrm{d}\boldsymbol{u} + \boldsymbol{F}_0(\mathrm{d}f, \mathrm{d}\boldsymbol{u}) = \boldsymbol{0} \tag{2}$$

Here $J = \partial \boldsymbol{F}/\partial \boldsymbol{u}$ denotes the tangential stiffness matrix and $\boldsymbol{F}_0$ indicates a nonlinear vector.

Consider a group G made up of a series of geometric transformation g, such as reflections and rotations, in describing the symmetry of the equilibrium equation. For example, an element g of a group G transforms an N-dimensional vector $\boldsymbol{u}$ (respectively, $\boldsymbol{F}$) into $g(\boldsymbol{u})$ (respectively, $g(\boldsymbol{F})$). The mechanism of such transformation can be defined by an $N \times N$ representation matrix $T(g)$, such that

$$T(g)\boldsymbol{u} = g(\boldsymbol{u}), \quad T(g)\boldsymbol{F} = g(\boldsymbol{F}), \quad {}^\forall g \in G \tag{3}$$

The representation matrix, which represents the coordinate transformation of an element g in the relevant vector space, is assumed to be unitary. The equilibrium equation is said to be equivariant to a group G when

$$T(g)\boldsymbol{F}(f, \boldsymbol{u}) = \boldsymbol{F}(f, T(g)\boldsymbol{u}), \quad {}^\forall g \in G \tag{4}$$

is satisfied. Eq. (4), which is a general symmetry condition, means that the transforming of the independent variable $\boldsymbol{u}$ by $T(g)$ turns out to be identical with the transforming of the whole set of equations $\boldsymbol{F}$ by $T(g)$. The invariance of the solution $\boldsymbol{u}$, which is different concept from the equivariance, is defined by $T(g)\boldsymbol{u} = \boldsymbol{u}$ (${}^\forall g \in G$). For a G-invariant $\boldsymbol{u}$, the tangential stiffness matrix $J = \partial \boldsymbol{F}/\partial \boldsymbol{u}$ satisfies a symmetry condition $T(g)J = JT(g)$ (${}^\forall g \in G$) by Eq. (4), and hence can be put into a block-diagonal form by means of a suitable transformation. The equilibrium equations of a group-equivariant system, satisfying Eq. (4), is known to be partitioned into a series of equations associated with the irreducible representations of the group [14)]. The type and the number of irreducible representations, dependent on the group, will be shown later.

Define by

$$T^\mu(g) \equiv T_i^\mu(g), \tag{5}$$
$$i = 1, \cdots, a^\mu, \quad {}^\forall g \in G, \ \mu \in R(G)$$

the irreducible representations of a group G. Here μ indicates an irreducible representation of G, $R(G)$ denotes the whole set of irreducible representations, a^μ denotes the multiplicity of μ in $T(g)$. These representations are not dependent on particular systems but solely on the group G. It is a fundamental strategy of the "group-representation theory" to derive general rules for those matrices, and, in turn, to describe the symmetry of a particular system by obtaining the transformation matrix H between $T(g)$ and the irreducible representations.

This corresponds to finding H that block-diagonalizes $T(g)$ into the components associated with the irreducible representations, that is,

$$H^\mathrm{T} T(g) H = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} T_i^\mu(g), \quad g \in G \tag{6}$$

where $\bigoplus$ denotes the direct sum. The form of H, which depends on the definition of $\boldsymbol{u}$ and G, will be given in the next section for particular examples.

The space X for $\boldsymbol{u}$ can be decomposed into the direct sum of subspaces X^μ

$$X = \bigoplus_{\mu \in R(G)} X^\mu \tag{7}$$

by means of the isotypic or standard decomposition in a form corresponding to Eq. (6)[14]. Each subspace is associated with the solution for the main or bifurcation path. The transformation matrix can also be decomposed into the form of $H \equiv [\cdots, H^{\mu}, \cdots]$ made up of blocks H^{μ} associated with the irreducible representations. With the use of this H the tangential stiffness matrix can be block-diagonalized, that is,

$$\tilde{J} = H^{\mathrm{T}} J H = \mathrm{diag}[\cdots, \tilde{J}^{\mu}, \cdots] \qquad (8)$$

where diag[$\cdots$] denotes a block-diagonal matrix. Block-diagonalization, which reduces the size of matrices for analysis, is numerically efficient.

Define a coordinate system $\boldsymbol{w} = [\cdots, (\boldsymbol{w}^{\mu})^{\mathrm{T}}, \cdots]^{\mathrm{T}}$ associated with the representations by

$$\boldsymbol{u} = H\boldsymbol{w} = \sum_{\mu} H^{\mu} \boldsymbol{w}^{\mu} \qquad (9)$$

The equilibrium equations (2) can be transformed by means of (9) into a block-diagonal form

$$\tilde{J}^{\mu^*} \mathrm{d}\boldsymbol{w}^{\mu^*} + (H^{\mu^*})^{\mathrm{T}} \boldsymbol{F}_0 = \boldsymbol{0} \qquad (10\mathrm{a})$$

$$\tilde{J}^{\mu} \mathrm{d}\boldsymbol{w}^{\mu} = \boldsymbol{0}, \qquad \mu \neq \mu^* \qquad (10\mathrm{b})$$

in compatibly with the representations. Here μ^* in Eq. (10a) indicates the unit irreducible representation, and is related to the main path. Eq. (10b) yields a trivial solution when $\tilde{J}^{\mu}$ is regular, and a bifurcation mode when singular. The block-diagonalization method is advantageous in that the main path can be obtained from Eq. (10a) even at a bifurcation point, where $\tilde{J}^{\mu}$ is singular[2),10)], and that the block-diagonal form of Eq. (10) corresponds to the categorization of singular points[11)].

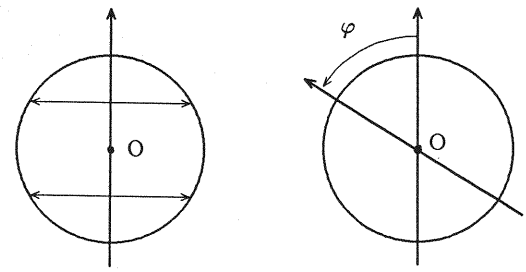
A system equivariant to a group G is known to lose symmetry through symmetry-breaking bifurcation[2),3),4),9)]. Such bifurcation can be characterized by a nested set of subgroups $G \to G_1 \to G_2 \to \cdots$, where $G_i \to G_{i+1}$ represents the emergence a G_{i+1}-invariant solution from a G_i-invariant one. This equation means that the symmetry of the system is reduced from a G-invariant state into G_1, G_2-invariant ones. The framework of the bifurcation of the system can be known a *priori* by investigating the bifurcation structure of the groups G, G_1, $G_2, \cdots$.

