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A FREE SURFACE ANALYSIS FOR TRANSIENT CREEPING FLOW PROBLEMS
—MARKER METHOD FOR A SECOND ORDER FINITE ELEMENT—

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A finite element method for analyzing transient incompressible creeping flows is presented. Marker particles are used to represent free surfaces and to visualize flow patterns. A six-node triangular isoparametric (second-order) element is used for the analysis. Marker positions are determined by the use of triangular area co-ordinate system. A simple solution algorithm is obtained and a higher order accuracy, which is difficult with the finite difference method, can be expected with the present method. To verify the scheme, a simple punch indentation problem is solved.

Key Words : *finite element method, transient flow, creeping flow, freesurface flow, marker particles*

1. INTRODUCTION

In this paper we describe a calculating scheme to figure out free surface configurations and to visualize flow patterns of transient incompressible creeping flows. Marker particles are used to represent the flows. A six-node triangular isoparametric element of the finite clement method is used for the calculation to make simple solution algorithm and to obtain better approximation.

It is one of the most important areas in technological and engineering fields to calculate transient flows, such as those found in flows of soils, cements, metals, and polymers etc. There are so called particle-in-cell (PIC) method[1] and marker-and-cell (MAC) method[2] for analyzing the flows. These methods make use of marker particles to show free surfaces. The flow patterns can be visualized if the markers are also arranged inside the transient flow. These methods, however, are difficult to apply under complicated fixed boundary conditions. Such conditions can easily be covered if the finite element method is used. Using the finite element method, markers can be also used for the analysis. Shiojima et al[3),4)]. have introduced the area co-ordinate system of the linear triangular element, which is utilized for finding the new marker position, into a four-node rectangular isoparametric element. The two kinds of interpolation systems, however, give complicated solution algorithm and may give unexpected errors on calculated results. Also, this method requires finer mesh arrangement to obtain an accurate result. This implies that a large amount of extra storage may be required for the analysis.

In this study, We present a finite element method using six-node triangular isoparametric element for the analysis, in which marker particles are introduced to represent the transient creeping fluid flow motion. For determining the marker positions in an element, area co-ordinate system of the triangular element is used. With the six-node element, the determination of the new marker position in an element becomes very simple compared with those using the linear triangular element. It is because that we can make use of three mid-node points in each side of the triangle to the corresponding three area co-ordinates. Also the velocities at the six nodes obtained by the calculation can be directly used for the interpolation to the marker velocity. The program becomes simple and higher order accuracy can be expected, and coarse mesh arrangement can be available with the present method.

To verify the scheme, we present a simple punch indentation problem for both plane and axisymmetric regions.

2. GOVERNING EQUATIONS

The time dependent creeping flows of incompressible Newtonian fluids without body forces in rectangular Cartesian co-ordinates are

equilibrium $\rho u_{i,t}+\sigma_{ij,j}=0,$ ··············· (1)

continuity (incompressible fluids)

$$u_{i,i}=\varepsilon_{ii}=0, \quad (2)$$

constitutive relationships

$$\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2, \quad (3)$$

$$\sigma_{ij}=\sigma_{ij}'-\delta_{ij}p, \quad (4)$$

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$$\sigma_{ij}'=2\mu\varepsilon_{ij}, \quad\cdots\cdots(5)$$

boundary conditions

$$u_i=\bar{u}_i, \quad\cdots\cdots(6)$$

$$\nu_j\sigma_{ij}=\bar{T}_i. \quad\cdots\cdots(7)$$

where ρ is the density, μ is the viscosity coefficient, u_i is the velocity component in the x_i-direction, p is the pressure, σ_{ij} is the total stress, σ_{ij}' is the deviatoric stress. Equations (6) and (7) represent the boundary conditions. ν_j is the component of unit outward normal vector on the boundary.

3. FINITE ELEMENT METHOD

(1) Velocity calculation

We interpolate the velocity and the pressure as

$$u_i\cong N_k\hat{u}_{ki}, \quad p\cong M_m\hat{p}_m. \quad\cdots\cdots(8a, 8b)$$

where $\hat{u}_{ki}$, $\hat{p}_m$ are the velocity and the pressure at nodal points, respectively. N_k and M_m are the shape functions. Galerkin's method applied to Equation (1), with the boundary conditions of Equation (7), gives

$$\rho\int_v Nu_{i,t}dV+\int_v N_{,j}\sigma_{ij}dV=\int_s NT_i ds \quad\cdots\cdots(9)$$

