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AXIALLY ASYMMETRIC STRESSES IN A TRANSVERSELY ISOTROPIC, SHORT CYLINDER SUBJECTED TO SECTORIAL PRESSURES ON THE END FACES

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An analysis of axially asymmetric stresses in a transversely isotropic, short cylinder subjected to sectorial pressures on the end faces is presented. The generalized Elliott solution is used for the analysis. The solution yields two different elasticity solutions to be necessary for satisfying boundary conditions at the end faces and the side surface. Magnesium and cadmium crystals and an isotropic material are treated. Numerical results for displacements and stresses in these materials are illustrated. The effect of anisotropy on the displacements and stresses is examined through a comparison with the isotropic material.

Key Words : *elasticity, transverse isotropy, stress analysis, asymmetric problem, short cylinder*

1. INTRODUCTION

Recent studies on three-dimensional elasticity problems have turned to those of anisotropic solids. Although there are various classes of anisotropy, practical necessity is concerned with orthotropy, cylindrical anisotropy or transverse isotropy. Studies on orthotropic and cylindrically anisotropic solids are few in number at the present time, because it is hard to find the three-dimensional elasticity solutions to these anisotropic solids. However, there are a lot of studies on transversely isotropic solids, because the three-dimensional elasticity solutions to this anisotropy have been found. Levine and Klosner[1] analyzed axially symmetric stresses in a long cylindrical shell subjected to radial band loads. Atsumi and Itou[2] analyzed axially symmetric stresses in an infinite cylinder with a spherical cavity. Mirsky[3] and Chen[4] analyzed the wave propagation in an infinite cylinder and the concentrated force moving with uniform velocity in an infinite solid, respectively. Zureick[5] analyzed axially asymmetric stresses in an infinite solid containing a spheroidal cavity.

These studies as stated above are concerned with infinite solids and use Elliott's and Lodge's solutions. Although Elliott's and Lodge's solutions are simple in applications, they seem to have been hardly applied to three-dimensional problems of finite solids, for instance, short rectangular prisms, short cylinders or short hollow cylinders. The writer proposed the generalized Elliott solution and analyzed axially symmetric stresses in a short cylinder subjected to a radial band load in a previous paper[6]. The solution was also applied to an axially symmetric stress analysis of a short hollow cylinder subjected to an outer band load[7]. However, studies on axially asymmetric stresses in a short cylinder or in a short hollow cylinder have not, to the writer's knowledge, been carried out.

This paper is concerned with an analysis of axially asymmetric stresses in a transversely isotropic, short cylinder subjected to sectorial pressures on the end faces. The generalized Elliott solution is used for the analysis. The three-dimensional problem of a short cylinder is much more complicated than that of an infinite or a long cylinder, because the method of solution requires two different elasticity solutions to satisfy boundary conditions at the end faces and the side surface. The generalized Elliott solution yields the two elasticity solutions. Additional solutions are also used to treat initial terms in Fourier and Bessel expansions. Magnesium and cadmium crystals, as examples of transversely isotropic materials, as well as an isotropic material, are treated in numerical calculations.

2. THE GENERALIZED ELLIOTT SOLUTION

Using cylindrical coordinates (r, θ, z) such that the z-axis is taken parallel to the axis of elastic symmetry, the generalized Elliott solution[6] is expressed in terms of displacement components, i.e., u_r, u_θ and u_z as

$$u_r=\frac{\partial}{\partial r}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\cdots\cdots\cdots\cdots\cdots\cdots\text{(1a)}$$

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$$u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)\right.$$
$$\left.-\gamma_2\phi_1-\gamma_3\phi_3\right]-\frac{\partial\psi}{\partial r} \quad\cdots\cdots(1b)$$

$$u_z=\frac{\partial}{\partial z}\left[k_1(\phi_{01}-\gamma_3\phi_1)+k_2(\phi_{03}-\gamma_2\phi_3)\right.$$
$$\left.+\gamma_1\left(k_1r\frac{\partial\phi_1}{\partial r}+k_2z\frac{\partial\phi_3}{\partial z}\right)\right] \quad\cdots\cdots(1c)$$

in which

$$\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}=0,\ \nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}=0 \quad\cdots\cdots(2a,\ b)$$

$$\nabla_1^2\phi_1+\nu_2\frac{\partial^2\phi_1}{\partial z^2}=0,\ \nabla_1^2\phi_3+\nu_1\frac{\partial^2\phi_3}{\partial z^2}=0 \quad\cdots(2c,\ d)$$

$$\nabla_1^2\psi+\nu_3\frac{\partial^2\psi}{\partial z^2}=0,\ \nabla_1^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad\cdots\cdots(2e)$$

$$\gamma_1=\begin{cases}1 \text{ for } \nu_1=\nu_2\\ 0 \text{ for } \nu_1\neq\nu_2\end{cases} \quad\cdots\cdots(3a)$$

$$\gamma_2=\begin{cases}\dfrac{2c_{11}\nu_2}{c_{11}\nu_2-c_{44}} \text{ for } \nu_1=\nu_2\\ \dfrac{2\nu_2}{\nu_1-\nu_2}\dfrac{c_{11}\nu_1-c_{44}}{c_{11}\nu_2-c_{44}} \text{ for } \nu_1\neq\nu_2\end{cases} \quad\cdots\cdots(3b)$$

$$\gamma_3=\begin{cases}0 \text{ for } \nu_1=\nu_2\\ \dfrac{2\nu_2}{\nu_1-\nu_2} \text{ for } \nu_1\neq\nu_2\end{cases} \quad\cdots\cdots(3c)$$

$$k_1=\frac{c_{11}\nu_1-c_{44}}{c_{13}+c_{44}},\ k_2=\frac{c_{11}\nu_2-c_{44}}{c_{13}+c_{44}} \quad\cdots\cdots(4a,\ b)$$

$$\nu_3=\frac{c_{44}}{c_{66}}=\frac{2c_{44}}{c_{11}-c_{12}} \quad\cdots\cdots(4c)$$

and c_{ij} denotes the elastic constant of transversely isotropic solids, and ν_1 and ν_2 are the roots of

$$c_{11}c_{44}\nu^2+[c_{13}(c_{13}+2c_{44})-c_{11}c_{33}]\nu+c_{33}c_{44}=0 \quad\cdots\cdots(5)$$

Determining the coordinate system of a short cylinder such as **Fig.1** and regarding that the displacement and stress field is even in z, potential functions are obtained from Eqs.(2a-e) in the form

$$\phi_{03}=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}A_{ms}\cos m\theta J_m(\alpha_{ms}r)\cosh\frac{\alpha_{ms}z}{\sqrt{\nu_2}} \quad\cdots\cdots(6a)$$

$$\phi_3=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}C_{ms}\cos m\theta J_m(\alpha_{ms}r)\cosh\frac{\alpha_{ms}z}{\sqrt{\nu_1}} \quad\cdots\cdots(6b)$$

$$\phi_{01}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}D_{mm}\cos m\theta\cos\beta_nzI_m(\sqrt{\nu_1}\beta_nr) \quad\cdots\cdots(6c)$$

$$\phi_1=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}F_{mn}\cos m\theta\cos\beta_nzI_m(\sqrt{\nu_2}\beta_nr) \quad\cdots\cdots(6d)$$

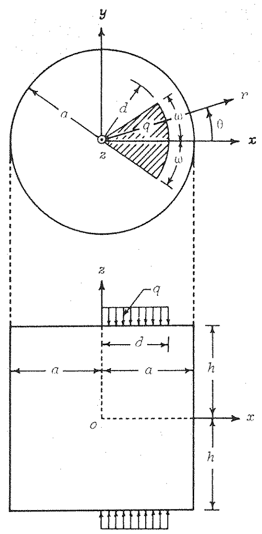

Fig.1 Coordinate system of short cylinder.