3. BIFURCATION STRUCTURE OF D_{∞}-EQUIVARIANT SYSTEM

We focus on the following bifurcation structure of a system equivariant to a group D_{∞}:

$$D_{\infty} \to D_n \to \cdots \to C_1 \qquad (11)$$

The group D_{∞} in (11) expresses the symmetry of a circle, and is defined by $D_{\infty} \equiv \langle s, r(\varphi) \rangle$ Here



(*a*) *s* (*b*) *r*

Fig. 2 The actions of transformation s and r

$\langle\ \rangle$ denotes a group generated by the elements therein, s means a reflection, and $r(\varphi)$ indicates a counter-clockwise φ $(0 \leq \varphi < 2\pi)$ rotation around the origin shown in **Fig.2**. In addition, $D_n \equiv \langle s, r(2\pi/n) \rangle$ is a dihedral group of the degree n expressing the regular $n$gonal symmetry. The subgroups of D_n in Eq.(11) are expressed as

$$D_m^j \equiv \langle r(2\pi/m), sr(2\pi(j-1)/m) \rangle \qquad (12\mathrm{a})$$
$$j = 1, \cdots, m-1$$

$$C_m \equiv \langle r(2\pi/m) \rangle \qquad (12\mathrm{b})$$

where $D_m = D_m^1$ and $C_1 = \langle 1 \rangle$, The dihedral group D_m^j of degree m expresses the reflection symmetry with respect to m straight lines, and the cyclic group C_m denotes the rotation symmetry with respect to an angle of $2\pi/m$, and C_1 indicates an asymmetric mode.

(1) D_{∞}-invariant solution

We investigate the direct bifurcation from a D_{∞}-invariant solution[3),4)]. The whole set of the irreducible representations of a group D_{∞} reads

$$R(D_{\infty}) = \begin{array}{l} ((1,1)_{D_{\infty}}, (1,2)_{D_{\infty}}, \\ (2,1)_{D_{\infty}}, (2,2)_{D_{\infty}}, \cdots) \end{array} \qquad (13)$$

where $(1,1)_{D_{\infty}}$ and $(1,2)_{D_{\infty}}$ correspond to the one-dimensional irreducible representations and their representation matrices are 1×1 and are defined by

$$T^{(1,1)_{D_{\infty}}}(r(\varphi)) = 1, \quad T^{(1,1)_{D_{\infty}}}(s) = 1 \quad (14\mathrm{a})$$

$$T^{(1,2)_{D_{\infty}}}(r(\varphi)) = 1, \quad T^{(1,2)_{D_{\infty}}}(s) = -1 \quad (14\mathrm{b})$$

By contrast, $(2,n)_{D_{\infty}}$ $(n = 1, 2, \cdots)$ correspond to the two-dimensional ones and their representation matrices are 2×2 and are defined by

$$T^{(2,n)_{D_{\infty}}}(r(\varphi)) = \begin{pmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{pmatrix} (15\mathrm{a})$$

$$T^{(2,n)_{D_{\infty}}}(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad n = 1, 2, \cdots (15\mathrm{b})$$

The isotypic (standard) decomposition of the

Table 1 Categorization of critical points of a D_∞-equivariant system

μ satisfying $\det \tilde{J}^\mu = 0$	$(1,1)_{D_\infty}$	$(1,2)_{D_\infty}$	$(2,n)_{D_\infty}$
Category of points	Limit points	Simple, symmetric bifurcation point	Double bifurcation point
Symmetry of solutions	D_∞	C_∞	D_n

space X with respect to a group D_∞ reads

$$X = X^{(1,1)_{D_\infty}} \bigoplus X^{(1,2)_{D_\infty}} \bigoplus \left(\bigoplus_{n=1}^{\infty} X^{(2,n)_{D_\infty}} \right) \tag{16}$$

Here the subspace $X^{(2,n)_{D_\infty}}$ can be further decomposed into a direct sum of two subspaces:

$$X^{(2,n)_{D_\infty}} = X^{(2,n)^+_{D_\infty}} \bigoplus X^{(2,n)^-_{D_\infty}} \tag{17}$$

The coordinate transformation matrix H for this decomposition is defined by

$$H = \Big[H^{(1,1)_{D_\infty}}, H^{(1,2)_{D_\infty}}, H^{(2,1)^+_{D_\infty}}, H^{(2,1)^-_{D_\infty}}, H^{(2,2)^+_{D_\infty}}, H^{(2,2)^-_{D_\infty}}, \cdots \Big] \tag{18}$$

The blocks H^μ associated with the irreducible representations can be chosen to be labeled by the following symmetry groups:

$$\Sigma(H^{(1,1)_{D_\infty}}) = D_\infty, \quad \Sigma(H^{(1,2)_{D_\infty}}) = C_\infty \tag{19a}$$

$$\Sigma(H^{(2,n)^+_{D_\infty}}) = D_n, \quad \Sigma(H^{(2,n)^-_{D_\infty}}) = C_n \tag{19b}$$

where $\Sigma(\cdot)$ denotes the symmetry of the column vectors of the matrix therein. The block-diagonal from of the tangential stiffness matrix becomes

$$\tilde{J} = \mathrm{diag}\Big[\tilde{J}^{(1,1)_{D_\infty}}, \tilde{J}^{(1,2)_{D_\infty}}, \tilde{J}^{(2,1)_{D_\infty}}, \tilde{J}^{(2,1)_{D_\infty}}, \tilde{J}^{(2,2)_{D_\infty}}, \tilde{J}^{(2,2)_{D_\infty}}, \cdots \Big] \tag{20}$$

A simple critical point is defined as a point where the one-dimensional irreducible representation becomes singular. In particular, the unit irreducible representation $(1,1)_{D_\infty}$ is associated with a limit point of the loading parameter f, and $(1,2)_{D_\infty}$ to a simple, symmetric bifurcation point. A double (group-theoretic) bifurcation point is associated with the two-dimensional one, which corresponds to two identical diagonal blocks. According to the type and the number of the diagonal blocks that become singular, we can categorize the critical points as listed in **Table 1**.

(2) D_n-invariant solution

We investigate the mechanism of the bifurcation from a D_n-invariant solution, which emerges as a bifurcated solution of a D_∞-invariant system [9]. The whole set of the irreducible representations of D_n is given by[10]

$$R(D_n) = \{\mu \equiv (d,j)_{D_n} \mid j = 1, \cdots, m_d; d = 1,2\} \tag{21}$$

where $(d,j)_G$ means the jth d-dimensional irreducible representations of a group G, and m_d denotes the number of d-dimensional irreducible representations, and is equal to

$$\begin{cases} m_1 = 4, & m_2 = n/2 - 1, \quad \text{when } n = \text{even} \\ m_1 = 2, & m_2 = (n-1)/2, \quad \text{when } n = \text{odd} \end{cases} \tag{22}$$

The irreducible representation matrices of one-dimensional irreducible representations $(1,j)_{D_n}$ $(j = 1,2,3,4)$ are given by

$$\begin{cases} T^{(1,1)_{D_n}}(r) = 1, & T^{(1,1)_{D_n}}(s) = 1 \\ T^{(1,2)_{D_n}}(r) = 1, & T^{(1,2)_{D_n}}(s) = -1 \\ T^{(1,3)_{D_n}}(r) = -1, & T^{(1,3)_{D_n}}(s) = 1 \\ T^{(1,4)_{D_n}}(r) = -1, & T^{(1,4)_{D_n}}(s) = -1 \end{cases} \tag{23}$$

