Structural Eng./Earthquake Eng. Vol. 11, No. 3, 141s—144s, October 1994
Japan Society of Civil Engineers (J. Struct. Mech. Earthquake Eng. No. 501/ I—29)

A CONSIDERATION ON THE SLACKENED AND TIGHTENED CABLES

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On the exact equilibrium configuration of an elastic cable spanned, the potential of uniform self-weight and the strain energy in stretching are analytically estimated. Those quantities are related to the work done by the chord force into an energy conservation. By the differentiation of the energy relation with respect to the span elongation, the chord force is separated into two components reflecting the sag effect and the elastic elongation. In a numerical analysis, it is shown that a characteristic magnitude of the stretching exists between the slackened and the tightened states.

Key Words : *elastic catenary, sag effect, chord force*

1. INTRODUCTION

As a practical analysis of cable structures, a cable element is dealt with as the *simple-tension member* : straight in configuration and irresistible to compression. At the same time, we have the sagged elastic curve under the uniform self-weight per unit natural length.[1),2)] The state of an actual cable is continuously changed in this exact configuration called *elastic catenary*. In this study, the potential of self-weight and the strain energy existing in that catenary are derived in analytical form. Then, by the use of a relation of energy conservation, it is presented that the cable force can be divided into two components which are associated with the sag change and the elastic elongation, respectively. Their relative magnitudes in the cable force can be of a practical use to estimate to what extent a cable is stretched.

2. ELASTIC CATENARY

Consider that one end of an isolated cable of length l is anchored at the origin of two-dimensional Cartesian coordinates $\{x, y\}$. The exact equilibrium configuration under the self-weight is written as

$$x(\boldsymbol{T}_0, s)=\frac{T_{0x}s}{EA}+\frac{T_{0x}}{w}\log\left[\frac{T_{0y}+\sqrt{T_{0x}^2+T_{0y}^2}}{(T_{0y}-ws)+\sqrt{T_{0x}^2+(T_{0y}-ws)^2}}\right]$$

$$y(\boldsymbol{T}_0, s)=\frac{T_{0y}s-ws^2/2}{EA}+\frac{1}{w}\left\{\sqrt{T_{0x}^2+T_{0y}^2}-\sqrt{T_{0x}^2+(T_{0y}-ws)^2}\right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1.\text{a,b})$$

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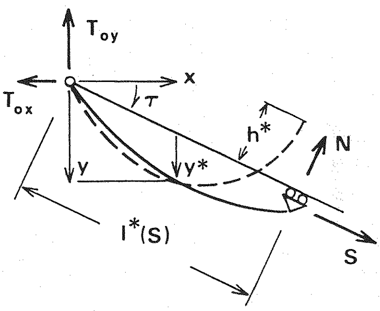

Fig.1 Elastic Cable Spanned on a Slope

where s=material coordinate along natural length ($0\leq s\leq l$); EA=extension rigidity; w=weight per unit natural length; and $\boldsymbol{T}_0$ ($=\{T_{0x}, T_{0y}\}$)=tension components at the initial end. In this expression, components $\{T_{0x}, T_{0y}\}$ can be regarded as parameters to determine the catenary curve : If the other end is not anchored, but is subjected to a known force, they are directly obtained as the sum of the external forces acting on the cable. On the other hand, if that end is anchored at another $\{x^*, y^*\}$, a numerical iteration is necessary to find $\{T_{0x}, T_{0y}\}$ satisfying $x(\boldsymbol{T}_0, l)=x^*$ and $y(\boldsymbol{T}_0, l)=y^*$.

3. ENERGY QUANTITIES IN SPAN STRETCHING

Let an elastic cable be spanned on a slope with angle τ from the horizontal, as shown in **Fig.1**. At the roller end, an external force S is applied into the slope direction, with the normal reaction being denoted by N. The equilibrium curve for a given S is obtained by the following iteration : First, an initial value is given to N. Then, the distribution of tension components $\{T_x, T_y\}$ is statically determined : $\{T_{lx}, T_{ly}\}=\{S\cos\tau+N\sin\tau, S\sin\tau-N\cos\tau\}$ at the roller support, and $\{T_{0x}, T_{0y}\}=\{T_{lx}, T_{ly}+wl\}$ at the fixed end. In a catenary curve drawn by (1.a,b), as the result of N being not true, we have normal incompatibility h^* at the roller end. Then, by the use of the derivatives of (1.a,b) with respect to $\{T_{0x}, T_{0y}\}$, we can estimate the correction of N to diminish the incompatibility. The calculation is continued by the Newton-Raphson method. If the convergence is difficult by reason of the initial N assumed far from its true value, the iteration can be switched to the method given in Ref. 3), for instance.