$$\psi=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}B_{ms}\sin m\theta J_m(\alpha_{ms}r)\cosh\frac{\alpha_{ms}z}{\sqrt{\nu_3}}$$
$$+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}E_{mn}\sin m\theta\cos\beta_nzI_m(\sqrt{\nu_3}\beta_nr) \quad\cdots\cdots(6e)$$

in which $A_{ms},\cdots,E_{mn}$ are arbitrary constants to be determined from boundary conditions. Furthermore, $J_m(\alpha_{ms}r)$ and $I_m(\sqrt{\nu_j}\beta_nr)$ denote Bessel function and the modified Bessel function of the first kind, of order m, respectively, and

$$\alpha_{ms}=\frac{\lambda_{ms}}{a}(m=0,1,\cdots,s=1,2,\cdots) \quad\cdots\cdots(7a)$$

$$\beta_n=\frac{n\pi}{h}(n=1,2,\cdots) \quad\cdots\cdots(7b)$$

in which λ_{ms} is the root of a transcendental equation as stated later. Eqs. (6a, b) and a part of Eq. (6e) and Eqs. (6c, d) and another part of Eq. (6e) are the solutions satisfying the boundary conditions at the end faces and at the side surface, respectively.

In order to satisfy the boundary conditions as stated below, Fourier and Bessel expansions are needed. Since the expansions bring initial terms in Fourier or Bessel series, additional solutions are needed to treat the initial terms. They are as follows :

$$\phi_{01,0}=\sum_{m=2}^{\infty}D_{m0}\cos m\theta r^m-\frac{\nu_1\gamma_4}{\nu_3(\gamma_2-\gamma_3)}$$
$$\cdot\sum_{m=1}^{\infty}(\gamma_1m-\gamma_3)F_{m0}\cos m\theta[r^{m+2}/2$$
$$-(m+1)r^mz^2/\nu_1] \quad\cdots\cdots(8a)$$

$$\phi_{1,0}=\frac{\nu_2\gamma_4}{\nu_3(\gamma_2-\gamma_3)}\sum_{m=1}^{\infty}F_{m0}\cos m\theta[r^{m+2}/2-(m+1)r^m z^2/\nu_2] \quad \text{(8b)}$$

$$\phi_0=\gamma_4\sum_{m=1}^{\infty}F_{m0}\sin m\theta[r^{m+2}/2-(m+1)r^m z^2/\nu_3] \quad \text{(8c)}$$

in which

$$\gamma_4=\frac{4c_{11}}{c_{11}+c_{12}} \quad \text{(9)}$$

and

$$\phi_{3,00}=C_{00}\left(\frac{r^2}{2}-\frac{z^2}{\nu_1}\right) \quad \text{(10a)}$$

$$\phi_{1,00}=F_{00}\left(\frac{r^2}{2}-\frac{z^2}{\nu_2}\right) \quad \text{(10b)}$$

The generalized Hooke's law of transversely isotropic solids is

$$\sigma_{rr}=c_{11}\varepsilon_{rr}+c_{12}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz} \quad \text{(11a)}$$

$$\sigma_{\theta\theta}=c_{12}\varepsilon_{rr}+c_{11}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz} \quad \text{(11b)}$$

$$\sigma_{zz}=c_{13}\varepsilon_{rr}+c_{13}\varepsilon_{\theta\theta}+c_{33}\varepsilon_{zz} \quad \text{(11c)}$$

$$\sigma_{\theta z}=2c_{44}\varepsilon_{\theta z},\ \sigma_{zr}=2c_{44}\varepsilon_{zr} \quad \text{(11d, e)}$$

$$\sigma_{r\theta}=(c_{11}-c_{12})\varepsilon_{r\theta} \quad \text{(11f)}$$

in which σ_{ij} and ε_{ij} denote stress and strain components, respectively. If we substitute the potential functions of Eqs.(6a-e), (8a-c) and (10a, b) into Eqs.(1a-c), we obtain expressions for the displacement components. To make the explanation brief, we affix superscripts (1), (2), (0, 1) and (0, 0) successively to the displacement and stress components induced from Eqs. (6a, b, e), (6c, d, e), (8a-c) and (10a, b). Then, the displacement and stress components are expressed in the sum of four quantities as

$$u_r=u_r^{(1)}+u_r^{(2)}+u_r^{(0,1)}+u_r^{(0,0)};$$

$$\cdots\cdots;$$

$$\sigma_{r\theta}=\sigma_{r\theta}^{(1)}+\sigma_{r\theta}^{(2)}+\sigma_{r\theta}^{(0,1)}+\sigma_{r\theta}^{(0,0)} \quad \text{(12)}$$

If we find the strain components from the displacement components obtained from Eqs.(1a-c) and use Eqs.(11a-f), we obtain expressions for the stress components. For example, the expressions for $\sigma_{\theta z}$ and σ_{zr} are

$$\sigma_{\theta z}^{(1)}=-\frac{c_{44}}{2\sqrt{\nu_2}}\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\alpha_{ms}^2\sin m\theta\Big\langle(1+k_2)\cdot[J_{m-1}(\alpha_{ms}r)+J_{m+1}(\alpha_{ms}r)]\cdot\Big\{A_{ms}\sinh\frac{\alpha_{ms}z}{\sqrt{\nu_2}}+C_{ms}\sqrt{\frac{\nu_2}{\nu_1}}\Big[\gamma_1\frac{\alpha_{ms}z}{\sqrt{\nu_1}}\cosh\frac{\alpha_{ms}z}{\sqrt{\nu_1}}+\Big(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\Big)\sinh\frac{\alpha_{ms}z}{\sqrt{\nu_1}}\Big]\Big\}+\sqrt{\frac{\nu_2}{\nu_3}}B_{ms}[J_{m-1}(\alpha_{ms}r)-J_{m+1}(\alpha_{ms}r)]\sinh\frac{\alpha_{ms}z}{\sqrt{\nu_3}}\Big\rangle \quad \text{(13a)}$$

$$\sigma_{zr}^{(1)}=\frac{c_{44}}{2\sqrt{\nu_2}}\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\alpha_{ms}^2\cos m\theta\Big\langle(1+k_2)\cdot[J_{m-1}(\alpha_{ms}r)-J_{m+1}(\alpha_{ms}r)]\cdot\Big\{A_{ms}\sinh\frac{\alpha_{ms}z}{\sqrt{\nu_2}}+C_{ms}\sqrt{\frac{\nu_2}{\nu_1}}\Big[\gamma_1\frac{\alpha_{ms}z}{\sqrt{\nu_1}}\cdot\cosh\frac{\alpha_{ms}z}{\sqrt{\nu_1}}+\Big(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\Big)\sinh\frac{\alpha_{ms}z}{\sqrt{\nu_1}}\Big]\Big\}+\sqrt{\frac{\nu_2}{\nu_3}}B_{ms}[J_{m-1}(\alpha_{ms}r)+J_{m+1}(\alpha_{ms}r)]\sinh\frac{\alpha_{ms}z}{\sqrt{\nu_3}}\Big\rangle \quad \text{(14a)}$$

$$\sigma_{\theta z}^{(2)}=\frac{c_{44}\sqrt{\nu_1}}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\beta_n^2\sin m\theta\sin\beta_n z\cdot\langle D_{mn}(1+k_1)[I_{m-1}(\sqrt{\nu_1}\beta_n r)-I_{m+1}(\sqrt{\nu_1}\beta_n r)]+F_{mn}\sqrt{\nu_2/\nu_1}\cdot\{[\gamma_1 m(1+k_1)-(\gamma_2+k_1\gamma_3)]\cdot I_{m-1}(\sqrt{\nu_2}\beta_n r)+[\gamma_1 m(1+k_1)+\gamma_2+k_1\gamma_3]I_{m+1}(\sqrt{\nu_2}\beta_n r)\}+E_{mn}\sqrt{\nu_3/\nu_1}[I_{m-1}(\sqrt{\nu_3}\beta_n r)+I_{m+1}(\sqrt{\nu_3}\beta_n r)]\rangle \quad \text{(13b)}$$