Those for two-dimensional ones $(2,j)_{D_n}$ are

$$T^{(2,j)_{D_n}}(r) = R^j, \quad T^{(2,j)_{D_n}}(s) = S \tag{24}$$

$(j = 1, 2, \cdots)$. Here

$$R = \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{25}$$

The isotypic (standard) decomposition of the space X with respect to D_n reads

$$X = \left(\bigoplus_{j=1}^{m_1} X^{(1,j)_{D_n}} \right) \bigoplus \left(\bigoplus_{j=1}^{m_2} X^{(2,j)_{D_n}} \right) \tag{26}$$

The subspace $X^{(2,j)_{D_n}}$ can be further decomposed as follows:

$$X^{(2,j)_{D_n}} = X^{(2,j)^+_{D_n}} \bigoplus X^{(2,j)^-_{D_n}} \tag{27}$$

The transformation matrix is expressed as

$$H = \Big[H^{(1,1)_{D_n}}, \cdots, H^{(1,m_1)_{D_n}}, H^{(2,1)^+_{D_n}}, H^{(2,1)^-_{D_n}}, \cdots, H^{(2,m_2)^+_{D_n}}, H^{(2,m_2)^-_{D_n}} \Big] \tag{28}$$

We can choose the blocks H^μ of H such that

$$\Sigma(H^{(1,1)_{D_n}}) = D_n, \quad \Sigma(H^{(1,2)_{D_n}}) = C_n \tag{29a}$$

$$\Sigma(H^{(1,3)_{D_n}}) = D_{n/2}, \quad \Sigma(H^{(1,4)_{D_n}}) = D^2_{n/2} \tag{29b}$$

$$\Sigma(H^{(2,j)^+_{D_n}}) = D^k_{\mathrm{gcd}(j,n)} \tag{29c}$$

$$\Sigma(H^{(2,j)^-_{D_n}}) = \begin{cases} D^{k+n'/2}_{\mathrm{gcd}(j,n)}, & \text{when } n' = \text{even} \\ C_{\mathrm{gcd}(j,n)}, & \text{when } n' = \text{odd} \end{cases} \tag{29d}$$

Table 2 Categorization of critical points of a D_n-equivariant system

μ satisfying $\det \tilde{J}^{\mu}=0$	$(1,1)_{D_n}$	$(1,2)_{D_n}$	$(1,3)_{D_n}$	$(1,4)_{D_n}$	$(2,j)_{D_n}$
Category of points	Limit point	Simple, symmetric bifurcation point	Simple, symmetric bifurcation point	Simple, symmetric bifurcation point	Double bifurcation point
Symmetry of solutions	D_n	C_n	$D_{n/2}$	$D^2_{n/2}$	$D^k_{\mathrm{gcd}(j,n)}$

Table 3 Categorization of critical points of a C_n-equivariant system

μ satisfying $\det \tilde{J}^{\mu}=0$	$(1,1)_{C_n}$	$(1,2)_{C_n}$	$(2,j)_{C_n}$
Category of points	Limit point	Simple, symmetric bifurcation point	Double bifurcation point
Symmetry of solutions	C_n	$C_{n/2}$	$C_{\mathrm{gcd}(j,n)}$

$$n' = n/\mathrm{gcd}(j,n), \quad 1 \le k \le n', \quad j = 1, \cdots, m_2$$

where $\mathrm{gcd}(j,n)$ expresses the greatest common divisor of j and n.

The block-diagonal form of the tangential stiffness matrix reads

$$\tilde{J} = \mathrm{diag}\left[\tilde{J}^{(1,1)_{D_n}}, \cdots, \tilde{J}^{(1,m_1)_{D_n}}, \tilde{J}^{(2,1)_{D_n}}, \tilde{J}^{(2,1)_{D_n}}, \cdots, \tilde{J}^{(2,m_2)_{D_n}}, \tilde{J}^{(2,m_2)_{D_n}}\right] \tag{30}$$

According to the diagonal block that becomes singular, we can categorize critical points as listed in **Table 2**. The unit irreducible representation $(1,1)_{D_n}$ is associated with the limit point of the loading parameter, and the two-dimensional ones to double bifurcation points. Although the symmetry of the space $X^{(2,j)}$ is labeled by $C_{\mathrm{gcd}(j,n)}$, the bifurcated solution is labeled by a higher symmetry $D^k_{\mathrm{gcd}(j,n)}$.

(3) C_n-invariant solution

The mechanism of the bifurcation from a C_n-invariant solution, which branches from a D_n-invariant one, is presented[9]. The number of the irreducible representaions of a cyclic group C_n is equal to

$$\begin{cases} m_1 = 2\ ,\ m_2 = n/2-1, & \text{when } n = \text{even} \\ m_1 = 1\ ,\ m_2 = (n-1)/2, & \text{when } n = \text{odd} \end{cases} \tag{31}$$

The irreducible representation matrices are

$$T^{(1,1)_{C_n}}(r) = 1, \quad T^{(1,2)_{C_n}}(r) = -1 \\ T^{(2,j)_{C_n}}(r) = R^j \tag{32}$$

Similar to the case of D_n, the space X can be decomposed into the form of Eq. (26), the transformation matrix H into that of Eq. (28), the tangential stiffness matrix J into Eq. (30), respectively. Yet caution must be exercised on the fact that the number of the one-dimensional irreducible representaions differs from that for D_n, as can be seen from Eqs. (22) and (31). The symmetry of the blocks H^{μ} of H is labeled by

$$\Sigma(H^{(1,1)_{C_n}}) = C_n, \ \Sigma(H^{(1,2)_{C_n}}) = C_{n/2} \tag{33a}$$

$$\Sigma(H^{(2,j)^+_{C_n}}) = \Sigma(H^{(2,j)^-_{C_n}}) = C_{\mathrm{gcd}(j,n)} \tag{33b}$$

($j = 1, \cdots, m_2$). **Table 3** categorizes the critical points. It is to be noted that the bifurcated solutions of the double critical points for C_n exist only for the potential system.

4. DEGENERATION OF STRUCTURE OF BIFURCATION

Consider a (partial) differential equation

$$F(f, v) = 0 \tag{34}$$

on a domain of the length of a ($0 \le x \le a$), where v denotes the displacement. We assume that this equation undergoes bifurcation with the trivial solution of $v = 0$, and that, due to the boundary conditions, the displacement v can be written as the Fourier series of the sine, that is,

$$v = \sum_{i=1}^{\infty} u_i \sin\left(n\frac{x}{a}\pi\right) \tag{35}$$

As is clear from these assumptions, the problem formulation in this section focuses on a particular boundary condition; nonetheless, it is easily extendable to other type of boundaries. Defining a displacement vector $\boldsymbol{u} = (u_1, u_2, \cdots)^{\mathrm{T}}$ and discretizing Eq. (34), we can obtain a nonlinear equilibrium equation (1).