In the equilibrium curve (1.a,b) with $\{T_{0x}, T_{0y}\}$ determined, we can develop the following integrations for the strain energy and the potential of self-weight, respectively :

$$U(S)=\int_0^l \frac{T^2}{2EA}ds=\int_0^l \frac{T_{0x}^2+(T_{0y}-ws)^2}{2EA}ds$$

$$=\frac{1}{2EA}\left\{T_{0x}^2 l+\frac{1}{3w}(T_{0y}^3-T_{ly}^3)\right\} \cdots\cdots(2.a)$$

$$-\Phi_w(S)=\int_0^l wy(s)ds=\frac{w}{EA}\left(\frac{T_{0y}l^2}{2}-\frac{wl^3}{6}\right)+T_0 l$$

$$+\frac{T_{0x}^2}{2w}\left\{\frac{T_{ly}T_l}{T_{lx}^2}-\frac{T_{0y}T_0}{T_{0x}^2}+\frac{1}{2}\log\left(\frac{T_l+T_{ly}}{T_l-T_{ly}}\cdot\frac{T_0-T_{0y}}{T_0+T_{0y}}\right)\right\} \cdots\cdots(2.b)$$

where $T_0=\sqrt{T_{0x}^2+T_{0y}^2}$, and $T_l=\sqrt{T_{lx}^2+T_{ly}^2}$. The system of our simply-supported elastic cable is conservative in energy. The work done by the end force S is stored into strain energy $U(S)$ and self-weight potential $\Phi_w(S)$. Denoting the work in stretching from $S=0$ by $\Phi_S(S)$, we have the following energy relation :

$$\Phi_S(S)=(U(S)-U(0))+(\Phi_w(S)-\Phi_w(0))$$

$$=U(S)+\Phi_w(S)+const. \cdots\cdots\cdots(3)$$

In the cable spanned on a slope ($\tau\neq 0$), since the compatible configuration is determined by a numerical procedure, we can not obtain the explicit expression for $\Phi_S(S)$. But, if supported horizontally as shown in **Fig.2**, we directly have $N=wl/2$ and $\{T_{0x}, T_{0y}\}=\{S, wl/2\}$. Then, the following integration can be carried out for a change of span length $l^*(S)$ $(=x(S,l))$:

$$\Phi_S(S)=\int_0^{l^*} Sdl^*=\int_0^S S\frac{dl^*(S)}{dS}dS$$

$$=\frac{S^2 l}{2EA}+\frac{wl^2}{4}-\frac{l}{2}\sqrt{S^2+\left(\frac{wl}{2}\right)^2}$$

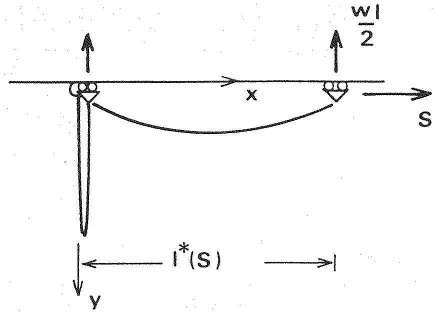

Fig.2 Horizontally-Supported Cable

$$+\frac{S^2}{2w}\log\left(\frac{\sqrt{S^2+\left(\frac{wl}{2}\right)^2}+\frac{wl}{2}}{\sqrt{S^2+\left(\frac{wl}{2}\right)^2}-\frac{wl}{2}}\right) \cdots(4)$$

By the use of this $\Phi_S(S)$ and the former (2.a,b), relation (3) is actually confirmed.

For end force S given, our analysis has been developed. We now have this force again from the differentiation of its potential $\Phi_S(S)$ with respect to span length l^*. Applying this differentiation to the energy relation (3), we have the following separation of end force S :

$$S\left(=\frac{d\Phi_S(S)}{dl^*}\right)=S_U(S)+S_w(S) \cdots\cdots\cdots(5)$$

where

$$S_U=\frac{dU(S)}{dl^*}=\frac{\frac{dU(S)}{dS}}{\frac{dl^*(S)}{dS}}$$

$$S_w=\frac{d\Phi_w(S)}{dl^*}=\frac{\frac{d\Phi_w(S)}{dS}}{\frac{dl^*(S)}{dS}} \cdots\cdots\cdots\cdots(6.a,b)$$

Those S_U *(strain force)* and S_w *(sag force)* come physically from the changes of the strain energy and the self-weight potential, respectively, in displacement into l^*. The relative ratio of S_U and S_w is a direct parameter for how end force S is exerted in the elastic cable.