$$\sigma_{zr}^{(2)}=-\frac{c_{44}\sqrt{\nu_1}}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\beta_n^2\cos m\theta\sin\beta_n z\cdot\langle D_{mn}(1+k_1)[I_{m-1}(\sqrt{\nu_1}\beta_n r)+I_{m+1}(\sqrt{\nu_1}\beta_n r)]+F_{mn}\sqrt{\nu_2/\nu_1}\cdot\{[\gamma_1 m(1+k_1)-(\gamma_2+k_1\gamma_3)]I_{m-1}(\sqrt{\nu_2}\beta_n r)-[\gamma_1 m(1+k_1)+\gamma_2+k_1\gamma_3]I_{m+1}(\sqrt{\nu_2}\beta_n r)+2\gamma_1(1+k_1)\sqrt{\nu_2}\beta_n r I_m(\sqrt{\nu_2}\beta_n r)\}+E_{mn}\sqrt{\nu_3/\nu_1}[I_{m-1}(\sqrt{\nu_3}\beta_n r)-I_{m+1}(\sqrt{\nu_3}\beta_n r)]\rangle \quad \text{(14b)}$$

$$\sigma_{\theta z}^{(0,1)}=\sigma_{\theta z}^{(0,0)}=0 \quad \text{(13c)}$$

$$\sigma_{zr}^{(0,1)}=\sigma_{zr}^{(0,0)}=0 \quad \text{(14c)}$$

3. BOUNDARY CONDITIONS

We consider a short cylinder whose both end faces are subjected to uniformly distributed sectorial loads and whose side surface is free from surface tractions.

The boundary conditions for that case become at $r=a$,

$$\sigma_{rr}=0,\ \sigma_{r\theta}=0,\ \sigma_{rz}=0 \qquad (15a\text{-}c)$$

at $z=\pm h$,

$$\sigma_{zr}=0,\ \sigma_{z\theta}=0,\ \sigma_{zz}=-p(r,\theta) \qquad (16a\text{-}c)$$

in which

$$p(r,\theta)=\begin{cases} q \text{ for } 0<r<d,\ 0\leq\theta<\omega \text{ and} \\ \qquad 2\pi-\omega<\theta\leq 2\pi \\ 0 \text{ for the other domain}\end{cases} \qquad (17)$$

From Eqs. (13b) and (14b), we obtain

$$(\sigma_{\theta z}^{(2)})_{z=\pm h}=(\sigma_{zr}^{(2)})_{z=\pm h}=0 \qquad (18)$$

Regarding Eq.(18) and imposing boundary conditions (16a, b) on Eqs.(14a) and (13a), we obtain the following relationships:

$$B_{ms}=0,\ A_{ms}=a_{ms}\frac{\sinh\zeta_1 h}{\sinh\zeta_2 h}C_{ms} \qquad (19a, b)$$

in which

$$\zeta_1=\frac{\alpha_{ms}}{\sqrt{\nu_1}},\ \zeta_2=\frac{\alpha_{ms}}{\sqrt{\nu_2}} \qquad (20a, b)$$

$$a_{ms}=-\sqrt{\frac{\nu_2}{\nu_1}}\left[\gamma_1(1+\zeta_1 h\coth\zeta_1 h)-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right] \qquad (20c)$$

Substituting relationships (19a, b) into Eq.(14a), the expression for $\sigma_{zr}^{(1)}$ is rewritten in the form

$$\sigma_{zr}^{(1)}=c_{44}\frac{1+k_2}{\sqrt{\nu_2}}\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\alpha_{ms}^2 C_{ms}\cos m\theta J_m'(\alpha_{ms}r)$$
$$\cdot\left\{a_{ms}\sinh\zeta_1 h\frac{\sinh\zeta_2 z}{\sinh\zeta_2 h}+\sqrt{\frac{\nu_2}{\nu_1}}\left[\gamma_1\zeta_1 z\cosh\zeta_1 z\right.\right.$$
$$\left.\left.+\left(\gamma_1-\frac{k_2\gamma_2+\gamma_3}{1+k_2}\right)\sinh\zeta_1 z\right]\right\} \qquad (21)$$

in which

$$J_m'(\alpha_{ms}r)=\frac{mJ_m(\alpha_{ms}r)}{\alpha_{ms}r}-J_{m+1}(\alpha_{ms}r) \qquad (22)$$

Setting $r=a$ in Eq.(21) and taking λ_{ms} as the positive root of the following transcendental equation:

$$J_m'(\alpha_{ms}a)=J_m'(\lambda_{ms})=0 \qquad (23)$$

we obtain

$$(\sigma_{rz}^{(1)})_{r=a}=0 \qquad (24)$$

Regarding Eq.(24) and imposing boundary condition (15c) on Eq.(14b), we obtain the following relationship:

$$D_{mn}=b_{mn}^{(1)}\frac{I_m(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}F_{mn}+b_{mn}^{(2)}\frac{I_m(\sqrt{\nu_3}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}E_{mn} \qquad (25)$$

in which

$$b_{mn}^{(1)}=-\frac{1}{\Delta_{mn}}\left\{\gamma_1[m^2+(\sqrt{\nu_2}\beta_n a)^2]-\frac{\gamma_2+k_1\gamma_3}{1+k_1}\right.$$
$$\left.\cdot\left[m+\sqrt{\nu_2}\beta_n a\frac{I_{m+1}(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_2}\beta_n a)}\right]\right\} \qquad (26a)$$

$$b_{mn}^{(2)}=-\frac{m}{\Delta_{mn}(1+k_1)} \qquad (26b)$$

$$\Delta_{mn}=m+\sqrt{\nu_1}\beta_n a\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)} \qquad (26c)$$

Thus, boundary conditions (15c) and (16a, b) were rigorously satisfied. By making use of relationships (19a, b) and (25), we can make the expressions for the displacement and stress components that eliminate arbitrary constants B_{ms}, A_{ms} and D_{mn} from the original expressions. For example, we have

$$\sigma_{rr}^{(1)}=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\alpha_{ms}^2 C_{ms}\cos m\theta\left\langle(c_{11}-c_{12})\frac{1}{\alpha_{ms}r}\right.$$
$$\cdot\left[J_{m+1}(\alpha_{ms}r)+\frac{m(m-1)}{\alpha_{ms}r}J_m(\alpha_{ms}r)\right]$$
$$\cdot\left[a_{ms}\sinh\zeta_1 h\frac{\cosh\zeta_2 z}{\sinh\zeta_2 h}+\gamma_1\zeta_1 z\sinh\zeta_1 z\right.$$
$$\left.-\gamma_3\cosh\zeta_1 z\right]-J_m(\alpha_{ms}r)$$
$$\cdot\left\{\left(c_{11}-c_{13}\frac{k_2}{\nu_2}\right)a_{ms}\sinh\zeta_1 h\frac{\cosh\zeta_2 z}{\sinh\zeta_2 h}\right.$$
$$+\gamma_1\left(c_{11}-c_{13}\frac{k_2}{\nu_1}\right)\zeta_1 z\sinh\zeta_1 z$$
$$\left.\left.-\left[c_{11}\gamma_3+c_{13}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\right]\cosh\zeta_1 z\right\}\right\rangle \qquad (27a)$$

$$\sigma_{zz}^{(1)}=-\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\alpha_{ms}^2 C_{ms}\cos m\theta J_m(\alpha_{ms}r)$$
$$\cdot\left\{\left(c_{13}-c_{33}\frac{k_2}{\nu_2}\right)a_{ms}\sinh\zeta_1 h\frac{\cosh\zeta_2 z}{\sinh\zeta_2 h}\right.$$
$$+\gamma_1\left(c_{13}-c_{33}\frac{k_2}{\nu_1}\right)\zeta_1 z\sinh\zeta_1 z$$
$$\left.-\left[c_{13}\gamma_3+c_{33}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\right]\cosh\zeta_1 z\right\} \qquad (28a)$$