In describing the symmetry of the solutions of this problem, we double the interval $[0, a]$ into $[0, 2a]$[2]. As can be seen from **Fig.3**, since we can transform the domain $[0, 2a]$ into an imaginary circle, the trivial solution $v = 0$ has a circular

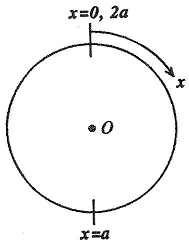


Fig. 3 An imaginary circle connecting the ends of the interval $[0, 2a]$

symmetry (D_∞-invariance), and hence Eq. (34) is D_∞-invariant. Denote by X the space for the solutions which can satisfy the boundary conditions, and by $\widehat{X}$ which cannot. We call the latter space an extended solution space. These spaces are defined respectively as

$$X = \text{span}\left[\sin\left(n\frac{x}{a}\pi\right)\right]_{n=1}^{\infty} \tag{36a}$$

$$\widehat{X} = \text{span}\left[\sin\left(n\frac{x}{a}\pi\right),\ \cos\left(n\frac{x}{a}\pi\right)\right]_{n=1}^{\infty} \tag{36b}$$

($0 \le x \le 2a$). Here span$[\cdot]$ indicates that the relevant space is spanned by the functions therein. The structure of bifurcation for the space $\widehat{X}$ has already been presented in Chapter 2. In this section, we investigate the way this mechanism is inherited to the space X, which is restricted by the boundary conditions, to obtain the Fourier series spanning the subspaces X^μ of X.

(1) D_∞-invariant solution

The direct bifurcation from a D_∞-invariant, trivial solution $v = 0$ is investigated. The action of the transformation $r(\varphi)$ and of s are expressed respectively as

$$r(\varphi) : x \to x + \frac{\varphi}{\pi}L, \qquad s : x \to -x \tag{37}$$

Their action on the Fourier series satisfies

$$r(\varphi)\begin{pmatrix}\cos(n\frac{x}{a}\pi)\\ \sin(n\frac{x}{a}\pi)\end{pmatrix} = T^{(2,n)_{D_\infty}}(r(\varphi))\begin{pmatrix}\cos(n\frac{x}{a}\pi)\\ \sin(n\frac{x}{a}\pi)\end{pmatrix}$$
$$= \begin{pmatrix}\cos(n\varphi) & -\sin(n\varphi)\\ \sin(n\varphi) & \cos(n\varphi)\end{pmatrix}\begin{pmatrix}\cos(n\frac{x}{a}\pi)\\ \sin(n\frac{x}{a}\pi)\end{pmatrix} \tag{38}$$

$$s\begin{pmatrix}\cos(n\frac{x}{a}\pi)\\ \sin(n\frac{x}{a}\pi)\end{pmatrix} = T^{(2,n)_{D_\infty}}(s)\begin{pmatrix}\cos(n\frac{x}{a}\pi)\\ \sin(n\frac{x}{a}\pi)\end{pmatrix}$$
$$= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos(n\frac{x}{a}\pi)\\ \sin(n\frac{x}{a}\pi)\end{pmatrix} \tag{39}$$

[cf. Eq. (15)]. Note that the nth modes $\cos(n\frac{x}{a}\pi)$ and $\sin(n\frac{x}{a}\pi)$ are related to the two-dimensional irreducible representation $(2,n)_{D_\infty}$. The trivial solution $v = 0$ is associated with $(1,1)_{D_\infty}$, and no solution is related to $(1,2)_{D_\infty}$.

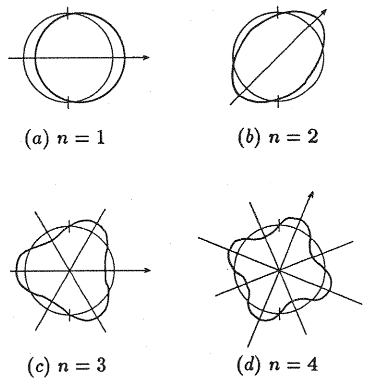


Fig. 4 The shift of the reflection planes due to the parity of n

By Eq. (16) the subspaces of the extended space $\widehat{X}$ decomposed by the isotypic decomposition is

$$\widehat{X}^{(1,1)_{D_\infty}} = \text{span}\,[\,0\,] \tag{40a}$$

$$\widehat{X}^{(2,n)^+_{D_\infty}} = \text{span}\left[\cos\left(n\frac{x}{a}\pi\right)\right] \tag{40b}$$

$$\widehat{X}^{(2,n)^-_{D_\infty}} = \text{span}\left[\sin\left(n\frac{x}{a}\pi\right)\right] \tag{40c}$$

The space X restricted by the boundary conditions, which lacks the cosine terms, are spanned by

$$X^{(1,1)_{D_\infty}} = \text{span}\,[\,0\,] \tag{41a}$$

$$X^{(2,n)_{D_\infty}} = \text{span}\left[\sin\left(n\frac{x}{a}\pi\right)\right] \tag{41b}$$

The decomposition in Eq. (17) ceases to exist due to the degeneration by the boundary conditions. Eq. (41) means that the main path is associated with the trivial solution $v = 0$, and that the bifurcation process $D_\infty \to D_n$ takes place at a simple, symmetric bifurcation point with a D_n-invariant bifurcation mode $\sin(n\frac{x}{a}\pi)$. A double bifurcation point is degenerated into a simple one due to the boundary conditions. The bifurcation process $D_\infty \to C_\infty$ ceases to exist due to the degeneration.

(2) D_n-invariant solution

Bifurcated solutions from a D_n-invariant solution are categorized. In view of the fact that the reflection planes of the D_n-invariant mode $\sin(n\frac{x}{a}\pi)$ depends on the parity of n, as shown in **Fig.4**, we define

$$r \equiv r\left(\frac{2\pi}{n}\right) : x \to x + \frac{2a}{n} \tag{42}$$

$$s : \begin{cases} x + \frac{a}{2} \to -\left(x + \frac{a}{2}\right), & n = \text{odd} \\ x + \frac{a}{2n} \to -\left(x + \frac{a}{2n}\right), & n = \text{even} \end{cases} \tag{43}$$

Consider a new coordinate system, which is chosen compatibly with the location of the reflection planes, that is,

$$x^* = \begin{cases} x + \frac{a}{2}, & n = \text{odd} \\ x + \frac{a}{2n}, & n = \text{even} \end{cases} \tag{44}$$

Then the action of r and of s, is rewritten as

$$r : x^* \to x^* + \frac{2a}{n}, \quad s : x^* \to -x^* \tag{45}$$

respectively. The mechanism of the bifurcation from the D_n-invariant system can be made clear by investigating the action of r and of s on the Fourier series of x^*, that is,

$$r\begin{pmatrix} \cos\left(i\frac{x^*}{a}\pi\right) \\ \sin\left(i\frac{x^*}{a}\pi\right) \end{pmatrix} = R^i \begin{pmatrix} \cos\left(i\frac{x^*}{a}\pi\right) \\ \sin\left(i\frac{x^*}{a}\pi\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{2i}{n}\pi\right) & -\sin\left(\frac{2i}{n}\pi\right) \\ \sin\left(\frac{2i}{n}\pi\right) & \cos\left(\frac{2i}{n}\pi\right) \end{pmatrix} \begin{pmatrix} \cos\left(i\frac{x^*}{a}\pi\right) \\ \sin\left(i\frac{x^*}{a}\pi\right) \end{pmatrix} \tag{46}$$

$$s\begin{pmatrix} \cos\left(i\frac{x^*}{a}\pi\right) \\ \sin\left(i\frac{x^*}{a}\pi\right) \end{pmatrix} = S \begin{pmatrix} \cos\left(i\frac{x^*}{a}\pi\right) \\ \sin\left(i\frac{x^*}{a}\pi\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(i\frac{x^*}{a}\pi\right) \\ \sin\left(i\frac{x^*}{a}\pi\right) \end{pmatrix} \tag{47}$$

