Structural Eng. /Earthquake Eng. Vol. 12, No. 2, 27s—35s, July 1995
Japan Society of Civil Engineers (J. Struct. Mech. Earthquake Eng. No. 519/ I—32)

# AN EXPLICIT GEOMETRICALLY-NONLINEAR DISCRETIZATION OF THE 3-D TETRAHEDRAL ELEMENT

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An actual discretization is presented for the 3-D tetrahedral element in large displacements. After a decomposition of the total freedom of that element into parameters of position-as-a-rigid-body and those of deformation, the strain-constant interpolation is applied to the defined deformation. The geometrical decompositions and the associated force transformations are developed physically in explicit form. While the material is assumed elastic for finite strains, any geometrically nonlinear effects are taken into account, systematically and rigorously.

***Key Words*** : *geometrically nonlinear, finite strain, strain-constant interpolation*

## 1. INTRODUCTION

After the strain-constant interpolation applied to the finite displacements of a solid continuum, the Green's strain components are still written in a quadratic form (double summation) of the nodal parameters. Based on this kind of sum expression or matrix notation, the existing FEM formulations called the B-notation and the N-notation methods are mathematically accomplished[1)-7)]. However, their expansions are not to be understood physically. The stiffness relations are estimated numerically after the summations executed.

On the other hand, the followings have been recognized in those finite-displacement problems in which strains result in a small range : Under a subdivision into small enough elements, each element is largely translated and rotated as a rigid body, but is deformed only to the extent of small strains. Then, if observed in a coordinate system which goes with the rigid displacement, the remaining deformation can be dealt with by the less nonlinear field equations.

As actual treatment based on this feature, there exist the two methods in principle. One is the updated Lagrangian description : in an incremental loading, the spatial coordinates for each element are moved step by step in close to its previous position as a rigid body[e.g.8)-10)]. The other is the method to separate the entire nodal freedom of an element into parameters of displacement as a rigid body and those of deformation[e.g.11)-16)]. In general, this separation method seems effective only to the small-strain problems under large displacements. In case of the large (elastic) strains of a beam or plate, for instance, the deformations even after the separation remain finite and so much nonlinear to be adequately interpolated. However, it is not true of the strain-constant interpolation applied to a solid continuum : regardless of small or large strains, that interpolation can give any feasible constant strain states. As a potential advantage in this case, it is enough for the discretization to be developed upon the reduced degrees of freedom.

The FEM procedure of separation-into-rigid-displacement-and-deformation has been described in general terms[13),14)]. And, guided by that study, an actual discretization is already given to the 2-D triangular element[16)]. In a complete accordance with those precedents, the 3-D tetrahedral element is considered in this study for its specific formulation. The material problems beyond the elastic range are disregarded, but the entire formulation is theoretical and rigorous as a geometrically nonlinear discretization.

## 2. DESCRIPTION OF GEOMETRY

In the 3-D Cartesian coordinates, $\{x, y, z\}$, we consider a four-node tetrahedral element $(e)$. The unit base vectors into $\{x, y, z\}$ are denoted by

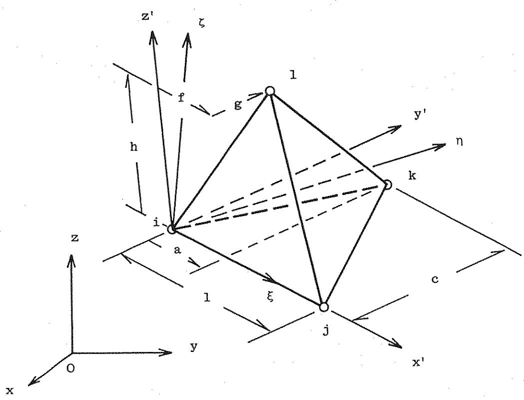


**Fig.1** Geometry of Tetrahedral Element

$\{i_{(xyz)}\}=\{\boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_z\}$. As the *element position* of $(e)$, we employ the spatial coordinates of its nodes :

$$\{\boldsymbol{x}\}_{(e)}=\{(x, y, z)_i, (\ )_j, (\ )_k, (\ )_l\} \qquad (1)$$

where subscript $i, j, k$ and $l$ stand for the four nodes $(i<j<k<l)$.

As the *element coordinates*, a rectangular $\{x', y', z'\}$ is taken in relation to the current configuration : as shown in **Fig.1**, $x'$ is directed from node $i$ to $j$; in the base plane spanned by node $i$, $j$ and $k$, coordinate $y'$ is taken from $i$ into the direction right-angled with $x'$ to see node $k$ on its positive side; and $z'$ is perpendicular to the base plane to make $\{x', y', z'\}$ right-handed. The unit vectors into $\{x', y', z'\}$ are denoted by $\{\boldsymbol{i}_{(x'y'z')}\}$. In the initial (stress-free) state, the $\{x', y', z'\}$-coordinates of material points are employed as Lagrangian coordinates $\{\xi, \eta, \zeta\}$.

The shape as a tetrahedron is characterized by $\{x', y', z'\}$ of the nodes. By collecting the six nonzero coordinates into a set (**Fig.1**), we define *shape* of $(e)$ :

$$\boldsymbol{g}_{(e)}=\{l, a, c, f, g, h\}_{(e)} \qquad (2)$$

Let the orthogonal matrix relating the element's $\{\boldsymbol{i}_{(x'y'z')}\}$ to the global $\{\boldsymbol{i}_{(xyz)}\}$ be denoted by

$$\{\boldsymbol{i}_{(x'y'z')}\}=[t(\{\boldsymbol{x}\})]_{(e)}\{\boldsymbol{i}_{(xyz)}\} \qquad (3)$$

Those $\boldsymbol{g}_{(e)}$ and $[t]_{(e)}$ are determined by $\{\boldsymbol{x}\}_{(e)}$. For short expression, the position vectors of node $j$, $k$ and $l$ relative to $i$ are introduced :

$$\{\bar{x}, \bar{y}, \bar{z}\}=\{x_j-x_i, y_j-y_i, z_j-z_i\},$$

$$\{\hat{x}, \hat{y}, \hat{z}\}=\{x_k-x_i, y_k-y_i, z_k-z_i\},$$

$$\{\overset{*}{x}, \overset{*}{y}, \overset{*}{z}\}=\{x_l-x_i, y_l-y_i, z_l-z_i\} \qquad (4.a\text{-}c)$$

From the geometry shown in **Fig.1**, we have

$$l(=|\bar{x}|)=\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2} \quad (>0)$$

$$a\left(=\hat{\boldsymbol{x}}\cdot\frac{\bar{\boldsymbol{x}}}{|\bar{x}|}\right)=\frac{1}{l}\{\bar{x}\hat{x}+\bar{y}\hat{y}+\bar{z}\hat{z}\}$$

$$c\left(=\sqrt{|\hat{x}|^2-a^2}\right)=\sqrt{\hat{x}^2+\hat{y}^2+\hat{z}^2-a^2} \quad (>0) \quad (5.a\text{-}c)$$