We now approximate Equation (2) using the small number κ (10^{-6}). The penalty function method[5] gives

$$\frac{1}{\kappa}\int_V Mu_{i,i}dv+\int_V Mpdv=0 \quad\cdots\cdots(10)$$

Substitution of p in Equation (10) into Equation (9) gives, in matrix form

$$[C]\{\hat{u}_{,t}\}+[K]\{\hat{u}\}=\{F\}, \quad\cdots\cdots(11)$$

where $[C]$ and $[K]$ are from the first and second term, and $\{F\}$ is from the right hand side of Equation (9), respectively. The θ-method for the time derivative in Equation (11) gives

$$\left[\frac{1}{\Delta t}[C]+\theta[K]\right]\{u\}_{t+\Delta t}$$
$$=\{F\}_{avg}+\left[\frac{1}{\Delta t}[C]-(1-\theta)[K]\right]\{u\}_t, \quad\cdots\cdots(12)$$

$$\{F\}_{avg}=\{(1-\theta)\{F\}_t+\theta\{F\}_{t+\Delta t}\}. \quad\cdots\cdots(13)$$

(2) Marker method

We introduce marker particles to figure out free surface configurations and to visualize flow patterns. **Fig.1** shows a position of the marker in a six-node triangular isoparametric element.

The marker position at the time $t+\Delta t$ becomes

$$(x_i)_{t+\Delta t}=(x_i)_t+\int_t^{t+\Delta t}(u_i)_t dt \quad\cdots\cdots(14)$$

$$(u_i)_t=[N_k]\{\hat{u}_{ki}\}_t \quad\cdots\cdots(15)$$

where, $(x_i)_t$ is the marker position and $(u_i)_t$ is the velocity at time t. With the shape functions N_k expressed by the area co-ordinate (ξ, η, ζ) and

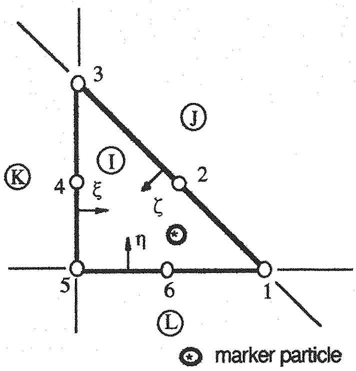

Fig.1 Six-node triangular isoparametric element

Table 1 Area co-ordinates and elements

	I	J	K	L
ξ(p)	≧0	≧0	<0	≧0
η(r)	≧0	≧0	≧0	<0
ζ(t)	≧0	<0	≧0	≧0

the nodal point position $\hat{x}_{ki}$ by the global co-ordinate, the marker position $(x_i)_{t+\Delta t}$ becomes

$$(x_i)_{t+\Delta t}=[N_k]\{\hat{x}_{ki}\} \quad\cdots\cdots(16)$$

Because $(x_i)_{t+\Delta t}$ is known from Equation (14) and $\hat{x}_{ik}$ is co-ordinate values of the six nodal points in an element, we can calculate the marker poisition in the triangular co-ordinate ξ, η, ζ by Newton-Raphson method. We now can find out the new marker position according to the signs of ξ, η, ζ. Referring to **Fig.1**, the marker stays within the element I, or moves into the element J, K or L.

(3) Array NPIX for Element *I, J, K, L*

The array NPIX is an extra storage area for finding the element in which the marker comes. We express the Nth marker in element I as

$$\mathrm{MKER}(N)=I. \quad\cdots\cdots(17)$$

Only two elements share the mid-node point in each side of triangle, for example, only the element I and J share the point 2 in **Fig.1**. We then add up these two elements and store it in the array NPIX.

$$\left.\begin{array}{l}\mathrm{NPIX}(2)=I+J,\\ \mathrm{NPIX}(4)=I+K,\\ \mathrm{NPIX}(6)=I+L.\end{array}\right\} \quad\cdots\cdots(18)$$

We now can figure out the new marker position in the following manner. Referring to **Table 1**,

if $\xi<0$

$\mathrm{MKER}(N)=\mathrm{NPIX}(4)-\mathrm{MKER}(N)=K$,

if $\eta<0$

$\mathrm{MKER}(N)=\mathrm{NPIX}(6)-\mathrm{MKER}(N)=L$,

if $\zeta<0$

$\mathrm{MKER}(N)=\mathrm{NPIX}(2)-\mathrm{MKER}(N)=J$.