When $2i$ is the multiple of n, for which R^i in Eq. (46) becomes diagonal, the ith cosine and sine terms correspond to two one-dimensional irreducible representaions. Otherwise, they are associated with a two-dimensional one.

a) When $n = $ odd

The displacement v associated with the one-dimensional irreducible representation $(1,1)_{D_n}$ satisfies the relationships $s \cdot v = v$ and $r \cdot v = v$ by Eq. (23). From Eqs. (46) and (47), v satisfying these relationships turns out to be associated with the cosine terms of x^* when the wave number i is the multiple of n, that is,

$$\cos\left(k'n\frac{x^*}{a}\pi\right), \quad k' = 1, 2, \cdots \tag{48}$$

With the use of Eq. (44), we can rewrite these terms into the functions in x ($k = 1, 2, \cdots$):

$$\cos\left(2kn\frac{x}{a}\pi\right) \quad \text{or} \quad \sin\left((2k-1)\,n\frac{x}{a}\pi\right) \tag{49}$$

By choosing the sine terms in this equation, we can obtain the terms spanning the space $X^{(1,1)_{D_n}}$

$$X^{(1,1)_{D_n}} = \text{span}\left[\sin\left((2k-1)\,n\frac{x}{a}\pi\right)\right]_{k=1}^{\infty} \tag{50}$$

This equation contains the sine series for n, $3n$, $\cdots$, while Eq. (41) has only the nth order term. It demonstrates that a D_n-invariant solution branching from a D_∞-invariant path possesses the pure nth sine mode in the vicinity of the bifurcation point but it is deformed by the mixing with higher order modes through "mode interaction."

The component v associated with one-dimensional irreducible representation $(1,2)_{D_n}$ satisfies the relationships $s \cdot v = -v$, $r \cdot v = v$ by Eq. (23). From Eqs. (46) and (47), v satisfying these relationships is the sine terms of x^* when the wave number i is the multiple of n. Through the rewriting of these terms by Eq. (44) into functions in x, one can see that

$$X^{(1,2)_{D_n}} = \text{span}\left[\sin\left(2kn\frac{x}{a}\pi\right)\right]_{k=1}^{\infty} \tag{51}$$

The Fourier series with the wave number $i \neq k'n$ correspond to the two-dimensional irreducible representaions $(2,j)_{D_n}$ ($j = 1, \cdots, \frac{n-1}{2}$). We categorize these wave numbers i into

$$\begin{aligned} &k'n + j, \quad (k'+1)n - j \\ &j = 1, \cdots, \tfrac{n-1}{2}, \quad k' = 0, 1, \cdots \end{aligned} \tag{52}$$

With the use of $R^j = T^{(2,j)_{D_n}}$ by Eq. (24), for the wave number $i = nk' + j$, Eq. (46) becomes

$$r\begin{pmatrix} \cos\left((k'n+j)\frac{x^*}{a}\pi\right) \\ \sin\left((k'n+j)\frac{x^*}{a}\pi\right) \end{pmatrix} = T^{(2,j)_{D_n}} \begin{pmatrix} \cos\left((k'n+j)\frac{x^*}{a}\pi\right) \\ \sin\left((k'n+j)\frac{x^*}{a}\pi\right) \end{pmatrix} \tag{53}$$

By Eqs. (46) and (53), the Fourier series for these wave numbers are associated with the jth two-dimensional irreducible representation $(2,j)$. The rewriting of these series into functions in x yields

$$\cos\left((k'n+j)\tfrac{x^*}{a}\pi\right) = \begin{cases} (-1)^{k+\frac{j-1}{2}} & \sin\left((2kn+j)\frac{x}{a}\pi\right) \\ (-1)^{k+\frac{n+j}{2}} & \cos\left(\{(2k+1)n+j\}\frac{x}{a}\pi\right) \\ & j = \text{odd} \\ (-1)^{k+\frac{j}{2}} & \cos\left((2kn+j)\frac{x}{a}\pi\right) \\ (-1)^{k+\frac{j}{2}+n} & \sin\left(\{(2k+1)n+j\}\frac{x}{a}\pi\right) \\ & j = \text{even} \end{cases} \tag{54}$$

($k = 0, 1, \cdots$). The terms for the sine of x^* can be given by replacing cosine with sine in this equation. In the sequel, the equations for sine terms of x^* are omitted for this reason.

Similarly, the Fourier series for the wave numbers $i = (k'+1)n - j$

$$\begin{pmatrix} \cos\left(\{(k'+1)n - j\}\frac{x^*}{a}\pi\right) \\ -\sin\left(\{(k'+1)n - j\}\frac{x^*}{a}\pi\right) \end{pmatrix} \tag{55}$$

Table 4 Categorization of critical points of a degenerated D_n-equivariant system

(a) n =odd

μ satisfying $\det \tilde{J}^{\mu}=0$	$(1,1)_{D_n}$	$(1,2)_{D_n}$	$(2,j)^{+}_{D_n}$	$(2,j)^{-}_{D_n}$
Category of points	Limit point	Simple, symmetric bifurcation point	Simple, symmetric bifurcation point	Simple, symmetric bifurcation point
Symmetry of solutions	D_n	C_n	$D_{\gcd(j,n)}$	$C_{\gcd(j,n)}$

(b) n =even

μ satisfying $\det \tilde{J}^{\mu}=0$	$(1,1)_{D_n}$	$(1,2)_{D_n}$	$(1,3)'_{D_n}$	$(2,j)_{D_n}$
Category of points	Limit point	Simple-symmetric bifurcation point	Simple-symmetric bifurcation point	Simple-symmetric bifurcation point
Symmetry of solution	D_n	C_n	$C_{n/2}$	$C_{\gcd(j,n)}$

are associated with the jth two-dimensional irreducible representaions. The rewriting of these Fourier series into functions in x yields

$$\cos\left(\{(k'+1)n-j\}\tfrac{x^*}{a}\pi\right) = \begin{cases} (-1)^{k+\frac{n-j}{2}} & \cos\left(\{(2k+1)n-j\}\tfrac{x}{a}\pi\right) \\ (-1)^{k+\frac{j-1}{2}} & \sin\left(\{2(k+1)n-j\}\tfrac{x}{a}\pi\right) \\ & j=\text{odd} \\ (-1)^{k-\frac{j}{2}+n} & \sin\left(\{(2k+1)n-j\}\tfrac{x}{a}\pi\right) \\ (-1)^{k+1-\frac{j}{2}} & \cos\left(\{2(k+1)n-j\}\tfrac{x}{a}\pi\right) \\ & j=\text{even} \end{cases} \tag{56}$$

$(k=0,1,\cdots)$. By Eqs. (54) and (56), the space $X^{(2,j)_{D_n}}$ for $(2,j)_{D_n}$ can be further decomposed into two subspaces, that is,

$$X^{(2,j)_{D_n}} = X^{(2,j)^{+}_{D_n}} \bigoplus X^{(2,j)^{-}_{D_n}} \tag{57}$$

which are spanned respectively by

$$X^{(2,j)^{+}_{D_n}} = \text{span}\left[\sin\left((2kn+j)\frac{x}{a}\pi\right), \sin\left(\{2(k+1)n-j\}\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{58a}$$

$$X^{(2,j)^{-}_{D_n}} = \text{span}\left[\sin\left(\{(2k+1)n-j\}\frac{x}{a}\pi\right), \sin\left(\{(2k+1)n+j\}\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{58b}$$