From relations $\boldsymbol{i}_{x'}=1/l\cdot\langle\bar{x}, \bar{y}, \bar{z}\rangle\{\boldsymbol{i}_{(xyz)}\}$, $\boldsymbol{i}_{y'}\cdot\bar{\boldsymbol{x}}=0$, $|\boldsymbol{i}_{y'}|=1$ and $\boldsymbol{i}_{z'}=\boldsymbol{i}_{x'}\times\boldsymbol{i}_{y'}$, the rotation matrix is derived as

$$[t(\{\boldsymbol{x}\})]_{(e)}=\begin{bmatrix} \dfrac{\bar{x}}{l}, & \dfrac{\bar{y}}{l}, & \dfrac{\bar{z}}{l} \\ \dfrac{\hat{x}}{c}-\dfrac{a\bar{x}}{cl}, & \dfrac{\hat{y}}{c}-\dfrac{a\bar{y}}{cl}, & \dfrac{\hat{z}}{c}-\dfrac{a\bar{z}}{cl} \\ \dfrac{\bar{y}\hat{z}-\bar{z}\hat{y}}{cl}, & \dfrac{\bar{z}\hat{x}-\bar{x}\hat{z}}{cl}, & \dfrac{\bar{x}\hat{y}-\bar{y}\hat{x}}{cl} \end{bmatrix} \qquad (6)$$

The remaining lengths, $f$, $g$ and $h$, in $\boldsymbol{g}_{(e)}$ are now obtained as the $\{x', y', z'\}$-coordinates of node $l$ :

$$f(=\overset{*}{\boldsymbol{x}}\cdot\boldsymbol{i}_{x'})=\frac{1}{l}\{\bar{x}\overset{*}{x}+\bar{y}\overset{*}{y}+\bar{z}\overset{*}{z}\},$$

$$g(=\overset{*}{\boldsymbol{x}}\cdot\boldsymbol{i}_{y'})=\frac{1}{c}\{\hat{x}\overset{*}{x}+\hat{y}\overset{*}{y}+\hat{z}\overset{*}{z}\}-\frac{af}{c},$$

$$h(=\overset{*}{\boldsymbol{x}}\cdot\boldsymbol{i}_{z'})=\frac{1}{cl}\begin{vmatrix} \bar{x}, & \bar{y}, & \bar{z} \\ \hat{x}, & \hat{y}, & \hat{z} \\ \overset{*}{x}, & \overset{*}{y}, & \overset{*}{z} \end{vmatrix} \qquad (5.d\text{-}f)$$

Let the freedom of three degrees in the spatial rotation of system $\{x', y', z'\}$ be denoted by frd.$[t]_{(e)}$. Then, its spatial position is described by

$$\boldsymbol{v}_{(e)}=\{(x, y, z)_i, \text{frd.}[t]_{(e)}\} \qquad (7)$$

Now, it can be said that $\boldsymbol{v}_{(e)}$ and $\boldsymbol{g}_{(e)}$ are a separation of the total freedom $\{\boldsymbol{x}\}_{(e)}$ into the *position as a rigid body* and the shape.

Next, we consider the tangent relations for an infinitesimal variation of $\{\boldsymbol{x}\}_{(e)}$. Those tangent relations can be derived by the mathematical differentiations of the former relations. But, the expansions are much complicated. We here develop them under the geometrical decompositions. First, by the use of (6), we re-decompose the independent displacements into the $\{x', y', z'\}$-directions :

$$\delta\{\boldsymbol{x}'\}_{(e)}=[T(\{\boldsymbol{x}\})]_{(e)}\delta\{\boldsymbol{x}\}_{(e)},$$

$$[T]_{(e)}=\begin{bmatrix} [t] & & & \\ & [t] & & \\ & & [t] & \\ & & & [t] \end{bmatrix}_{(e)} \qquad (8.a, b)$$

Element system $\{x', y', z'\}$ itself is rotated by $\delta\{\boldsymbol{x}'\}_{(e)}$. The resolution of this infinitesimal rotation into components around $\{\boldsymbol{i}_{(x'y'z')}\}$ is developed in **Appendix** Ⅰ. The result is as follows :

$$\delta\theta_{x'}=\frac{a-l}{cl}\delta z_i'-\frac{a}{cl}\delta z_j'+\frac{1}{c}\delta z_k',$$

$$\delta\theta_{y'}=\frac{1}{l}\delta z_i'-\frac{1}{l}\delta z_j',$$

$$\delta\theta_{z'}=-\frac{1}{l}\,\delta y_i'+\frac{1}{l}\,\delta y_j' \tag{9.a-c}$$

By the use of those components, the disturbed $\{\boldsymbol{i}_{(x'y'z')}\}$ is written as

$$\{\boldsymbol{i}^{\delta}_{(x'y'z')}\}=([I]+[\delta\Phi]_{(e)})\{\boldsymbol{i}_{(x'y'z')}\}$$

$$[\delta\Phi]_{(e)}=\begin{bmatrix} 0, & \delta\theta_{z'}, & -\delta\theta_{y'} \\ -\delta\theta_{z'}, & 0, & \delta\theta_{x'} \\ \delta\theta_{y'}, & -\delta\theta_{x'}, & 0 \end{bmatrix} \tag{10.a, b}$$

where superscript $(\ )^{\delta}$ denotes a quantity after $\delta\{\boldsymbol{x}\}_{(e)}$.

Upon the preceding $\{\boldsymbol{i}_{(x'y'z')}\}$ to $\delta\{\boldsymbol{x}'\}_{(e)}$, vectors $\vec{ij}$, $\vec{ik}$ and $\vec{il}$ after $\delta\{\boldsymbol{x}'\}_{(e)}$ are written as

$$\begin{Bmatrix} \vec{ij}^{\delta} \\ \vec{ik}^{\delta} \\ \vec{il}^{\delta} \end{Bmatrix}=$$

$$\begin{bmatrix} l-\delta x_i'+\delta x_j', & -\delta y_i'+\delta y_j', & -\delta z_i'+\delta z_j' \\ a-\delta x_i'+\delta x_k', & c-\delta y_i'+\delta y_k', & -\delta z_i'+\delta z_k' \\ f-\delta x_i'+\delta x_l', & g-\delta y_i'+\delta y_l', & h-\delta z_i'+\delta z_l' \end{bmatrix}\begin{Bmatrix} \boldsymbol{i}_{x'} \\ \boldsymbol{i}_{y'} \\ \boldsymbol{i}_{z'} \end{Bmatrix} \tag{11}$$

Since $[I]+[\delta\Phi]_{(e)}$ is a matrix of diagonal 1 and (antisymmetric) differentials, relation (10) can be inverted by the transposition :

$$\{\boldsymbol{i}_{(x'y'z')}\}=([I]-[\delta\Phi]_{(e)})\{\boldsymbol{i}^{\delta}_{(x'y'z')}\} \tag{12}$$