In case of a sloped cable ($\tau\neq 0$), by the same reason stated before, it is difficult to develop the analytical differentiations, (6.a,b). But, by the following expressions in finite differences, we can estimate those strain and sag forces, numerically:

$$S_U(S)\doteqdot\frac{1}{2}\left(\frac{U(S+\Delta S)-U(S)}{l^*(S+\Delta S)-l^*(S)}+\frac{U(S)-U(S-\Delta S)}{l^*(S)-l^*(S-\Delta S)}\right)$$

$$S_w(S)\doteqdot\frac{1}{2}\left(\frac{\Phi_w(S+\Delta S)-\Phi_w(S)}{l^*(S+\Delta S)-l^*(S)}\right.$$

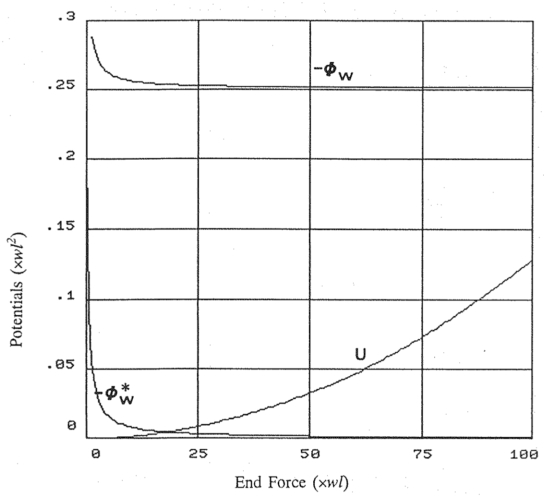

Fig.3 Strain Energy and Self-Weight Potential for $\tau=30°$

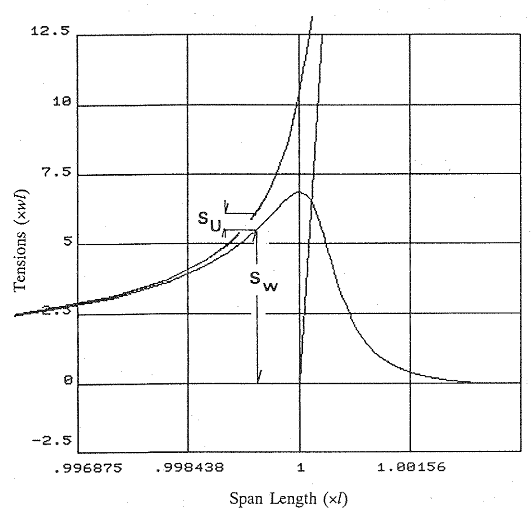

Fig.4 Sag Force and Strain Force for $\tau=30°$

$$+\frac{\Phi_w(S)-\Phi_w(S-\Delta S)}{l^*(S)-l^*(S-\Delta S)}\Bigg)\cdots\cdots(7.a,b)$$

4. NUMERICAL EXAMPLE

A 54-*mm*-diameter spiral-bridge cable is considered ($l=50.\ m$, $w=0.0144$ *ton f/m*, and $EA=28{,}016.$ *ton f*). In stretching from $S=0.$ to $100.\times wl$ on a slope $\tau=30°$, $U(S)$ and $\Phi_w(S)$ are obtained as shown in **Fig.3** (normalized by wl^2), in which Φ^*_w is a quantity defined by $-\int wy^*(s)ds$ as a potential of self-weight from the slope line. We can see a parabolic increase for U, and a plateauic increase for Φ_w. After the numerical differentiations by (7.a,b), the separation of S into S_U and S_w is plotted in **Fig.4** (normalized by wl). In the initial stretching, as is expected, the end force S almost comes from the sag force to balance with the self weight in a small elastic elongation. Then, after the plateau of Φ_w in **Fig.3** is approached, the strain force S_U becomes dominant in S with a steep descent of S_w. In the sag force, a clear peak ($6.86\times wl$) is seen at $S=10.38\times wl$. This value can be regarded as a critical point to divide the cable states into the two stages : slackened and tightened. In case the cable is stretched on a slope of $\tau=-30°$, a similar result is obtained as shown in **Fig.5**. But, since the two cables are approaching to the different slope lines, their values are different : the peak for $\tau=-30°$ is at $S=10.88\times wl$ ($S_w=7.36\times wl$).