$$\sigma_{r\theta}^{(1)}=-(c_{11}-c_{12})\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\alpha_{ms}^2 C_{ms}\sin m\theta\frac{m}{\alpha_{ms}r}$$
$$\cdot\left[\frac{m-1}{\alpha_{ms}r}J_m(\alpha_{ms}r)-J_{m+1}(\alpha_{ms}r)\right]$$
$$\cdot\left[a_{ms}\sinh\zeta_1 h\frac{\cosh\zeta_2 z}{\sinh\zeta_2 h}+\gamma_1\zeta_1 z\sinh\zeta_1 z\right.$$
$$\left.-\gamma_3\cosh\zeta_1 z\right] \qquad (29a)$$

$$\sigma_{rr}^{(2)}=\nu_1\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\beta_n^2\cos m\theta\cos\beta_n z$$
$$\cdot\left\langle F_{mn}\left(b_{mn}^{(1)}\frac{I_m(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\left\{I_m(\sqrt{\nu_1}\beta_n r)\right.\right.\right.$$
$$\cdot\left[(c_{11}-c_{12})\frac{m(m-1)}{(\sqrt{\nu_1}\beta_n r)^2}+c_{11}-c_{13}\frac{k_1}{\nu_1}\right]$$

$$-(c_{11}-c_{12})\frac{I_{m+1}(\sqrt{\nu_1}\beta_n r)}{\sqrt{\nu_1}\beta_n r}\Big\}$$

$$+\frac{\nu_2}{\nu_1}\Big\{I_m(\sqrt{\nu_2}\beta_n r)\Big[(c_{11}-c_{12})$$

$$\cdot(\gamma_1 m-\gamma_2)\frac{m(m-1)}{(\sqrt{\nu_2}\beta_n r)^2}+\gamma_1\Big(c_{11}(m+1)$$

$$+c_{12}-c_{13}\frac{k_1}{\nu_2}m\Big)-c_{11}\gamma_2+c_{13}\frac{k_1}{\nu_2}\gamma_3\Big]$$

$$+I_{m+1}(\sqrt{\nu_2}\beta_n r)\Big[(c_{11}-c_{12})\frac{\gamma_1 m^2+\gamma_2}{\sqrt{\nu_2}\beta_n r}$$

$$+\gamma_1\Big(c_{11}-c_{13}\frac{k_1}{\nu_2}\Big)\sqrt{\nu_2}\beta_n r\Big]\Big\}\Big)$$

$$+E_{mn}\Big(b_{mn}^{(2)}\frac{I_m(\sqrt{\nu_3}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}$$

$$\cdot\Big\{I_m(\sqrt{\nu_1}\beta_n r)\Big[(c_{11}-c_{12})\frac{m(m-1)}{(\sqrt{\nu_1}\beta_n r)^2}$$

$$+c_{11}-c_{13}\frac{k_1}{\nu_1}\Big]-(c_{11}-c_{12})\frac{I_{m+1}(\sqrt{\nu_1}\beta_n r)}{\sqrt{\nu_1}\beta_n r}\Big\}$$

$$+\frac{\nu_3}{\nu_1}(c_{11}-c_{12})\frac{m}{\sqrt{\nu_3}\beta_n r}\Big[\frac{m-1}{\sqrt{\nu_3}\beta_n r}I_m(\sqrt{\nu_3}\beta_n r)$$

$$+I_{m+1}(\sqrt{\nu_3}\beta_n r)\Big]\Big)\Big\rangle \quad \cdots\cdots (27b)$$

$$\sigma_{zz}^{(2)}=\nu_1\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\beta_n^2\cos m\theta\cos\beta_n z$$

$$\cdot\Big\langle F_{mn}\Big(b_{mn}^{(1)}\frac{I_m(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}$$

$$\cdot\Big(c_{13}-c_{33}\frac{k_1}{\nu_1}\Big)I_m(\sqrt{\nu_1}\beta_n r)+\frac{\nu_2}{\nu_1}\Big\{\Big(\gamma_1\Big[c_{13}(m+2)$$

$$-c_{33}\frac{k_1}{\nu_2}m\Big]-c_{13}\gamma_2+c_{33}\frac{k_1}{\nu_2}\gamma_3\Big)I_m(\sqrt{\nu_2}\beta_n r)$$

$$+\gamma_1\Big(c_{13}-c_{33}\frac{k_1}{\nu_2}\Big)\sqrt{\nu_2}\beta_n r I_{m+1}(\sqrt{\nu_2}\beta_n r)\Big\}\Big)$$

$$+E_{mn}b_{mn}^{(2)}\Big(c_{13}-c_{33}\frac{k_1}{\nu_1}\Big)\frac{I_m(\sqrt{\nu_3}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}$$

$$\cdot I_m(\sqrt{\nu_1}\beta_n r)\Big\rangle \quad \cdots\cdots (28b)$$

$$\sigma_{r\theta}^{(2)}=-(c_{11}-c_{12})\nu_1\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\beta_n^2\sin m\theta\cos\beta_n z\Big\langle mF_{mn}$$

$$\cdot\Big(b_{mn}^{(1)}\frac{I_m(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\frac{1}{\sqrt{\nu_1}\beta_n r}$$

$$\cdot\Big[\frac{m-1}{\sqrt{\nu_1}\beta_n r}I_m(\sqrt{\nu_1}\beta_n r)+I_{m+1}(\sqrt{\nu_1}\beta_n r)\Big]$$

$$+\frac{\nu_2}{\nu_1}\Big\{I_m(\sqrt{\nu_2}\beta_n r)\Big[\gamma_1+(\gamma_1 m-\gamma_2)$$

$$\cdot\frac{m-1}{(\sqrt{\nu_2}\beta_n r)^2}\Big]-(\gamma_1+\gamma_2)\frac{I_{m+1}(\sqrt{\nu_2}\beta_n r)}{\sqrt{\nu_2}\beta_n r}\Big\}\Big)$$

$$+E_{mn}\Big(b_{mn}^{(2)}\frac{I_m(\sqrt{\nu_3}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\frac{m}{\sqrt{\nu_1}\beta_n r}\Big[\frac{m-1}{\sqrt{\nu_1}\beta_n r}$$

$$\cdot I_m(\sqrt{\nu_1}\beta_n r)+I_{m+1}(\sqrt{\nu_1}\beta_n r)\Big]$$

$$+\frac{\nu_3}{\nu_1}\Big\{I_m(\sqrt{\nu_3}\beta_n r)\Big[\frac{1}{2}+\frac{m(m-1)}{(\sqrt{\nu_3}\beta_n r)^2}\Big]$$

$$-\frac{I_{m+1}(\sqrt{\nu_3}\beta_n r)}{\sqrt{\nu_3}\beta_n r}\Big\}\Big)\Big\rangle \quad \cdots\cdots (29b)$$

and the expressions for the stress components induced from additional solutions (8a-c) and (10a, b) are

$$\sigma_{rr}^{(0,1)}=(c_{11}-c_{12})\sum_{m=2}^{\infty}D_{m0}m(m-1)\cos m\theta r^{m-2}$$

$$+\sum_{m=1}^{\infty}F_{m0}(m+1)[m(c_{11}-c_{12})$$

$$+(c_{11}+c_{12})(2-\gamma_4)]\cos m\theta r^m \quad \cdots\cdots (27c)$$

$$\sigma_{zz}^{(0,1)}=2c_{13}(2-\gamma_4)\sum_{m=1}^{\infty}F_{m0}(m+1)\cos m\theta r^m$$