Introducing this inverse into (11), we have $\{\vec{ij}^{\delta}, \vec{ik}^{\delta}, \vec{il}^{\delta}\}$ represented in a matrix form upon the current $\{\boldsymbol{i}^{\delta}_{(x'y'z')}\}$. The nonzero elements in that matrix are to be $l+\delta l$, $a+\delta a$, $c+\delta c$, $f+\delta f$, $g+\delta g$ and $h+\delta h$. After the actual expansion, we have

$$\delta\boldsymbol{g}_{(e)}=[Q_x^I(\boldsymbol{g})]_{(e)}\delta\{\boldsymbol{x}'\}_{(e)} \tag{13.a}$$

$$[Q_x^I]_{(e)}=$$

$$\left[\begin{array}{cccccccccccc} -1, & 0, & 0, & 1, & 0, & 0, & 0, 0, & 0, & 0, 0, 0 \\ -1, & -\frac{c}{l}, & 0, & 0, & \frac{c}{l}, & 0, & 1, 0, & 0, & 0, 0, 0 \\ 0 & -\left(1-\frac{a}{l}\right), & 0, & 0, & -\frac{a}{l}, & 0, & 0, 1, & 0, & 0, 0, 0 \\ -1, & -\frac{g}{l}, & -\frac{h}{l}, & 0, & \frac{g}{l}, & \frac{h}{l}, & 0, 0, & 0, & 1, 0, 0 \\ 0, & -\left(1-\frac{f}{l}\right), & -\frac{h}{c}\left(1-\frac{a}{l}\right), & 0, & -\frac{f}{l}, & -\frac{ah}{lc}, & 0, 0, & \frac{h}{c}, & 0, 1, 0 \\ 0, & 0, & \left(-1+\frac{f}{l}\right)+\frac{g}{c}\left(1-\frac{a}{l}\right), & 0, & 0, & \frac{1}{l}\left(\frac{ag}{c}-f\right), & 0, 0, & -\frac{g}{c}, & 0, 0, 1 \end{array}\right] \tag{13.b}$$

## 3. INTERPOLATION OF DEFORMATION

Applying the strain-constant interpolation to the deformed $(e)$ in the $\{x', y', z'\}$-coordinates, we have

$$x'(\xi,\ \eta,\ \zeta)=\left\{\frac{\xi}{l_0}-\frac{a_0\eta}{l_0c_0}+\frac{1}{l_0}\left(\frac{a_0g_0}{c_0}-f_0\right)\frac{\zeta}{h_0}\right\}l+\left\{\frac{\eta}{c_0}-\frac{g_0\zeta}{c_0h_0}\right\}a+\left\{\frac{\zeta}{h_0}\right\}f,$$

$$y'(\xi,\ \eta,\ \zeta)=\left\{\frac{\eta}{c_0}-\frac{g_0\zeta}{c_0h_0}\right\}c+\left\{\frac{\zeta}{h_0}\right\}g,$$

$$z'(\xi,\ \eta,\ \zeta)=\left\{\frac{\zeta}{h_0}\right\}h \tag{14.a-c}$$

where $\{l_0, a_0, c_0, f_0, g_0, h_0\}$ are the lengths in the initial shape. The associated Green's strain components are obtained as follows :

$$e_{\xi\xi}=\frac{1}{2}\left\{\left(\frac{l}{l_0}\right)^2-1\right\},$$

$$e_{\eta\eta}=\frac{1}{2}\left\{\frac{1}{c_0^2}\left(a-\frac{a_0l}{l_0}\right)^2+\left(\frac{c}{c_0}\right)^2-1\right\},$$

$$e_{\zeta\zeta}=\frac{1}{2}\left[\frac{1}{h_0^2}\left\{f-\frac{g_0a}{c_0}+\frac{l}{l_0}\left(\frac{a_0g_0}{c_0}-f_0\right)\right\}^2+\frac{1}{h_0^2}\left(g-\frac{g_0c}{c_0}\right)^2+\left(\frac{h}{h_0}\right)^2-1\right],$$

$$e_{\eta\zeta}=\frac{1}{2c_0h_0}\left[\left(a-\frac{a_0l}{l_0}\right)\left\{f-\frac{g_0a}{c_0}+\frac{l}{l_0}\left(\frac{a_0g_0}{c_0}-f_0\right)\right\}+c\left(g-\frac{g_0c}{c_0}\right)\right],$$

$$e_{\zeta\xi}=\frac{l}{2l_0h_0}\left\{f-\frac{g_0a}{c_0}+\frac{l}{l_0}\left(\frac{a_0g_0}{c_0}-f_0\right)\right\},$$

$$e_{\xi\eta}=\frac{l}{2l_0c_0}\left(a-\frac{a_0l}{l_0}\right) \tag{15.a-f}$$

By employing $\gamma_{\eta\zeta}=2e_{\eta\zeta}$, $\gamma_{\zeta\xi}=2e_{\zeta\xi}$, $\gamma_{\xi\eta}=2e_{\xi\eta}$ as the alternative shear components, we define *deformation* of $(e)$ by

$$\varepsilon_{(e)}=\{e_{\xi\xi},\ e_{\eta\eta},\ e_{\zeta\zeta},\ \gamma_{\eta\zeta},\ \gamma_{\zeta\xi},\ \gamma_{\xi\eta}\} \tag{16}$$

This deformation is in a one-to-one correspondence to shape $\boldsymbol{g}_{(e)}$.

For short expression, the following lengths will be used in the later expansions :

$$b_0=l_0-a_0,\quad e_0=l_0-f_0,\quad d_0=\frac{g_0a_0}{c_0}-f_0,$$

$$d_0'=\frac{g_0b_0}{c_0}-e_0 \tag{17.a-d}$$

By the use of (15.a-f) together with the former (4.a-c) and (5.a-f), strain state $\varepsilon_{(e)}$ is estimated for element position $\{\boldsymbol{x}\}_{(e)}$. And, we already have (8) and (13) as the tangent relations from $\delta\{\boldsymbol{x}\}_{(e)}$ to $\delta\boldsymbol{g}_{(e)}$. By the mathematical differentiation of (15.a-f), we now have

$$\delta\varepsilon_{(e)}=[Q_x^{II}(\boldsymbol{g})]_{(e)}\delta\boldsymbol{g}_{(e)} \tag{18.a}$$