The symmetry of each subspace is labeled by

$$\Sigma(X^{(1,1)_{D_n}}) = D_n, \quad \Sigma(X^{(1,2)_{D_n}}) = C_n \tag{59a}$$

$$\Sigma(X^{(2,j)^{+}_{D_n}}) = D_{\gcd(j,n)} \tag{59b}$$

$$\Sigma(X^{(2,j)^{-}_{D_n}}) = C_{\gcd(j,n)} \tag{59c}$$

The block-diagonal form of the tangential stiffness matrix becomes

$$\tilde{J} = \text{diag}\left[\tilde{J}^{(1,1)_{D_n}}, \tilde{J}^{(1,2)_{D_n}}, \tilde{J}^{(2,1)^{+}_{D_n}}, \tilde{J}^{(2,1)^{-}_{D_n}}, \cdots, \tilde{J}^{(2,(n-1)/2)^{+}_{D_n}}, \tilde{J}^{(2,(n-1)/2)^{-}_{D_n}}\right] \tag{60}$$

Owing to the degeneration due to the boundary conditions, the blocks $\tilde{J}^{(2,j)^{+}_{D_n}}$ and $\tilde{J}^{(2,j)^{-}_{D_n}}$ are no longer identical, unlike to the case of Eq. (30). The bifurcation point associated with the two-dimensional irreducible representation $(2,j)_{D_n}$ is degenerated into a simple bifurcation point. Critical points are categorized in **Table 4**(a).

b) When n = even

When n =even,

$$\cos\left(k'n\frac{x^*}{a}\pi\right) = \begin{cases} (-1)^{k}\sin\left((2k+1)n\tfrac{x}{a}\pi\right) \\ (-1)^{k+1}\cos\left(2(k+1)n\tfrac{x}{a}\pi\right) \end{cases} \tag{61a}$$

$$\sin\left(k'n\frac{x^*}{a}\pi\right) = \begin{cases} (-1)^{k+1}\cos\left((2k+1)n\tfrac{x}{a}\pi\right) \\ (-1)^{k+1}\sin\left(2(k+1)n\tfrac{x}{a}\pi\right) \end{cases} \tag{61b}$$

$(k=0,1,\cdots,\ k'=1,2,\cdots)$ are associated with one-dimensional irreducible representations $(1,1)_{D_n}$ and $(1,2)_{D_n}$. Hence the spaces for these representations are spanned by

$$X^{(1,1)_{D_n}} = \text{span}\left[\sin\left((2k+1)n\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{62a}$$

$$X^{(1,2)_{D_n}} = \text{span}\left[\sin\left(2(k+1)n\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{62b}$$

The cosine and sine functions

$$\cos\left(\left(k+\frac{1}{2}\right)n\frac{x^*}{a}\pi\right) = \cos\left(\left(k+\frac{1}{2}\right)n\frac{x}{a}\pi + \underline{\left(\frac{k}{2}+\frac{1}{4}\right)\pi}\right) \tag{63}$$

$$\sin\left(\left(k+\frac{1}{2}\right)n\frac{x^*}{a}\pi\right) = \sin\left(\left(k+\frac{1}{2}\right)n\frac{x}{a}\pi + \underline{\left(\frac{k}{2}+\frac{1}{4}\right)\pi}\right) \tag{64}$$

$(k=0,1,\cdots)$, which are associated with one-dimensional irreducible representations $(1,3)_{D_n}$ and $(1,4)_{D_n}$, cannot be expressed only by either sine or cosine terms due to the presence of the underlined terms. Hence in the space X, which are

Table 5 Categorization of critical points of a degenerated C_n-equivariant system

μ satisfying $\det\tilde{J}^{\mu}=0$	$(1,1)_{C_n}$	$(1,2)_{C_n}$	$(2,j)_{C_n}$
Category of points	Limit point	Simple, symmetric bifurcation point	Simple, symmetric bifurcation point
Symmetry of solutions	C_n	$C_{n/2}$	$C_{\mathrm{gcd}(j,n)}$

restricted by the boundary conditions, the irreducible representations $(1,3)_{D_n}$ and $(1,4)_{D_n}$ cannot be identified, and are degenerated into another irreducible representation $\mu=(1,3)'_{D_n}$ satisfying an action $r\cdot v=-v$. The following space corresponds to this representation:

$$X^{(1,3)'_{D_n}} = \mathrm{span}\left[\sin\left(\left(k+\frac{1}{2}\right)n\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{65}$$

Wave numbers $i=kn+j$ and $(k+1)n-j$ correspond to two-dimensional irreducible representations $(2,j)_{D_n}$ $(j=1,\cdots,\frac{n}{2}-1)$. The cosine terms of x^* for these wave numbers can be rewritten as

$$\begin{aligned}&\cos\left((kn+j)\frac{x^*}{a}\pi\right)\\ &= \cos\left((kn+j)\frac{x}{a}\pi+\underline{\left(\frac{k}{2}+\frac{j}{2n}\right)\pi}\right)\end{aligned} \tag{66}$$

$$\begin{aligned}&\cos\left(\{(k+1)n-j\}\frac{x^*}{a}\pi\right)\\ &= \cos\left(\{(k+1)n-j\}\frac{x}{a}\pi+\underline{\left(\frac{k+1}{2}-\frac{j}{2n}\right)\pi}\right)\end{aligned} \tag{67}$$

These terms cannot be expressed only by either sine or cosine terms due to the presence of the underlined terms. Space $X^{(2,j)_{D_n}}$, accordingly, cannot be decomposed, unlike in Eq. (27). This space, accordingly, is spanned by

$$\begin{aligned}X^{(2,j)_{D_n}} = \mathrm{span}\,[\ &\sin((kn+j)\tfrac{x}{a}\pi),\\ &\sin(\{(k+1)n-j\}\tfrac{x}{a}\pi)]_{k=0}^{\infty}\end{aligned} \tag{68}$$

To sum up, the isotypic decomposition of the space X reads

$$X = X^{(1,1)_{D_n}} \bigoplus X^{(1,2)_{D_n}} \bigoplus X^{(1,3)'_{D_n}} \bigoplus\left(\bigoplus_{j=1}^{n/2-1} X^{(2,j)_{D_n}}\right) \tag{69}$$

The symmetry of each subspace is labeled by

$$\Sigma(X^{(1,1)_{D_n}})=D_n,\quad \Sigma(X^{(1,2)_{D_n}})=C_n \tag{70a}$$

$$\Sigma(X^{(1,3)'_{D_n}})=C_{n/2} \tag{70b}$$

$$\Sigma(X^{(2,j)_{D_n}})=C_{\mathrm{gcd}(j,n)} \tag{70c}$$

A block-diagonal form of J reads

$$\begin{aligned}\tilde{J}=\mathrm{diag}\ \Big[\ &\tilde{J}^{(1,1)_{D_n}},\tilde{J}^{(1,2)_{D_n}},\tilde{J}^{(1,3)'_{D_n}},\\ &\tilde{J}^{(2,1)_{D_n}},\cdots,\tilde{J}^{(2,n/2-1)_{D_n}}\Big]\end{aligned} \tag{71}$$

Owing to a degeneration due to the boundary conditions, the block $\tilde{J}^{(2,j)_{D_n}}$ cannot be decomposed into two blocks, unlike in Eq. (30). Critical points are categorized as listed in **Table 4**(b).