$$\cdots\cdots (28c)$$

$$\sigma_{r\theta}^{(0,1)}=-(c_{11}-c_{12})\Big[\sum_{m=1}^{\infty}D_{m0}m(m-1)\sin m\theta r^{m-2}$$

$$+\sum_{m=1}^{\infty}F_{m0}m(m+1)\sin m\theta r^m\Big] \quad \cdots\cdots (29c)$$

$$\sigma_{rr}^{(0,0)}=F_{00}\Big[(c_{11}+c_{12})(2\gamma_1-\gamma_2)+\frac{2k_1}{\nu_2}c_{13}\gamma_3\Big]-C_{00}$$

$$\cdot\Big[\gamma_3(c_{11}+c_{12})+\frac{2k_2}{\nu_1}c_{13}(2\gamma_1-\gamma_2)\Big] \quad \cdots (27d)$$

$$\sigma_{zz}^{(0,0)}=2F_{00}\Big[c_{13}(2\gamma_1-\gamma_2)+c_{33}\frac{k_1\gamma_3}{\nu_2}\Big]-2C_{00}$$

$$\cdot\Big[c_{13}\gamma_3+c_{33}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\Big] \quad \cdots\cdots (28d)$$

$$\sigma_{r\theta}^{(0,0)}=0 \quad \cdots\cdots (29d)$$

4. SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

In this article, we consider satisfying boundary conditions (15a, b) and (16c). The Fourier-Bessel expansion of load function (17), under the condition of Eq.(23), is as follows:

$$p(r,\theta)=e_{00}+\sum_{s=1}^{\infty}e_{0s}J_0(\alpha_{0s}r)+\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}e_{ms}\cos m\theta$$

$$\cdot J_m(\alpha_{ms}r) \quad \cdots\cdots (30)$$

in which

$$e_{00}=\frac{q\omega}{\pi}\Big(\frac{d}{a}\Big)^2 \quad \cdots\cdots (31a)$$

$$e_{0s}=\frac{2q\omega}{\pi}\Big(\frac{d}{a}\Big)\frac{J_1\Big(\lambda_{0s}\frac{d}{a}\Big)}{\lambda_{0s}J_0^2(\lambda_{0s})} \quad \cdots\cdots (31b)$$

$$e_{ms}=\frac{4q\sin m\omega}{m\pi(\lambda_{ms}^2-m^2)J_m^2(\lambda_{ms})}\left[\lambda_{ms}\left(\frac{d}{a}\right)J_{m+1}\left(\lambda_{ms}\frac{d}{a}\right)+2m\sum_{k=0}^{\infty}J_{m+2+2k}\left(\lambda_{ms}\frac{d}{a}\right)\right] \quad \text{(31c)}$$

To satisfy boundary conditions (15a, b), we have to expand $\sigma_{rr}^{(1)}$ and $\sigma_{r\theta}^{(1)}$ at $r=a$ of Eqs.(27a) and (29a) into Fourier series. Then, we obtain

$$(\sigma_{rr}^{(1)})_{r=a}=\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{C_{ms}}{a^2}\sinh\zeta_1 hJ_m(\lambda_{ms})\cos m\theta\cdot\left({}_st_{m0}^{(1)}+\sum_{n=1}^{\infty}{}_st_{mn}^{(1)}\cos\beta_n z\right) \quad \text{(32)}$$

$$(\sigma_{r\theta}^{(1)})_{r=a}=(c_{11}-c_{12})\sum_{m=0}^{\infty}\sum_{s=1}^{\infty}\frac{mC_{ms}}{a^2}\sinh\zeta_1 hJ_m(\lambda_{ms})\cdot\sin m\theta\left({}_st_{m0}^{(2)}+\sum_{n=1}^{\infty}{}_st_{mn}^{(2)}\cos\beta_n z\right) \quad \text{(33)}$$

in which ${}_st_{m0}^{(1)}, \cdots, {}_st_{mn}^{(2)}$ are Fourier coefficients and ${}_st_{00}^{(1)}=0$.

Using Eq.(32) and imposing boundary condition (15a) on Eqs.(27b-d), we obtain three systems of linear algebraic equations in the form

$$\sum_{s=1}^{\infty}\frac{C_{ms}}{a^2}\sinh\zeta_1 h_st_{mn}^{(1)}J_m(\lambda_{ms})+\beta_n^2F_{mn}I_m(\sqrt{\nu_2}\beta_n a)\cdot\Big\langle b_{mn}^{(1)}\nu_1\Big[(c_{11}-c_{12})\frac{m(m-1)}{(\sqrt{\nu_1}\beta_n a)^2}+c_{11}-c_{13}\frac{k_1}{\nu_1}-\frac{c_{11}-c_{12}}{\sqrt{\nu_1}\beta_n a}\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\Big]$$
$$+\nu_2\Big\{(c_{11}-c_{12})(\gamma_1 m-\gamma_2)\frac{m(m-1)}{(\sqrt{\nu_2}\beta_n a)^2}+\gamma_1\Big[c_{11}(m+1)+c_{12}-c_{13}\frac{k_1}{\nu_2}m\Big]-c_{11}\gamma_2+c_{13}\frac{k_1\gamma_3}{\nu_2}+\frac{I_{m+1}(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_2}\beta_n a)}$$
$$\cdot\Big[(c_{11}-c_{12})\frac{\gamma_1 m^2+\gamma_2}{\sqrt{\nu_2}\beta_n a}+\gamma_1\Big(c_{11}-c_{13}\frac{k_1}{\nu_2}\Big)\cdot\sqrt{\nu_2}\beta_n a\Big]\Big\}\Big\rangle+\beta_n^2E_{mn}I_m(\sqrt{\nu_3}\beta_n a)$$
$$\cdot\Big\{b_{mn}^{(2)}\nu_1\Big[(c_{11}-c_{12})\frac{m(m-1)}{(\sqrt{\nu_1}\beta_n a)^2}+c_{11}-c_{13}\frac{k_1}{\nu_1}-\frac{c_{11}-c_{12}}{\sqrt{\nu_1}\beta_n a}\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\Big]+\nu_3(c_{11}-c_{12})\frac{m}{\sqrt{\nu_3}\beta_n a}\Big[\frac{m-1}{\sqrt{\nu_3}\beta_n a}+\frac{I_{m+1}(\sqrt{\nu_3}\beta_n a)}{I_m(\sqrt{\nu_3}\beta_n a)}\Big]\Big\}=0,\ (m\geq 0) \quad \text{(34a)}$$

$$\sum_{s=1}^{\infty}\frac{C_{ms}}{a^2}\sinh\zeta_1 h_st_{m0}^{(1)}J_m(\lambda_{ms})+\delta_m' D_{m0}a^{m-2}\cdot(c_{11}-c_{12})m(m-1)+F_{m0}a^m(m+1)[m(c_{11}-c_{12})+(c_{11}+c_{12})(2-\gamma_4)]=0,\ (m\geq 1) \quad \text{(34b)}$$

in which

$$\delta_m'=\begin{cases}0 \text{ for } m=1\\ 1 \text{ for } m\geq 2\end{cases} \quad \text{(35)}$$

$$F_{00}\Big[(c_{11}+c_{12})(2\gamma_1-\gamma_2)+\frac{2k_1}{\nu_2}c_{13}\gamma_3\Big]-C_{00}\Big[\gamma_3(c_{11}+c_{12})+\frac{2k_2}{\nu_1}c_{13}(2\gamma_1-\gamma_2)\Big]=0 \quad \text{(34c)}$$