$$[Q_x^{II}(\boldsymbol{g})]_{(e)}$$

$$=\begin{bmatrix}
\frac{l}{l_0^2}, & 0, & 0, & 0, & 0, & 0 \\
\frac{a_0}{l_0^2c_0^2}(a_0l-l_0a), & \frac{1}{l_0c_0^2}(-a_0l+l_0a), & \frac{c}{c_0^2}, & 0, & 0, & 0 \\
\frac{d_0}{l_0h_0^2}\left(\frac{d_0l}{l_0}-\frac{g_0a}{c_0}+f\right), & \frac{g_0}{c_0h_0^2}\left(-\frac{d_0l}{l_0}+\frac{g_0a}{c_0}-f\right), & \frac{g_0}{c_0^2h_0^2}(g_0c-c_0g), & \frac{1}{h_0^2}\left(\frac{d_0l}{l_0}-\frac{g_0a}{c_0}+f\right), & \frac{1}{c_0h_0^2}(-g_0c+c_0g), & \frac{h}{h_0^2} \\
\frac{2}{l_0c_0h_0}\left\{-\frac{a_0d_0}{l_0}l+\left(d_0+\frac{f_0}{2}\right)a-\frac{a_0f}{2}\right\}, & \frac{2}{c_0h_0}\left\{\left(d_0+\frac{f_0}{2}\right)\frac{l}{l_0}-\frac{g_0a}{c_0}-\frac{f}{2}\right\}, & \frac{2}{c_0h_0}\left(-\frac{g_0c}{c_0}+\frac{g}{2}\right), & \frac{1}{l_0c_0h_0}(-a_0l+l_0a), & \frac{c}{c_0h_0}, & 0 \\
\frac{2}{l_0h_0}\left\{\frac{d_0l}{l_0}-\frac{g_0a}{2c_0}+\frac{f}{2}\right\}, & -\frac{g_0l}{l_0c_0h_0}, & 0, & \frac{l}{l_0h_0}, & 0, & 0 \\
\frac{2}{l_0c_0}\left(-\frac{a_0l}{l_0}+\frac{a}{2}\right), & \frac{l}{l_0c_0}, & 0, & 0, & 0, & 0
\end{bmatrix} \tag{18.b}$$

Then, by the chain rule, we have the entire relation from $\delta\{\boldsymbol{x}\}_{(e)}$ to $\delta\varepsilon_{(e)}$ :

$$\delta\varepsilon_{(e)}=[Q_X(\{\boldsymbol{x}\})]_{(e)}\delta\{\boldsymbol{x}\}_{(e)} \tag{19.a}$$

$$[Q_X(\{\boldsymbol{x}\})]_{(e)}(=[Q_x^{II}]_{(e)}[Q_x^{I}]_{(e)}[T]_{(e)})=$$

$$\left[\begin{array}{ccc:cc:cc:cc}
q_{\xi\xi}^{ix}, & q_{\xi\xi}^{iy}, & q_{\xi\xi}^{iz} & q_{\xi\xi}^{jx}, & \cdots & q_{\xi\xi}^{kx}, & \cdots & q_{\xi\xi}^{lx}, & \cdots \\
q_{\eta\eta}^{ix}, & q_{\eta\eta}^{iy}, & q_{\eta\eta}^{iz} & q_{\eta\eta}^{jx}, & \cdots & q_{\eta\eta}^{kx}, & \cdots & q_{\eta\eta}^{lx}, & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
q_{\xi\eta}^{ix}, & q_{\xi\eta}^{iy}, & q_{\xi\eta}^{iz} & q_{\xi\eta}^{jx}, & \cdots & q_{\xi\eta}^{kx}, & \cdots & q_{\xi\eta}^{lx}, & \cdots
\end{array}\right] \tag{19.b}$$

where

$$q_{\xi\xi}^{ix}=-\frac{\bar{x}}{l_0^2},\quad q_{\xi\xi}^{jx}=\frac{\bar{x}}{l_0^2},\quad q_{\xi\xi}^{kx}=0,\quad q_{\xi\xi}^{lx}=0,$$

$$q_{\eta\eta}^{ix}=\frac{b_0}{l_0c_0^2}\left(\frac{a_0}{l_0}\bar{x}-\hat{x}\right),\quad q_{\eta\eta}^{jx}=\frac{a_0}{l_0c_0^2}\left(\frac{a_0}{l_0}\bar{x}-\hat{x}\right),$$

$$q_{\eta\eta}^{kx}=\frac{1}{c_0^2}\left(-\frac{a_0}{l_0}\bar{x}+\hat{x}\right),\quad q_{\eta\eta}^{lx}=0,$$

$$q_{\zeta\zeta}^{ix}=\frac{d_0'}{l_0h_0^2}\left(\frac{d_0}{l_0}\bar{x}-\frac{g_0}{c_0}\hat{x}+\overset{*}{x}\right),$$

$$q_{\zeta\zeta}^{jx}=\frac{d_0}{l_0h_0^2}\left(\frac{d_0}{l_0}\bar{x}-\frac{g_0}{c_0}\hat{x}+\overset{*}{x}\right),$$

$$q_{\zeta\zeta}^{kx}=\frac{g_0}{c_0h_0^2}\left(-\frac{d_0}{l_0}\bar{x}+\frac{g_0}{c_0}\hat{x}-\overset{*}{x}\right),$$

$$q_{\zeta\zeta}^{lx}=\frac{1}{h_0^2}\left(\frac{d_0}{l_0}\bar{x}-\frac{g_0}{c_0}\hat{x}+\overset{*}{x}\right),$$

$$q_{\eta\zeta}^{ix}=\frac{1}{l_0c_0h_0}\left\{\left(a_0-f_0-\frac{2b_0d_0}{l_0}\right)\bar{x}+(e_0+2d_0')\hat{x}-b_0\overset{*}{x}\right\},$$

$$q_{\eta\zeta}^{jx}=\frac{1}{l_0c_0h_0}\left\{-\frac{2a_0d_0}{l_0}\bar{x}+(f_0+2d_0)\hat{x}-a_0\overset{*}{x}\right\},$$

$$q_{\eta\zeta}^{kx}=\frac{1}{c_0h_0}\left\{\frac{1}{l_0}(f_0+2d_0)\bar{x}-\frac{2g_0}{c_0}\hat{x}+\overset{*}{x}\right\},$$

$$q_{\eta\zeta}^{lx}=\frac{1}{c_0h_0}\left(-\frac{a_0}{l_0}\bar{x}+\hat{x}\right),$$

$$q_{\zeta\xi}^{ix}=\frac{1}{l_0h_0}\left\{\left(-1+\frac{g_0}{c_0}-\frac{2d_0}{l_0}\right)\bar{x}+\frac{g_0}{c_0}\hat{x}-\overset{*}{x}\right\},$$

$$q_{\zeta\xi}^{jx}=\frac{1}{l_0h_0}\left\{\frac{2d_0}{l_0}\bar{x}-\frac{g_0}{c_0}\hat{x}+\overset{*}{x}\right\},\quad q_{\zeta\xi}^{kx}=-\frac{g_0}{l_0c_0h_0}\bar{x},$$

$$q_{\zeta\xi}^{lx}=\frac{1}{l_0h_0}\bar{x},\quad q_{\xi\eta}^{ix}=\frac{1}{l_0c_0}\left(\frac{a_0-b_0}{l_0}\bar{x}-\hat{x}\right),$$

$$q_{\xi\eta}^{jx}=\frac{1}{l_0c_0}\left(-\frac{2a_0}{l_0}\bar{x}+\hat{x}\right),\quad q_{\xi\eta}^{kx}=\frac{1}{l_0c_0}\bar{x},\quad q_{\xi\eta}^{lx}=0 \tag{20.a-x}$$

Given above are only the coefficients in $\delta x$-columns of $[Q_X]_{(e)}$. The remainings in $\delta y$- and $\delta z$-columns are obtained by the replace of $(\bar{x}, \hat{x}, \overset{*}{x})$ by $(\bar{y}, \hat{y}, \overset{*}{y})$ and $(\bar{z}, \hat{z}, \overset{*}{z})$, respectively.