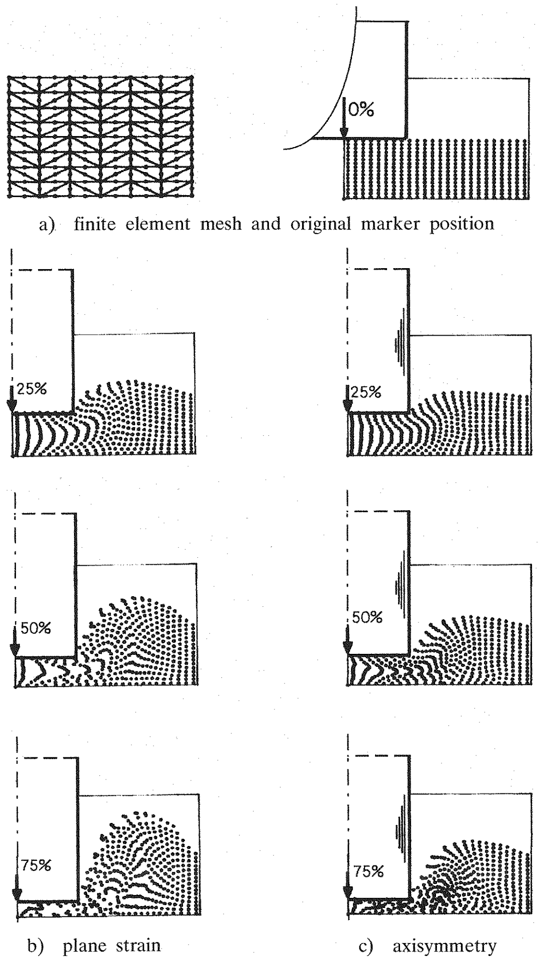

Fig.2 Punch indentation problem

To calculate the marker position at time $t+\Delta t$, we divide our time step Δt into n equal intervals. The new marker position at time $\Delta t/n$ is determined according to the relationship in **Table 1**. This process is then continued until the step $t+\Delta t$ is completed. The n-interval helps not only to find the new element but also to move the marker along a streamline with better accuracy. The calculation of NPIX array can be done easily by checking the connectivity of element with nodal points. The marker velocity can be interpolated directly from the velocities of six nodal points. These imply that the use of six-node element gives simpler solution algorithm than that of four-node element.

4. EXAMPLE PROBLEMS

We now solve the punch indentation problem shown in **Fig.2** a). **Fig.2** b) shows the plane strain problem and **Fig.2** c) shows the axisymmetric problem. We divide the region of analysis into 96 elements (221 points). The total number of markers is 384. The material properties used in the analysis are $\rho=1.0$, $\mu=1.0$. For the analysis we choose $\theta=0.75$ and $\Delta t=0.1$. The Δt is divided into from 20 to 50 intervals for better accuracy. To proceed with the transient analysis, we assume the free surface is physically a line consisting of the forefront markers and also the place to specify the boundary conditions is located on the boundary line between the element having a forefront markers and the element having not. It should be noted that the traction force along the side is zero. Then we do not specify any boundary values and the assumption does not affect the visualization of the free surface as a marker front.

From the figures, we can investigate the patterns of fluid flow. The result agrees well with those in reference (3).

5. CONCLUSIONS

The analysis of transient incompressible creeping flows with free surfaces by the finite element method is described. Markers are introduced to represent the fluid flow motions. The six-node triangular (second order) element is used to obtain better accuracy and to simplify the solution algorithm.

With the second order element, it is shown that the storage scheme of elements for marker movement becomes simple and a rough mesh arrangement to the region of analysis can give a reasonable accuracy. It should be noted that the present method can be applied to the problems of complicated fix-boundary conditions, which are difficult to solve by the finite difference method. The application of the present method to other important engineering problems remains for further theme.

REFERENCES

1) Harlow, F.H. : The particle-in-cell computing method for fluid dynamics, Methods Comput. Phys., Vol.3, pp.319～343, 1964.
2) Nichols, B.D. and Hirt, C.W. : Improved free surfact boundary conditions for numerical incompressible-flow calculations, J. Comp. Phys., Vol.8, pp.434～448, 1971.
3) Shiojima, T., Shimazaki, Y. and Daiguji, H. : Finite element analysis of creeping flows using marker particles, Int. J. Num. Meth. Fluids, Vol.11, pp.397～404, 1990.
4) Shiojima, T., Shimazaki, Y. and Daiguji, H. : Analyses of transient cviscous fluid creeping flows with free surfaces, Proc. JSME, No.86-1356 A, pp.2365～2369, 1987.
5) Kikuchi, F. and Navarro, M.P. : An iteration method for the mixed formulation of parameter dependent problems related to the Stokes equations, Comp. Mech. Vol.1, pp.141～151, 1986.

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