(3) C_n-invariant solution

Bifurcated solutions of a C_n-invariant solutions are categorized. The action of r, which serves as a generating element of the cyclic group C_n, on the Fourier series of x is expressed as

$$\begin{aligned}r\begin{pmatrix}\cos(i\frac{x}{a}\pi)\\ \sin(i\frac{x}{a}\pi)\end{pmatrix} &= R^i\begin{pmatrix}\cos(i\frac{x}{a}\pi)\\ \sin(i\frac{x}{a}\pi)\end{pmatrix}\\ &= \begin{pmatrix}\cos\left(\frac{2i}{n}\pi\right) & -\sin\left(\frac{2i}{n}\pi\right)\\ \sin\left(\frac{2i}{n}\pi\right) & \cos\left(\frac{2i}{n}\pi\right)\end{pmatrix}\begin{pmatrix}\cos(i\frac{x}{a}\pi)\\ \sin(i\frac{x}{a}\pi)\end{pmatrix}\end{aligned} \tag{72}$$

This equation indicates that the representation matrices R^i becomes diagonal for wave numbers i for which $2i/n$ is an integer. These wave numbers, accordingly, are associated with one-dimensional irreducible representaions. Each subspace is spanned by

$$X^{(1,1)_{C_n}} = \mathrm{span}\left[\sin\left((k+1)n\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{73a}$$

$$X^{(1,2)_{C_n}} = \mathrm{span}\left[\sin\left((k+1/2)\,n\frac{x}{a}\pi\right)\right]_{k=0}^{\infty} \tag{73b}$$

$$\begin{aligned}X^{(2,j)_{C_n}} = \mathrm{span}\Big[&\sin\left((kn+j)\frac{x}{a}\pi\right),\\ &\sin\left(\{(k+1)n-j\}\frac{x}{a}\pi\right)\Big]_{k=0}^{\infty}\end{aligned} \tag{73c}$$

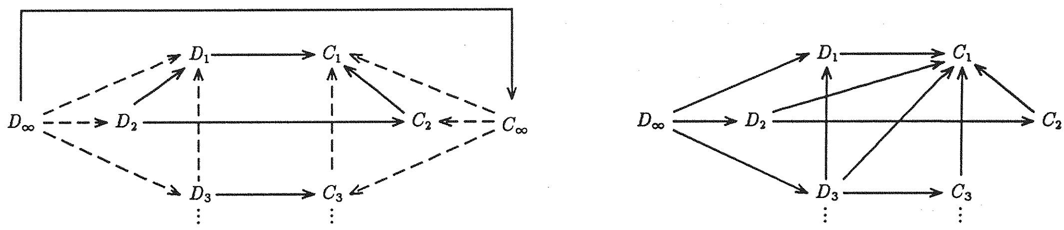
The block-diagonal form of J reads:

$$\tilde{J}=\mathrm{diag}[\tilde{J}^{(1,1)_{C_n}},\tilde{J}^{(1,2)_{C_n}},\tilde{J}^{(2,1)_{C_n}},\cdots,\tilde{J}^{(2,m_2)_{C_n}}] \tag{74}$$

where $X^{(1,2)_{C_n}}$ and $\tilde{J}^{(1,2)_{C_n}}$ exist when n is even. Owing to a degeneration due to the boundary conditions, the block $\tilde{J}^{(2,j)_{C_n}}$ cannot be decomposed into two blocks, unlike in Eq. (30). Critical points are categorized in **Table 5**.

5. BIFURCATION HIERARCHY

The rules of bifurcation from D_∞-, D_n-, and C_n-invariant paths have been presented in the previous sections. A repeated use of these rules



(a) General (b) Degenerated

Fig. 5 Diagram of the bifurcation hierarchy of a D_∞-invariant system

Table 6 Correspondence of subspaces and modes ($N_x = 6$)

(a) D_3

μ	$(1,1)_{D_3}$	$(1,2)_{D_3}$	$(2,1)^+_{D_3}$	$(2,1)^-_{D_3}$
order	3	6	1, 5	2, 4

(b) D_2

μ	$(1,1)_{D_2}$	$(1,2)_{D_2}$	$(1,3)'_{D_2}$
order	2, 6	4	1, 3, 5

(c) D_1

μ	$(1,1)_{D_1}$	$(1,2)_{D_1}$
order	1, 3, 5	2, 4, 6

(d) C_3

μ	$(1,1)_{C_3}$	$(2,1)_{C_3}$
order	3, 6	1, 2, 4, 5

(e) C_2

μ	$(1,1)_{C_2}$	$(1,2)_{C_2}$
order	2, 4, 6	1, 3, 5

(f) C_1

μ	$(1,1)_{C_1}$
order	1, 2, 3, 4, 5, 6

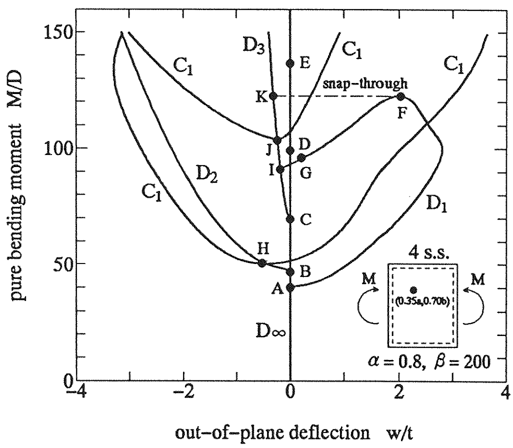


Fig. 6 Equilibrium paths

leads to a hierarchial bifurcation structure of D_∞-equivariant system shown in **Fig.5**. The solid lines denote bifurcation process associated with simple bifurcation points, while the dashed ones denote that with double ones. The general bifurcation structure in **Fig.5**(a) is considerably different from the one in (b) which is degenerated due to the boundary conditions. It shows the importance of the consideration of the mechanism of the degeneration. The bifurcation structure of D_∞-invariant system is quite complex but has a firm rule. It is, therefore, desirable to carry out bifurcation analysis with a knowledge on this rule.

6. BIFURCATION ANALYSIS OF A SIMPLY-SUPPORTED PLATE

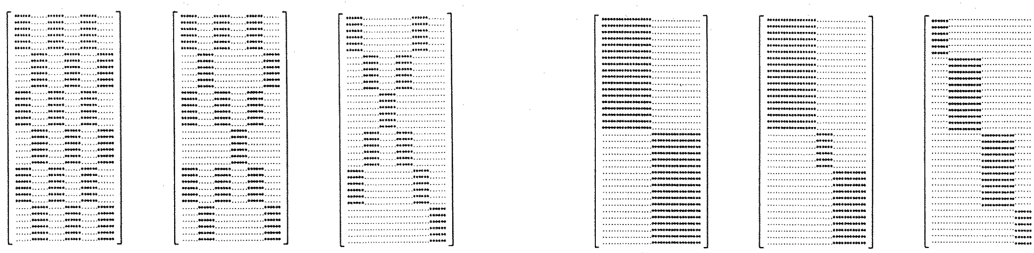
A bifurcation analysis was carried out on a simply-supported plate in **Fig.1**. Details on the derivation of equations and numerical analysis can be found in references[15),16),17)]. We employ the governing equation for the out-of-plane deformation by von Kármán, that is,

$$\nabla^4 w = \frac{t}{D}\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right] \tag{75}$$

$$\nabla^4 F = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{76}$$

where $w \equiv w(x,y)$ denotes the out-of-plane deflection, $F(x,y)$ indicates the stress function, t means the thickness of the plate, $D \equiv Et^3/(12(1-\nu^2))$ indicates the flexural rigidity, E is the modulus of elasticity, and ν is Poisson's ratio. As can be seen from **Fig.1**, this plate lacks the symmetry in the y-direction due to the presence of the bending moment, and hence is a D_∞-equivariant system with reflection and rotation symmetries in the x-direction.