## 4. DEFORMATION FORCE

We here consider an elastic finite-strain problem in which a strain-energy-density function $A(\boldsymbol{e})$ is prescribed in terms of the Green's strain components. The *deformation force*

$$\boldsymbol{f}_{(e)}=\{f_{\xi\xi}, f_{\eta\eta}, f_{\zeta\zeta}, f_{\eta\zeta}, f_{\zeta\xi}, f_{\xi\eta}\} \tag{21}$$

is defined in the following manner : for variation $\delta\boldsymbol{\varepsilon}_{(e)}$ from a current $\boldsymbol{\varepsilon}_{(e)}$, the change of strain energy of $(e)$ is expressed by inner product $\boldsymbol{f}_{(e)}\cdot\delta\boldsymbol{\varepsilon}_{(e)}$. Then, the present $\boldsymbol{f}_{(e)}-\boldsymbol{\varepsilon}_{(e)}$ relation is written as

$$f_{\xi\xi}\,(=V_0\sigma_{\xi\xi})=V_0\frac{\partial A(\boldsymbol{e})}{\partial e_{\xi\xi}},$$

$$\vdots$$

$$f_{\xi\eta}\,(=V_0\sigma_{\xi\eta})=V_0\frac{\partial A(\boldsymbol{e})}{\partial\gamma_{\xi\eta}} \tag{22.a-f}$$

where $\{\sigma_{\xi\xi}, \cdots, \sigma_{\xi\eta}\}$ are components of the second Piola-Kirchhoff stress tensor; and $V_0$ is the initial volume of $(e)$ :

$$V_0=\frac{l_0c_0|h_0|}{6},\quad\text{or}\quad=\frac{1}{6}\left|\,\begin{vmatrix}\bar{x}, & \bar{y}, & \bar{z}\\ \hat{x}, & \hat{y}, & \hat{z}\\ \overset{*}{x}, & \overset{*}{y}, & \overset{*}{z}\end{vmatrix}_0\,\right| \tag{23}$$

## 5. RELATIONS FROM $\boldsymbol{f}_{(e)}$ TO NODAL FORCES

Consider the 3-D simple support shown in **Fig.2** : node $i$ is fixed; node $j$ is allowed to move only into $x'$; and node $k$ is constrained onto the $\{x', y'\}$-plane. In the following sense, this statically-determinate support is associated with our separation of $\{\boldsymbol{x}\}_{(e)}$ into $\boldsymbol{v}_{(e)}$ and $\boldsymbol{g}_{(e)}$ : if that support is fixed in the space, the change of $\boldsymbol{v}_{(e)}$ are constrained, but any deformations are possible by the variety of $\boldsymbol{g}_{(e)}=\{l, a, c, f, g, h\}$.

Let the nodal forces be resolved into the global $\{x, y, z\}$-directions. Those components conjugate to $\{\boldsymbol{x}\}_{(e)}$ of (1) is called *element force* :

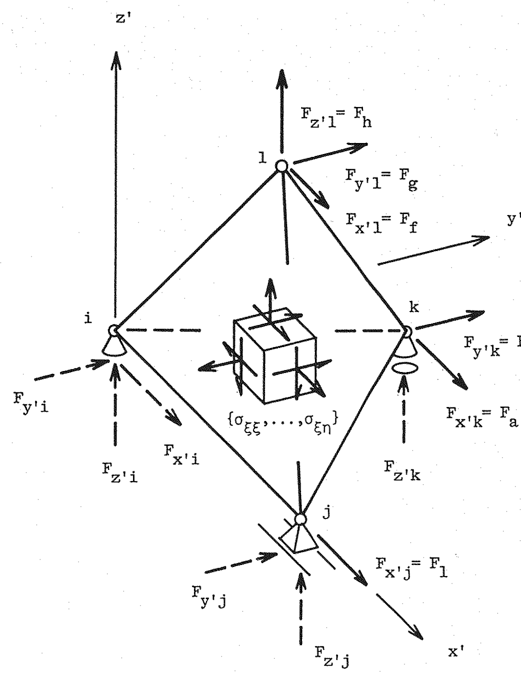


**Fig.2** Nodal Forces in 3-D Simple Support

$$\{\boldsymbol{F}\}_{(e)}=\{(F_x, F_y, F_z)_i, (\ )_j, (\ )_k, (\ )_l\} \tag{24}$$

The transformation from $\boldsymbol{f}_{(e)}$ to $\{\boldsymbol{F}\}_{(e)}$ can be derived in accordance with the former geometrical decompositions from $\{\boldsymbol{x}\}_{(e)}$ to $\boldsymbol{\varepsilon}_{(e)}$.

Under the simple support fixed, the force components into $\boldsymbol{g}_{(e)}$ are now denoted by $\boldsymbol{G}_{(e)}=\{F_l, F_a, F_c, F_f, F_g, F_h\}$ (**Fig.2**). By substituting (18.a) into the virtual-work equation

$$\delta\boldsymbol{g}_{(e)}\cdot\boldsymbol{G}_{(e)}=\delta\boldsymbol{\varepsilon}_{(e)}\cdot\boldsymbol{f}_{(e)} \tag{25}$$

we have $\boldsymbol{G}_{(e)}$ related to the deformation force :

$$\boldsymbol{G}_{(e)}=[Q_F^{II}(\boldsymbol{g})]_{(e)}\boldsymbol{f}_{(e)},$$

$$[Q_F^{II}(\boldsymbol{g})]_{(e)}=[Q_X^{II}(\boldsymbol{g})]_{(e)}^T \tag{26.a,b}$$

Next, we consider the entire nodal forces resolved into the element's $\{x', y', z'\}$-directions : $\{\boldsymbol{F}'\}_{(e)}=\{(F_{x'}, F_{y'}, F_{z'})_i, (\ )_j, (\ )_k, (\ )_l\}$. Trivially, $F_{x'j}=F_l$, $F_{x'k}=F_a$, $F_{y'k}=F_c$, $F_{x'l}=F_f$, $F_{y'l}=F_g$ and $F_{z'l}=F_h$. And, by the equilibrium conditions in $\{\boldsymbol{F}'\}_{(e)}$ upon the deformed $(e)$, or as the reactive forces in the 3-D simple support, the remaining six components in $\{\boldsymbol{F}'\}_{(e)}$ are determined. Then, $\{\boldsymbol{F}'\}_{(e)}$ is written in the form

$$\{\boldsymbol{F}'\}_{(e)}=[Q_F^I(\boldsymbol{g})]_{(e)}\boldsymbol{G}_{(e)} \tag{27.a}$$

In the actual result, we can see the contragredience between this (27.a) and the former (13) :

$$[Q_F^I(\boldsymbol{g})]_{(e)}=[Q_X^I(\boldsymbol{g})]_{(e)}^T \tag{27.b}$$

Finally, by the inverse rotation to (8), we have the element force :

$$\{\boldsymbol{F}\}_{(e)}=[T(\{x\})]_{(e)}^{T}\{\boldsymbol{F}'\}_{(e)} \tag{28}$$