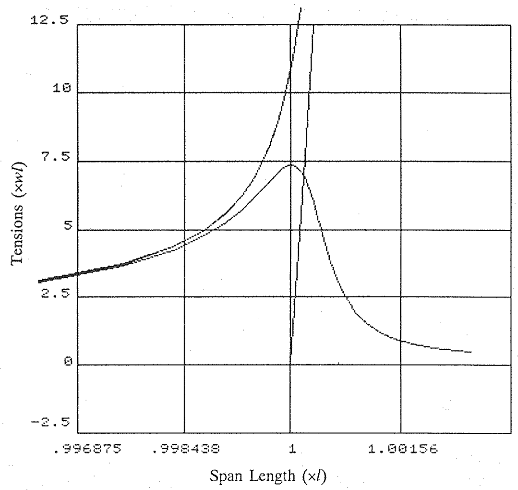

Fig.5 Sag Force and Strain Force for $\tau=-30°$

Here, we re-decompose the tensile force at the both ends into the slope and vertical directions : $\{T^*, V_0\}$ and $\{T^*, V_l\}$. In this expression, T^* is the same at the both ends, and then the same in the two cases of opposite chord angles. Let this T^* be called *chord force*, related to the former S and N by $T^*=S+N\tan\tau$. The following is confirmed in the numerical analysis : any two critical states in opposite chord angles have the same chord force ($T^*=10.62\times wl$ for the former $\tau=\pm 30°$). In **Fig.6**, those critical chord forces in various τ are shown for a logarithmic range of EA/wl.

5. CONCLUDING REMARKS

Presented in this study is a theoretical separation of the cable force into the two components: one is the *strain force*, associated with the elastic elongation, and the other is the *sag force*, balanced with the uniform self-weight. In the numerical analysis, it is shown further that there exists a clear peak value of the sag force in increase of the cable force. This magnitude of stretching can be employed as a theoretical boundary to define the *slackened* and

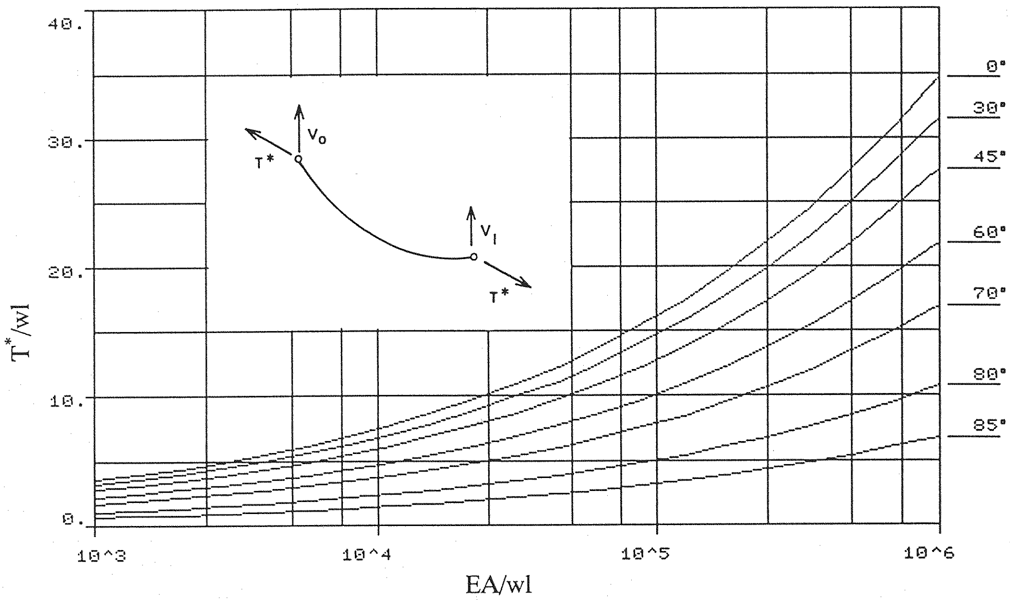

Fig.6 Critical Chord Force

tightened states of a cable. If a good accommodation is found with the existing practical analyses of the sag effect such as concerning the vibration modes of a cable member,[e.g.4)] the present result can be used for various purposes, for instance to confirm sufficient tensions as the *simple-tension members* in the analysis of a tightened cable structure.

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(Received December 27, 1993)