Approximate the out-of-plane deflection w satisfying the four sides simply-supported boundary



(i) D_1 (ii) D_2 (iii) D_3

(a) J (before block-diagonalization)

(i) D_1 (ii) D_2 (iii) D_3

(b) $\tilde{J}$ (after block-diagonalization)

Fig. 7 Distribution of nonzero components of the tangential stiffness matrix

conditions by the double Fourier series

$$w = t\sum_{i=1}^{N_x}\sum_{j=1}^{N_y} w_{ij} \sin\left(i\frac{x}{a}\pi\right) \sin\left(j\frac{y}{b}\pi\right) \quad (77)$$

with N_x terms in the x-direction, and N_y in the y-direction. The present method employs the Galerkin method in Eq. (75), and solves the resulting cubic algebraic equation by the Newton-Raphson method. Its geometrical nonlinearity is similar to that of the beam-column equation.

Rearrange the order of the variables w_{ij} to define a vector

$$\boldsymbol{u} = \left(\boldsymbol{w}_1^{\mathrm{T}}, \cdots \boldsymbol{w}_{N_x}^{\mathrm{T}}\right)^{\mathrm{T}}, \quad \boldsymbol{w}_i = \left(w_{\rho_i 1}, \cdots w_{\rho_i N_y}\right)^{\mathrm{T}} \quad (78)$$

where ρ_i is the permutation, being defined by

$$\begin{pmatrix} 1 & 2 & \cdots & N_x \\ \rho_1 & \rho_2 & \cdots & \rho_{N_x} \end{pmatrix} \quad (79)$$

In the numerical simulation, the numbers of the Fourier series in Eq. (77) were chosen to be $N_x = N_y = 6$. For example, on a D_1-invariant bifurcation path, from Eqs. (50) and (51), one can see that the 1st, 3rd, and 5th modes correspond to the space $X^{(1,1)D_1}$, and the 2nd, 4th, and 6th ones to $X^{(1,2)D_1}$. **Table 6** shows the relationship between the mode number and the subspaces. The permutation in Eq. (79) was chosen to be [2]

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 2 & 6 \end{pmatrix} \quad (80)$$

based on **Table 6**, so as to be compatible with the functions spanning the subspaces X^{μ} obtained in Chapter 4. This permutation can block-diagonalize the tangential stiffness matrix for $\boldsymbol{u}$.

Fig.6 shows a result of the bifurcation analysis. Since the bifurcated solutions branching toward the positive and negative directions of deflection correspond to the identical physical behavior, only one of them was plotted in this figure for simplicity. The abscissa denotes the deflection at $(x, y) = (0.35a, 0.70b)$, and the ordinate indicates the bending moment. The symmetry group of each path is shown in the figure, and critical points are expressed by ($\bullet$). The points A,$\cdots$,E on the trivial solution $w = 0$ are bifurcation points with bifurcation modes of the 1,$\cdots$,5th sine. The singular point F is a maximum (limit) point of load f. We obtained the secondary bifurcation paths from the bifurcation points A, B, and C that are D_1-, D_2-, and D_3-invariant, respectively (bifurcation paths from the bifurcation points D and E are omitted). A D_1-invariant path further branches from the D_3-invariant bifurcation path at a bifurcation point I, and is connceted with another D_1-invariant one branching from the main path. C_1-invariant paths branch from a bifurcation point H on a D_2-invariant one and a bifurcation point J on a D_3-invariant one. No bifurcation takes place on C_1-invariant ones without symmetry. The very complex bifurcation process presented above does follow the rules in **Fig.5**. This may be suffice to show the importance and the validity of the present theory. Although C_2- and C_3-invariant modes appeared in **Fig.5**, but not in **Fig.6**. It is based on the fact that the group-theoretic bifurcation theory can indicate all possible bifurcation process, but the presence of each process is dependent on cases.

Fig.7 shows the tangential stiffness matrices J and $\tilde{J}$, before and after the block-diagonalization, respectively. Here ($\cdot$) stands for zero components, and ($\bullet$) for nonzero ones.

The deformation modes on D_1-, D_2-, and D_3-invariant paths respectively are equal to the 1, 2, 3th sine modes in the vicinity of the bifurcation points. However, for example, on the D_1-invariant bifurcation path, the deformation mode is deformed through "mode interaction," as shown in **Fig.8**. The mechanism of the mode

[2] This permutation is known as the chain adapted basis[10].

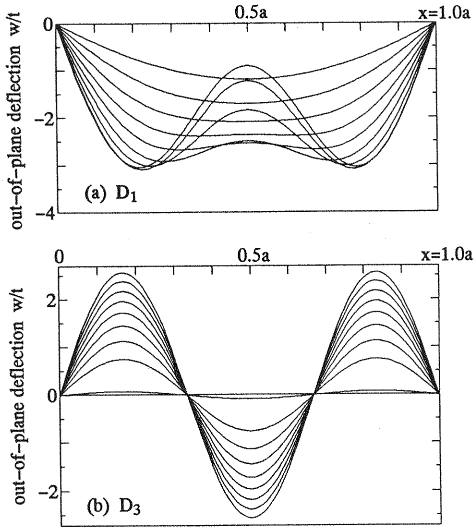


Fig. 8 The progress of out-of-plane deflection subject to mode interaction ($y = 0.7b$)

interaction is described by Eq. (50) in a complete manner. The deflection mode of the D_1-invariant bifurcation path is the superposition of the 1, 3, 5, $\cdots$th sine modes. In this numerical example, in which considered up to the 6th order term, the deflection mode is to be deformed through the mode interaction among the 1st, 3rd and 5th sine modes. Similarly, by Eq. (62a), on the D_2-invariant bifurcation path a mode interaction takes place between the 2nd and 6th sine modes. By contrast, the 3rd sine mode is completely preserved on the D_3-invariant path. Although by Eq. (50) the deformation modes on a D_3-invariant path, in general, is a superposition of the 3, 9, 15, $\cdots$th sine modes, the mode interaction cannot take place since only up to the 6th sine modes are involved. This is a kind of a discretizing error.

7. CONCLUSION

The structure of bifurcation process of system with symmetries can be known *a priori* by means of the group-theoretic bifurcation theory. While the bifurcation analysis technique is something like a car, this theory is something like a map. With the use of the combination of these, the mechanism of the complex bifurcation behavior presented in this paper can be understood.

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