Collecting (26), (27) and (28) into a unified matrix form, we have the transformation from $\boldsymbol{f}_{(e)}$ to $\{\boldsymbol{F}\}_{(e)}$ :

$$\{\boldsymbol{F}\}_{(e)}=[Q_F(\{x\})]_{(e)}\boldsymbol{f}_{(e)},$$

$$[Q_F]_{(e)}=[T]_{(e)}^{T}[Q_F^{I}]_{(e)}[Q_F^{II}]_{(e)} \tag{29.a,b}$$

Apparently, matrix $[Q_F]_{(e)}$ is related to $[Q_X]_{(e)}$ of (19) by the contragredience

$$[Q_F(\{x\})]_{(e)}=[Q_X(\{x\})]_{(e)}^{T} \tag{30}$$

## 6. TANGENT STIFFNESS

By the use of (4), (5), (15), (22) and (29), we can estimate element force $\{\boldsymbol{F}\}_{(e)}$ for element position $\{x\}_{(e)}$. We here consider the associated tangent stiffness matrix upon freedom $\{x\}_{(e)}$.

By differentiating (29.a), we have

$$\delta\{\boldsymbol{F}\}_{(e)}=[Q_F]_{(e)}\delta\boldsymbol{f}_{(e)}+(\delta[Q_F]_{(e)})\boldsymbol{f}_{(e)} \tag{31}$$

In terms of the independent $\delta\{x\}_{(e)}$, this $\delta\{\boldsymbol{F}\}_{(e)}$ is to be developed into the form

$$\delta\{\boldsymbol{F}\}_{(e)}=[k(\{x\})]_{(e)}\delta\{x\}_{(e)},$$

$$[k]_{(e)}=[k_M(\{x\})]_{(e)}+[k_G(\boldsymbol{f},\ \{x\})]_{(e)} \tag{32.a,b}$$

Those $[k_M]_{(e)}$ and $[k_G]_{(e)}$ are associated with the first and the second term of (31), called *deformation stiffness* and *geometrical stiffness* matrix, respectively.

Under strain-energy-density function $A(\boldsymbol{e})$ given, differential $\delta\boldsymbol{f}_{(e)}$ is related to $\delta\varepsilon_{(e)}$ by

$$\delta\boldsymbol{f}_{(e)}=[\kappa(\varepsilon)]_{(e)}\delta\varepsilon_{(e)},$$

$$[\kappa]_{(e)}=V_0\begin{bmatrix}\dfrac{\partial^2 A}{\partial e_{\xi\xi}^2}, & \cdots, & \dfrac{\partial^2 A}{\partial e_{\xi\xi}\partial\gamma_{\xi\eta}}\\ \vdots & & \vdots\\ \dfrac{\partial^2 A}{\partial\gamma_{\xi\eta}\partial e_{\xi\xi}}, & \cdots, & \dfrac{\partial^2 A}{\partial\gamma_{\xi\eta}^2}\end{bmatrix}_{(e)} \tag{33.a,b}$$

And, $\delta\varepsilon_{(e)}$ is related to $\delta\{x\}_{(e)}$ by (19). Then, the deformation stiffness is written as

$$[k_M]_{(e)}=[Q_F(x)]_{(e)}[\kappa(\varepsilon)]_{(e)}[Q_X(\{x\})]_{(e)} \tag{34}$$

By the use of (19) and (30), matrix $[Q_F(\{x\})]_{(e)}$ itself can be rewritten as

$$[Q_F(\{x\})]_{(e)}=[Q_X(\{x\})]_{(e)}^{T}=\left[\frac{\partial\varepsilon}{\partial\{x\}}\right]_{(e)}^{T} \tag{35}$$

Hence, the second term of (31) is developed as follows :

$$(\delta[Q_F(\{x\})]_{(e)})\boldsymbol{f}_{(e)}$$

$$=\left[\left[\!\left[\frac{\partial}{\partial\{x\}}\left[\frac{\partial\varepsilon}{\partial\{x\}}\right]^{T}\right]\!\right]\delta\{x\}\right]_{(e)}\boldsymbol{f}_{(e)},$$

$$=\left[\frac{\partial}{\partial\{x\}}\left\langle\frac{\partial(\varepsilon\cdot\boldsymbol{f})}{\partial\{x\}}\right\rangle^{T}\right]_{(e)}\Bigg|_{\boldsymbol{f}_{(e)}=\mathrm{const.}}\delta\{x\}_{(e)} \tag{36}$$

where notation $[\![\quad]\!]$ denotes a three-dimensional matrix; and subscript $\Big|_{\boldsymbol{f}_{(e)}=\mathrm{const.}}$ means that $\boldsymbol{f}_{(e)}$ is not subject to the differentiation.

As the actual expansion of (36) : first, we have the six matrices by differentiating each column of $[Q_F(\{x\})]_{(e)}$ (each transposed row of (19.b)) with respect to $\{x\}_{(e)}$; and then as the sum of those matrices multiplied by respective $f_{\xi\xi},\ \cdots,\ f_{\xi\eta}$, we have the geometrical stiffness matrix

$$[k_G(\boldsymbol{f},\ \{x\})]_{(e)}=$$

$$f_{\xi\xi}\left[\frac{\partial^2 e_{\xi\xi}}{\partial\{x\}^2}\right]_{(e)}+f_{\eta\eta}\left[\frac{\partial^2 e_{\eta\eta}}{\partial\{x\}^2}\right]_{(e)}+f_{\zeta\zeta}\left[\frac{\partial^2 e_{\zeta\zeta}}{\partial\{x\}^2}\right]_{(e)}$$

$$+f_{\eta\zeta}\left[\frac{\partial^2\gamma_{\eta\zeta}}{\partial\{x\}^2}\right]_{(e)}+f_{\zeta\xi}\left[\frac{\partial^2\gamma_{\zeta\xi}}{\partial\{x\}^2}\right]_{(e)}+f_{\xi\eta}\left[\frac{\partial^2\gamma_{\xi\eta}}{\partial\{x\}^2}\right]_{(e)} \tag{37}$$

where

$$\left[\frac{\partial^2 e_{\xi\xi}}{\partial(x)^2}\right]_{(e)}=\begin{bmatrix}\dfrac{1}{l_0^2}, & -\dfrac{1}{l_0^2}, & 0, & 0,\\ & \dfrac{1}{l_0^2}, & 0, & 0\\ \text{Sym.} & & 0, & 0\\ & & & 0\end{bmatrix},$$

$$\left[\frac{\partial^2 e_{\eta\eta}}{\partial(x)^2}\right]_{(e)}=\begin{bmatrix}\dfrac{b_0^2}{l_0^2c_0^2}, & \dfrac{a_0b_0}{l_0^2c_0^2}, & -\dfrac{b_0}{l_0c_0^2}, & 0\\ & \dfrac{a_0^2}{l_0^2c_0^2}, & -\dfrac{a_0}{l_0c_0^2}, & 0\\ \text{Sym.} & & \dfrac{1}{c_0^2}, & 0\\ & & & 0\end{bmatrix},$$