Using Eq.(33) and imposing boundary condition (15b) on Eqs.(29b, c), we obtain two systems of linear algebraic equations in the form

$$\sum_{s=1}^{\infty}\frac{mC_{ms}}{a^2}\sinh\zeta_1 h_st_{mn}^{(2)}J_m(\lambda_{ms})-\beta_n^2mF_{mn}I_m(\sqrt{\nu_2}\beta_n a)\cdot\Big\{\frac{b_{mn}^{(1)}\nu_1}{\sqrt{\nu_1}\beta_n a}\Big[\frac{m-1}{\sqrt{\nu_1}\beta_n a}+\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\Big]$$
$$+\nu_2\Big[\gamma_1+(\gamma_1 m-\gamma_2)\frac{m-1}{(\sqrt{\nu_2}\beta_n a)^2}-\frac{\gamma_1+\gamma_2}{\sqrt{\nu_2}\beta_n a}\frac{I_{m+1}(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_2}\beta_n a)}\Big]\Big\}-\beta_n^2E_{mn}I_m(\sqrt{\nu_3}\beta_n a)$$
$$\cdot\Big\{b_{mn}^{(2)}\frac{\nu_1 m}{\sqrt{\nu_1}\beta_n a}\Big[\frac{m-1}{\sqrt{\nu_1}\beta_n a}+\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\Big]+\nu_3\Big[\frac{1}{2}+\frac{m(m-1)}{(\sqrt{\nu_3}\beta_n a)^2}-\frac{1}{\sqrt{\nu_3}\beta_n a}\frac{I_{m+1}(\sqrt{\nu_3}\beta_n a)}{I_m(\sqrt{\nu_3}\beta_n a)}\Big]\Big\}=0,$$
$$(m\geq 0) \quad \text{(36a)}$$

$$\sum_{s=1}^{\infty}\frac{mC_{ms}}{a^2}\sinh\zeta_1 h_st_{m0}^{(2)}J_m(\lambda_{ms})-\delta_m' D_{m0}a^{m-2}m(m-1)-F_{m0}a^m m(m+1)=0,\ (m\geq 1) \quad \text{(36b)}$$

To satisfy boundary condition (16c), we have to expand $\sigma_{zz}^{(2)}$ at $z=\pm h$ of Eq.(28b) and $\sigma_{zz}^{(0,1)}$ of Eq.(28c) into Bessel series, under Eq.(23). Then, we obtain

$$(\sigma_{zz}^{(2)})_{z=\pm h}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^n\beta_n^2\cos m\theta\{F_{mn}I_m(\sqrt{\nu_2}\beta_n a)\cdot\Big[{}_nt_{00}^{(3)}+\sum_{s=1}^{\infty}{}_nt_{ms}^{(3)}J_m(\alpha_{ms}r)\Big]+E_{mn}\cdot I_m(\sqrt{\nu_3}\beta_n a)\Big[{}_nt_{00}^{(4)}+\sum_{s=1}^{\infty}{}_nt_{ms}^{(4)}J_m(\alpha_{ms}r)\Big]\Big\} \quad \text{(37a)}$$

$$\sigma_{zz}^{(0,1)}=2c_{13}(2-\gamma_4)\sum_{m=1}^{\infty}F_{m0}a^m(m+1)\cos m\theta\cdot\sum_{s=1}^{\infty}t_{ms}^{(5)}J_m(\alpha_{ms}r) \quad \text{(37b)}$$

in which ${}_nt_{00}^{(3)}={}_nt_{00}^{(4)}=0, \cdots, t_{ms}^{(5)}$ are Fourier coefficients.

Using Eqs.(30) and (37a, b) and imposing boundary condition (16c) on Eqs.(28a, d), we obtain two systems of linear algebraic equations in the form

$$\frac{C_{ms}}{a^2}\lambda_{ms}^2\sinh\zeta_1 h\Big\{\Big(c_{13}-c_{33}\frac{k_2}{\nu_2}\Big)a_{ms}\coth\zeta_2 h+\gamma_1 \cdot\Big(c_{13}-c_{33}\frac{k_2}{\nu_1}\Big)\zeta_1 h-\Big[c_{13}\gamma_3+c_{33}\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\Big] \cdot\coth\zeta_1 h\Big\}-\sum_{n=1}^{\infty}(-1)^n\beta_n^2[F_{mn}I_m(\sqrt{\nu_2}\beta_n a)_n t_{ms}^{(3)} +E_{mn}I_m(\sqrt{\nu_3}\beta_n a)_n t_{ms}^{(4)}]-2c_{13}(2-\gamma_4)\delta_m^* F_{m0}a^m \cdot(m+1)t_{ms}^{(5)}=e_{ms},\ (m\geq 0) \quad \text{(38a)}$$

$$2F_{00}\Big[c_{13}(2\gamma_1-\gamma_2)+c_{33}\frac{k_1\gamma_3}{\nu_2}\Big]-2C_{00}\Big[c_{13}\gamma_3+c_{33} \cdot\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)\Big]=-e_{00} \quad \text{(38b)}$$

in which

$$\delta_m^*=\begin{cases}0 \text{ for } m=0\\ 1 \text{ for } m\geq 1\end{cases} \quad \text{(39a)}$$

$$ {}_n t_{ms}^{(3)}=\frac{2\lambda_{ms}^2}{J_m(\lambda_{ms})(\lambda_{ms}^2-m^2)}\Big\langle b_{mn}^{(1)}\nu_1\Big(c_{13}-c_{33}\frac{k_1}{\nu_1}\Big) \cdot\frac{1}{\lambda_{ms}^2+(\sqrt{\nu_1}\beta_n a)^2}\Big[m+\sqrt{\nu_1}\beta_n a\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\Big] +\frac{\nu_2}{\lambda_{ms}^2+(\sqrt{\nu_2}\beta_n a)^2}\Big(\Big\{\gamma_1\Big[c_{13}(m+2)-c_{33}\frac{k_1}{\nu_2}m\Big] -c_{13}\gamma_2+c_{33}\frac{k_1}{\nu_2}\gamma_3\Big\}\Big[m+\sqrt{\nu_2}\beta_n a\frac{I_{m+1}(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_2}\beta_n a)}\Big] +\gamma_1\Big(c_{13}-c_{33}\frac{k_1}{\nu_2}\Big)\sqrt{\nu_2}\beta_n a\Big\{\sqrt{\nu_2}\beta_n a-(m+2) \cdot\frac{I_{m+1}(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_2}\beta_n a)}-\frac{2}{\lambda_{ms}^2+(\sqrt{\nu_2}\beta_n a)^2}\Big[m\sqrt{\nu_2}\beta_n a -\lambda_{ms}^2\frac{I_{m+1}(\sqrt{\nu_2}\beta_n a)}{I_m(\sqrt{\nu_2}\beta_n a)}\Big]\Big\}\Big)\Big\rangle \quad \text{(39b)}$$

$$ {}_n t_{ms}^{(4)}=\frac{2\lambda_{ms}^2}{J_m(\lambda_{ms})(\lambda_{ms}^2-m^2)}b_{mn}^{(2)}\nu_1\Big(c_{13}-c_{33}\frac{k_1}{\nu_1}\Big) \cdot\frac{1}{\lambda_{ms}^2+(\sqrt{\nu_1}\beta_n a)^2}\Big[m+\sqrt{\nu_1}\beta_n a\frac{I_{m+1}(\sqrt{\nu_1}\beta_n a)}{I_m(\sqrt{\nu_1}\beta_n a)}\Big] \quad \text{(39c)}$$

$$t_{ms}^{(5)}=\frac{2m}{J_m(\lambda_{ms})(\lambda_{ms}^2-m^2)} \quad \text{(39d)}$$