$$\left[\frac{\partial^2 e_{\zeta\zeta}}{\partial(x)^2}\right]_{(e)}=$$

$$\begin{bmatrix}\dfrac{d_0'}{l_0h_0^2}\left(-1+\dfrac{g_0}{c_0}-\dfrac{d_0}{l_0}\right), & \dfrac{d_0d_0'}{l_0^2h_0^2}, & -\dfrac{g_0d_0'}{l_0c_0h_0^2}, & \dfrac{d_0'}{l_0h_0^2}\\ & \dfrac{d_0^2}{l_0^2h_0^2}, & -\dfrac{g_0d_0}{l_0c_0h_0^2}, & \dfrac{d_0}{l_0h_0^2}\\ \text{Sym.} & & \dfrac{g_0^2}{c_0^2h_0^2}, & -\dfrac{g_0}{l_0h_0^2}\\ & & & \dfrac{1}{h_0^2}\end{bmatrix},$$

$$\left[\frac{\partial^2 \gamma_{\eta\zeta}}{\partial (x)^2}\right]_{(e)} =$$

$$\begin{bmatrix} \frac{2b_0}{l_0c_0h_0}\left(1-\frac{g_0}{c_0}+\frac{d_0}{l_0}\right), & \frac{1}{l_0c_0h_0}\left(a_0-f_0-\frac{2b_0d_0}{l_0}\right), & \frac{e_0+2d_0'}{l_0c_0h_0}, & -\frac{b_0}{l_0c_0h_0} \\ & -\frac{2a_0d_0}{l_0^2c_0h_0}, & \frac{f_0+2d_0}{l_0c_0h_0}, & -\frac{a_0}{l_0c_0h_0} \\ \text{Sym.} & & -\frac{2g_0}{c_0^2h_0}, & \frac{1}{c_0h_0} \\ & & & 0 \end{bmatrix},$$

$$\left[\frac{\partial^2 \gamma_{\zeta\xi}}{\partial (x)^2}\right]_{(e)} =$$

$$\begin{bmatrix} \frac{2}{l_0h_0}\left(1-\frac{g_0}{c_0}+\frac{d_0}{l_0}\right), & \frac{1}{l_0h_0}\left(-1+\frac{g_0}{c_0}-\frac{2d_0}{l_0}\right), & \frac{g_0}{l_0c_0h_0}, & -\frac{1}{l_0h_0} \\ & \frac{2d_0}{l_0^2h_0}, & -\frac{g_0}{l_0c_0h_0}, & \frac{1}{l_0h_0} \\ \text{Sym.} & & 0, & 0 \\ & & & 0 \end{bmatrix},$$

$$\left[\frac{\partial^2 \gamma_{\xi\eta}}{\partial (x)^2}\right]_{(e)} = \begin{bmatrix} \frac{2b_0}{l_0^2c_0}, & \frac{1}{l_0^2c_0}(a_0-b_0), & -\frac{1}{l_0c_0}, & 0 \\ & -\frac{2a_0}{l_0^2c_0}, & \frac{1}{l_0c_0}, & 0 \\ \text{Sym.} & & 0, & 0 \\ & & & 0 \end{bmatrix}$$

(38.a-f)

In those matrices of order four, for short expression, only the second derivatives into $\{x_i, x_j, x_k, x_l\}$ are presented. In the actual matrices of order 12, all the cross derivatives into $(y, z)$, $(z, x)$ and $(x, y)$ result into zero, and the remaining ones into $\{y_i, y_j, y_k, y_l\}$ and $\{z_i, z_j, z_k, z_l\}$ are the same to (38.a-f). That is, the full matrices are obtained by replacing each element of matrices (38.a-f) into diagonal $3\times3$ sub-matrix.

## 7. NUMERICAL EXAMPLE

As an illustrative example, an elastic body of square cross-section shown in **Fig.3** is analyzed. Each cubic unit is divided in the same way into five tetrahedral elements. The linear isotropic elastic relation is assumed : Young's modulus$=2,100.$ tonf/cm$^2$ and Possion's ratio$=0.3$. With node 1 to 9 being fixed, coupled 3,000. tonf are applied in 10 steps at node 28 and 36 into $\pm y$-directions. After the Newton-Raphson method applied to each load increment, the final equilibrium configuration is obtained as shown in **Fig.4**, where the largest strain component is $e_{\zeta\zeta}=0.730$ in the two elements

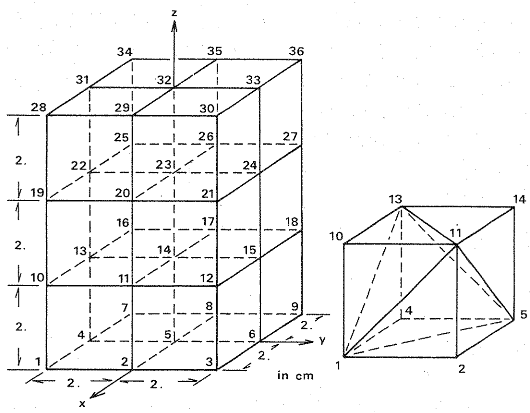


**Fig.3** Initial Configuration

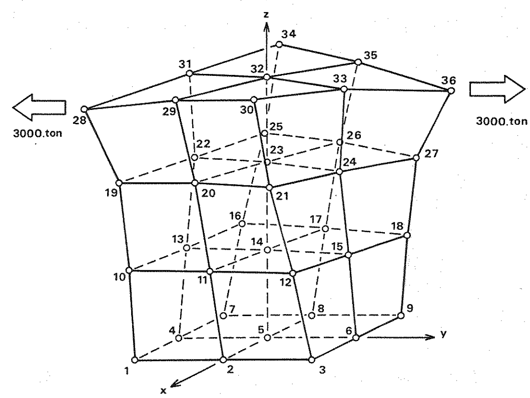


**Fig.4** Deformed Configuration

**Table 1** Nodal Positions
for $P_{28y}=-3{,}000.$ and $P_{36y}=3{,}000.$ tonf

| Node | Coordinates | | | Node | Coordinates | | |
|---|---|---|---|---|---|---|---|
| | x | y | z | | x | y | z |
| 1 | 2. | -2. | 0. | 19 | 1.140 | -2.780 | 3.795 |
| 2 | 2. | 0. | 0. | 20 | 1.832 | -0.728 | 3.966 |
| 3 | 2. | 2. | 0. | 21 | 2.448 | 1.272 | 4.010 |
| 4 | 0. | -2. | 0. | 22 | -0.651 | -1.959 | 3.922 |
| 5 | 0. | 0. | 0. | 23 | 0. | 0. | 3.969 |
| 6 | 0. | 2. | 0. | 24 | 0.651 | 1.959 | 3.922 |
| 7 | -2. | -2. | 0. | 25 | -2.448 | -1.272 | 4.010 |
| 8 | -2. | 0. | 0. | 26 | -1.832 | 0.728 | 3.966 |
| 9 | -2. | 2. | 0. | 27 | -1.140 | 2.780 | 3.795 |
| 10 | 1.552 | -2.358 | 1.929 | 28 | 0.797 | -3.741 | 5.381 |
| 11 | 1.925 | -0.337 | 1.997 | 29 | 1.664 | -1.235 | 5.807 |
| 12 | 2.229 | 1.684 | 2.011 | 30 | 2.455 | 0.933 | 5.999 |
| 13 | -0.335 | -1.998 | 1.962 | 31 | -0.734 | -2.117 | 5.793 |
| 14 | 0. | 0. | 1.985 | 32 | 0. | 0. | 5.873 |
| 15 | 0.335 | 1.998 | 1.962 | 33 | 0.734 | 2.117 | 5.793 |
| 16 | -2.229 | -1.684 | 2.011 | 34 | -2.455 | -0.933 | 5.999 |
| 17 | -1.925 | 0.337 | 1.997 | 35 | -1.664 | 1.235 | 5.807 |
| 18 | -1.552 | 2.358 | 1.929 | 36 | -0.797 | 3.741 | 5.381 |

with node 19-28-29-31 and 27-33-35-36. The nodal positions are given in **Table 1**.

In this example, the two different methods of deriving the stiffness relations are adopted, the B-notation method[3] and our method of separation-into-rigid-displacement-and-deformation. Their numerical results are the same in five digits. But, the computation time are different : the *cpu* time in the entire *executions* including 32 times inversion of the global stiffness matrix of order $81 \times 81$ are 5.42 and 4.92 seconds, in which 0.82 and 0.32 seconds are estimated for dealing with the element stiffness relations, for the B-notation method and the present method, respectively.

## 8. CONCLUDING REMARKS

The present FEM treatment is as follows : through the chained geometrical decompositions where shape $\boldsymbol{g}_{(e)}$ is defined as an intermediate parameter, element position $\{\boldsymbol{x}\}_{(e)}$ is analytically separated into position-as-a-rigid-body $\boldsymbol{v}_{(e)}$ and deformation $\boldsymbol{\varepsilon}_{(e)}$; deformation force $\boldsymbol{f}_{(e)}$ is related to $\boldsymbol{\varepsilon}_{(e)}$ under an assumed elastic constitutive relation; and element force $\{\boldsymbol{F}\}_{(e)}$ is derived from $\boldsymbol{f}_{(e)}$ by the force transformations associated with the geometrical decompositions. Those $\{\boldsymbol{F}\}_{(e)}$, $\boldsymbol{G}_{(e)}$ and $\boldsymbol{f}_{(e)}$ are defined as force components conjugate to $\{\boldsymbol{x}\}_{(e)}$, $\boldsymbol{g}_{(e)}$ and $\boldsymbol{\varepsilon}_{(e)}$, respectively. As the result, the geometrical relations and the force transformations are related into each pair by the contragredience. That is, the present FEM relations can be recognized as a potential problem on any stage of the formulation, i.e. between $\boldsymbol{f}_{(e)}$ and $\boldsymbol{\varepsilon}_{(e)}$, between $\boldsymbol{G}_{(e)}$ and $\boldsymbol{g}_{(e)}$, or between $\{\boldsymbol{F}\}_{(e)}$ and $\{\boldsymbol{x}\}_{(e)}$.

Since based upon the same strain-constant interpolation, our numerical results are not to be different from those by the N-notation and B-notation methods. In those existing methods, however, the significant relations presented in this study are buried in their sum expressions. The stiffness relations are estimated after those complicated summations executed numerically. On the other hand, our entire formulation is lengthy, but is carried out in explicit form. The numerical computation is less time-consuming than by the foregoing methods, for it is only the final relations to be adopted into the calculation.

## APPENDIX I. COMPONENTS OF ROTATION BY $\delta\{\boldsymbol{x}'\}_{(e)}$

Since any changes resulting from $\delta\{\boldsymbol{x}'\}_{(e)}$ are infinitesimal, the rotation of $\{\boldsymbol{i}_{(x'y'z')}\}$ is in a linear vector space. Under the rule of right-handed screw around the preceding $\{\boldsymbol{i}_{(x'y'z')}\}$, we consider the rotation resolved in the form

$$\delta\boldsymbol{\theta}=\delta\theta_{x'}\boldsymbol{i}_{x'}+\delta\theta_{y'}\boldsymbol{i}_{y'}+\delta\theta_{z'}\boldsymbol{i}_{z'}$$

As stated in **Sec.2**, the system $\{x', y', z'\}$ of

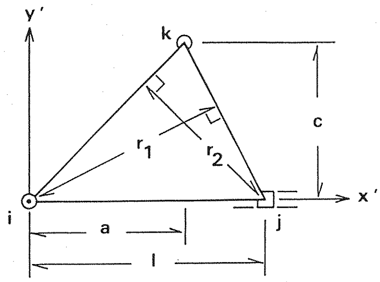


**Fig.A-1** $r_1$ and $r_2$

element $(e)$ is determined in the space by position-as-a-rigid-body $\boldsymbol{v}_{(e)}$, but is not affected by shape $\boldsymbol{g}_{(e)} = \{l, a, c, f, g, h\}$. Therefore, no rotation comes from variation $\delta x_j'=\delta l$, $\delta x_k'=\delta a$, $\delta y_k'=\delta c$, $\delta x_l'=\delta f$, $\delta y_l'=\delta g$ or $\delta z_l'=\delta h$. The effects of the remaining components in $\delta\{\boldsymbol{x}'\}_{(e)}$ are as follows :

$\delta x_i'$ → $\{\boldsymbol{i}_{(x'y'z')}\}$ is translated into $x'$, but is not rotated.

$\delta y_i'$ → rotation around $\boldsymbol{i}_{z'}$, by angle $-\delta y_i'/l$.

$\delta z_i'$ → rotation around $\overrightarrow{jk}$-axis by angle $\delta z_i'/r_1$ with $r_1=lc/\sqrt{c^2+(l-a)^2}$ (**Fig.A-1**). By the decomposition around $\boldsymbol{i}_{x'}$ and $\boldsymbol{i}_{y'}$, this rotation is written as

$$\delta\boldsymbol{\theta}=\frac{a-l}{cl}\delta z_i'\boldsymbol{i}_{x'}+\frac{1}{l}\delta z_i'\boldsymbol{i}_{y'}$$

$\delta y_j'$ → rotation around $\boldsymbol{i}_{z'}$ by angle $\delta y_j'/l$.

$\delta z_j'$ → rotation around $\overrightarrow{ki}$ by angle $\delta z_j'/r_2$ with $r_2=lc/\sqrt{c^2+a^2}$ (**Fig.A-1**). By the decomposition around $\boldsymbol{i}_{x'}$ and $\boldsymbol{i}_{y'}$, this rotation is written as

$$\delta\boldsymbol{\theta}=-\frac{a}{cl}\delta z_j'\boldsymbol{i}_{x'}-\frac{1}{l}\delta z_j'\boldsymbol{i}_{y'}$$

$\delta z_k'$ → rotation around $\boldsymbol{i}_{x'}$ by angle $\delta z_k'/c$.

By the superposition of those rotations, we have the result of (9.a-c).

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**(Received March 17, 1994)**