Solving the system of Eqs.(34b) and (36b) with D_{m0} and F_{m0}, we obtain

$$D_{m0}a^{m-2}=\frac{1}{\Delta_1(m-1)}\sum_{s=1}^{\infty}\frac{C_{ms}}{a^2}\sinh\zeta_1 h J_m(\lambda_{ms})\{{}_s t_{m0}^{(1)} +[m(c_{11}-c_{12})+(c_{11}+c_{12})(2-\gamma_4)]_s t_{m0}^{(2)}\}, (m\geq 2) \quad \text{(40a)}$$

$$F_{m0}a^m=-\frac{1}{\Delta_1(m+1)}\sum_{s=1}^{\infty}\frac{C_{ms}}{a^2}\sinh\zeta_1 h J_m(\lambda_{ms})[{}_s t_{m0}^{(1)} +m(c_{11}-c_{12})_s t_{m0}^{(2)}],\ (m\geq 1) \quad \text{(40b)}$$

in which

$$\Delta_1=(c_{11}+c_{12})(2-\gamma_4) \quad \text{(41)}$$

Solving the system of Eqs.(34c) and (38b) with C_{00} and F_{00}, we obtain

$$C_{00}=\frac{e_{00}}{2\Delta_2}\Big[(c_{11}+c_{12})(2\gamma_1-\gamma_2)+\frac{2k_1}{\nu_2}c_{13}\gamma_3\Big] \quad \text{(42a)}$$

$$F_{00}=\frac{e_{00}}{2\Delta_2}\Big[\gamma_3(c_{11}+c_{12})+\frac{2k_2}{\nu_1}c_{13}(2\gamma_1-\gamma_2)\Big] \quad \text{(42b)}$$

in which

$$\Delta_2=[2c_{13}^2-c_{33}(c_{11}+c_{12})] \cdot\Big[\frac{k_1\gamma_3^2}{\nu_2}-\frac{k_2}{\nu_1}(2\gamma_1-\gamma_2)^2\Big] \quad \text{(43)}$$

The system of Eqs.(34a), (36a), (38a) and (40b) with C_{ms}, F_{mn}, E_{mn} and F_{m0} can be numerically solved by an iterative method. Once all the arbitrary constants are determined, the values of the displacement and stress components of the short cylinder are completely determined. In order to facilitate numerical calculations of the system of linear algebraic equations, displacements and stresses, it is convenient to replace the arbitrary constants previously used with the following ones :

$$\frac{c_{44}C_{ms}}{qa^2}\sinh\zeta_1 h=\bar{C}_{ms};\ \frac{c_{44}F_{mn}}{qa^2}I_m(\sqrt{\nu_2}\beta_n a)=\bar{F}_{mn}; \frac{c_{44}E_{mn}}{qa^2}I_m(\sqrt{\nu_3}\beta_n a)=\bar{E}_{mn};\ \frac{c_{44}D_{m0}a^{m-2}}{q}=\bar{D}_{m0}; \frac{c_{44}F_{m0}a^m}{q}=\bar{F}_{m0};\ \frac{c_{44}F_{00}}{q}=\bar{F}_{00};\ \frac{c_{44}C_{00}}{q}=\bar{C}_{00} \quad \text{(44)}$$

5. NUMERICAL RESULTS

Numerical calculations were made for transversely isotropic and isotropic, short cylinders with $h/a=1.0$, $d/a=0.3$ and $\omega=0.3\pi$. Magnesium and cadmium crystals, as examples of transversely isotropic materials, and an isotropic material with Poisson's ratio $\nu=0.25$ were treated. The values of the elastic constants of these materials, as determined by Huntington[8], are given in **Table 1**. The roots of the transcendental equation were calculated by the Regula-Falsi method. Numerical results were obtained by taking the first 40 terms for m and 38 terms for s and n in the series. The check for the convergence of σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} at $\theta=0$ is given in **Table 2**. **Table 2** indicates that the convergence of the values in the interior points is very rapid and that the convergence of $\sigma_{\theta\theta}$ and σ_{zz} at $z=a$ is however slightly slow. For the magnesium crystal and the isotropic material, the distributions of σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{zr} in the planes of $\theta=0$ and $\theta=\pi$ are shown in **Figs.2**, **3**, **6** and **7**.

Table 1 Values of elastic constants c_{ij}.

(in units of 10GPa).

Material	c_{44}	c_{11}	c_{33}	c_{12}	c_{13}
Magnesium crystal	1.64	5.97	6.17	2.62	2.17
Cadmium crystal	1.56	11.0	4.69	4.04	3.83
Isotropy (ν=0.25)	1.0	3.0	3.0	1.0	1.0

Table 2 Check for convergence of σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} at $\theta=0$. (Magnesium, $h/a=1.0$, $d/a=0.3$, $\omega=0.3\pi$).

Number of terms			$-\sigma_{rr}/q$		$-\sigma_{\theta\theta}/q$		$-\sigma_{zz}/q$	
m	s	n	$r=0.4a$ $z=0.8a$	$r=a$ $z=0.8a$	$r=0.2a$ $z=0.8a$	$r=0.2a$ $z=a$	$r=0.2a$ $z=0.8a$	$r=0.2a$ $z=a$
20	20	20	0.0673	0.0001	0.0180	0.7084	0.5368	0.9074
30	30	30	0.0673	0.0002	0.0180	0.7257	0.5368	1.0421
40	38	38	0.0673	-0.0005	0.0180	0.7219	0.5368	1.0036
50	50	50	0.0673	0.0001	0.0180	0.7247	0.5368	1.0374

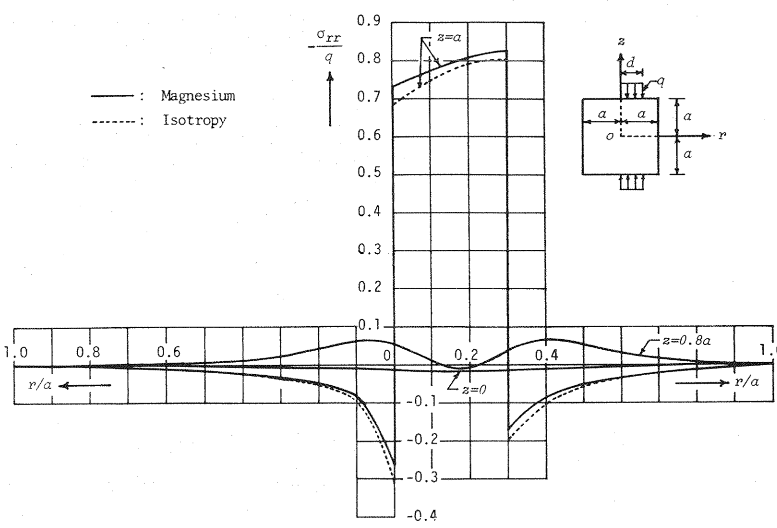

Fig.2 Distribution of σ_{rr} in planes of $\theta=0$ and $\theta=\pi$. (Magnesium and Isotropy, $h/a=1.0$, $d/a=0.3$, $\omega=0.3\pi$).

Fig.2 shows that the decay of σ_{rr} along the r-direction is rapid and that the value at the end face ($z=a$) becomes discontinuous at the boundary ($r=0$ and $r=0.3a$) of the sectorial load. The value at the end face was calculated by the method proposed by Saito[9]. Furthermore, the values in the points away from the end face are small due to the small loading area. **Fig.3** shows that the value of $\sigma_{\theta\theta}$ at $z=a$ becomes discontinuous at $r=0$ and $r=0.3a$ like σ_{rr} and that the values in the points away from $z=a$ are small. **Figs.4** and **5** show the distributions of σ_{rr} at $r=0.4a$ and $z=0.8a$ and $\sigma_{\theta\theta}$ at $r=0.2a$ and $z=0.8a$ along the θ-direction for the magnesium crystal. It is shown that the decay of σ_{rr} and $\sigma_{\theta\theta}$ along that direction is rapid. **Fig.6** shows that the value of σ_{zz} at $z=0.8a$ becomes small positive one at the side surface ($r=a$) and that the values in the points away from $z=a$ are comparatively large. **Fig.7** shows that the values of σ_{zr} are less than the normal stresses and that the values in the points near $z=a$ become larger at $r=0.3a$. In **Fig.7**, the values in the plane of $\theta=\pi$ are drawn in the inverse sign. **Tables 3** and **4** indicate comparisons of the displacement and stress values among the magnesium crystal, the cadmium crystal and the isotropic material, respectively. **Table 3** shows

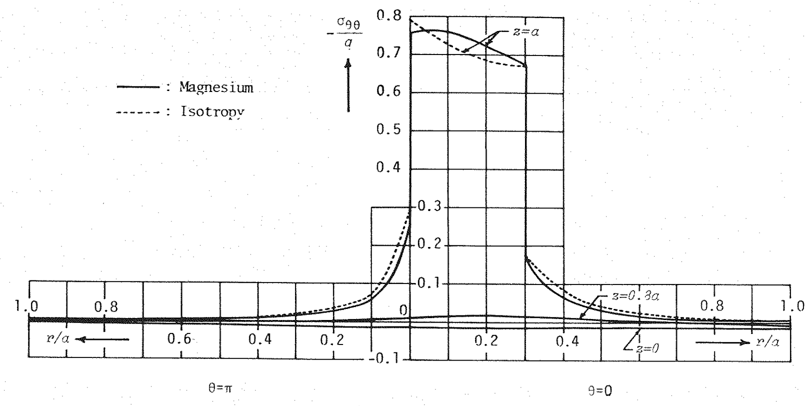

Fig.3 Distribution of $\sigma_{\theta\theta}$ in planes of $\theta=0$ and $\theta=\pi$. (Magnesium and Isotropy, $h/a=1.0$, $d/a=0.3$, $\omega=0.3\pi$).

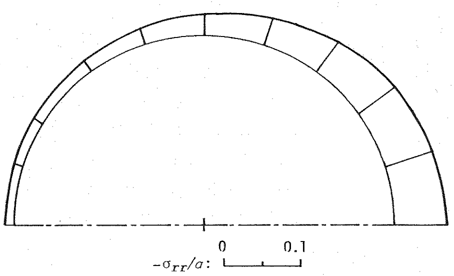

Fig.4 Distribution of σ_{rr} along θ-direction. (Magnesium, $r=0.4a$, $z=0.8a$).

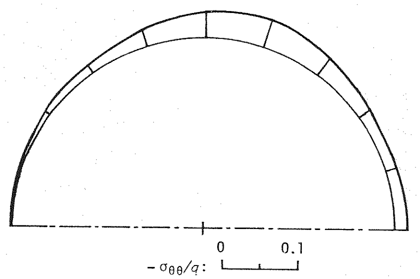

Fig.5 Distribution of $\sigma_{\theta\theta}$ along θ-direction. (Magnesium, $r=0.2a$, $z=0.8a$).

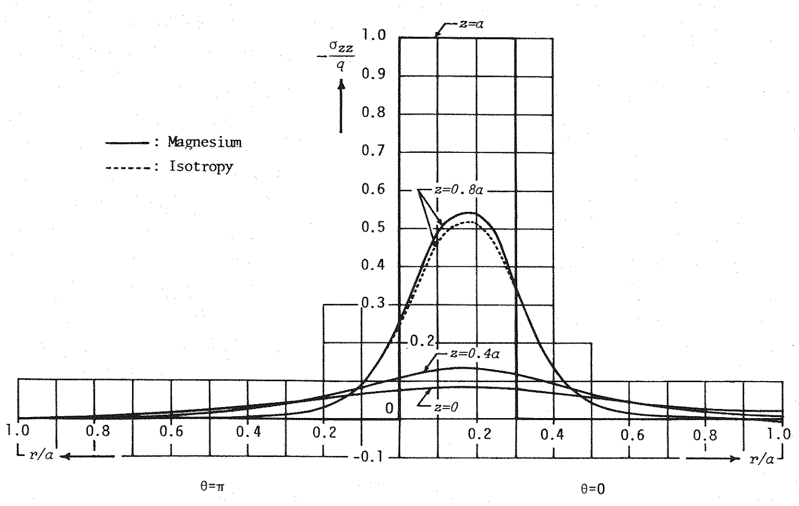

Fig.6 Distribution of σ_{zz} in planes of $\theta=0$ and $\theta=\pi$. (Magnesium and Isotropy, $h/a=1.0$, $d/a=0.3$, $\omega=0.3\pi$).

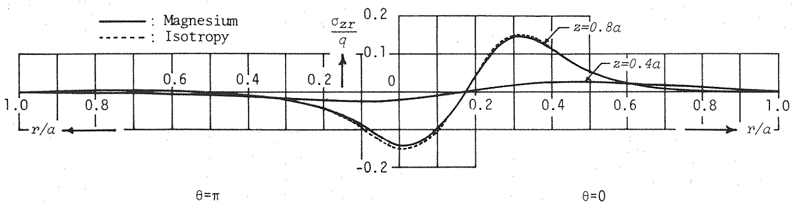

Fig.7 Distribution of σ_{zr} in planes of $\theta=0$ and $\theta=\pi$. (Magnesium and Isotropy, $h/a=1.0$, $d/a=0.3$, $\omega=0.3\pi$).

Table 3 Comparisons of displacement values ($r=0.2a$, $\theta=0.2\pi$, $z=0.8a$).

Material	$u_r/\frac{qa}{c_{11}}$	$u_\theta/\frac{qa}{c_{11}}$	$u_z/\frac{qa}{c_{11}}$
Magnesium crystal	0.00657	0.0125	-0.157
Cadmium crystal	0.0191	0.0259	-0.358
Isotropy	0.00634	0.0115	-0.146

Table 4 Comparisons of stress values ($r=0.2a$, $\theta=0$, $z=0.9a$).

Material	σ_{rr}/q	$\sigma_{\theta\theta}/q$	σ_{zz}/q	σ_{zr}/q
Magnesium crystal	-0.114	-0.163	-0.833	0.0628
Cadmium crystal	-0.203	-0.327	-0.749	0.0816
Isotropy	-0.121	-0.176	-0.826	0.0674

that the values of u_r, u_θ and u_z in the cadmium crystal are more than those in the isotropic material by 201.3%, 125.2% and 145.2%, respectively, but the values in the magnesium crystal differ slightly from those in the isotropic material. **Table 4** shows that the values of $\sigma_{\theta\theta}$ in the magnesium and cadmium crystals are less and more than that in the isotropic material by 7.4% and 85.8%, respectively.

6. CONCLUSIONS

Axially asymmetric stresses in a transversely isotropic, short cylinder subjected to sectorial pressures on the end faces were analyzed by the generalized Elliott solution. The method of analysis for the three-dimensional, asymmetric stress problem of the short cylinder was stated briefly. From the results of the numerical calculations for the short cylinder, the following conclusions may be drawn :

(1) The values of σ_{rr} and $\sigma_{\theta\theta}$ at the end face are discontinuous at the boundary of the sectorial load.

(2) The decay of σ_{rr} and $\sigma_{\theta\theta}$ along the radial and circumferential directions is rapid.

(3) The values of σ_{rr} and $\sigma_{\theta\theta}$ in the points away from the end face are small due to the small loading area.

(4) The values of σ_{zz} in the points near the end face are positively small at the side surface.

(5) The value of u_r at $r=0.2a$, $\theta=0.2\pi$ and $z=0.8a$ in the cadmium crystal is 201.3% more than that in the isotropic material with Poisson's ratio 0.25.

(6) The value of $\sigma_{\theta\theta}$ at $r=0.2a$, $\theta=0$ and $z=0.9a$ in the cadmium crystal is 85.8% more than that in the isotropic material.

(7) The differences in the displacement and stress values between the magnesium crystal and the isotropic material are narrow.

(8) The differences in the displacement and stress values between the transversely isotropic and isotropic materials grow wider as those of the values of the elastic constants become wider.

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