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REVIEW

# A REVIEW ON SHAPE OPTIMAL DESIGN AND SENSITIVITY ANALYSIS

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**ABSTRACT**

Presented is a review of shape optimal design and shape design sensitivity analysis with an emphasis on techniques dealing with shape of the boundaries of two- and three-dimensional bodies. Attention is focused on the continuum structural shape optimization based on numerical models by either finite elements or boundary elements. This requires sophisticated design sensitivity analysis techniques and a careful choice of design variables.

## 1. INTRODUCTION

Engineering design is an iterative process, in which the design is continuously modified until it meets the criteria set by engineers. A traditional design process is carried out by the so called *trial and error* method, in which the designer uses his experience and intuition to lead the design process. This manual design process has the advantage that the designer's knowledge can be utilized in the design. But as the design problem becomes more complex, design modification becomes more difficult, requiring a new tool.

For a particular design problem, there may exist a number of solutions that satisfy given conditions. The optimum design is a rational approach finding a solution which is *optimal* in the sense defined.

One category of problems attaining much attention recently is the optimal shape design. It is one order of magnitude more complex than the more classical size or parameter optimization. Due to variety of difficulties, it is not yet in the stage of practical applications, although theoretical base is well set up now. The importance of shape optimization is evident, since the first thing in a design problem is to determine the shape of the object to be designed when we look at any design problem. For determining optimal shape of elastic bodies in the general case, the main mathematical difficulty lies in that the domain of the governing equations is not specified beforehand, but is to be determined from conditions that the objective functional attains an extremal value possibly under many other constraints. These are so-called *problems with unknown boundaries*. As well as having a direct practical importance, such problems are of great interest from a mathematical point of view: to develop effective tools to such problems is a real challenge.

An accurate shape design sensitivity analysis (SDSA) is considered basic prerequisite to efficient handling of the shape optimization process. This has been a major topic of intensive research. The present paper reviews recent works in shape optimization and shape design sensitivity analysis. It is focused on the shape design sensitivity analysis methods and is limited mostly to design variables that control the boundary of two- and three-dimensional objects. It does not include works on topology optimization and sensitivity analysis of skeletal structures, because these problems belong to another category. However, the interested reader is referred to a survey by Topping[1] and Levy[2] on this topic.

Section 2 deals with procedures for properly defining shape design variables so that the finite element or boundary element mesh provides accurate state variables and accurate sensitivity results. Section 3 deals with shape optimal design problems treated up to now. Section 4 reviews the developments of shape design sensitivity analyses based on variational formulation and boundary integral formulation. Section 5 lists the shape design literature by areas of applications. The last section concludes the surveys.

## 2. SELECTION OF BOUNDARY REPRESENTATION METHOD AND DESIGN VARIABLES

The representation of the shape to be designed using a set of parameters or design variables is a key step in the process of formulating most shape optimal design problems, and remains one of the major difficulties. The optimum shape is highly dependent on the design parameterization selected. An inappropriate parameterization can lead to unacceptable shapes[3),4)]. And, changing the geometric shape of the design model to reflect successive changes in design parameters is a tedious, complicated, and inefficient process.

In general, structural shape design problems can be classified into three types, in terms of the characteristics of the design boundary. In the first type, the shape of an arbitrary open or closed boundary, such as a fillet[5)~16)] or a dam surface[17)~21)], is to be determined. In the second type of problem, dimensions of pre-defined shapes, e. g., the radius of a circular hole, the major and minor axes of an elliptic hole, dimensions of a slot, length of a rectangular membrane, or radius of a rounded corner, are to be found[22)~28)]. In the third type, locations of the design boundary, e. g., the locations of the center of a circular hole, an elliptic hole, hole of an arbitrary given shape, or slot that has either arbitrary or pre-defined shape relative to the global reference frame, are to be determined[29),30)]. Shape design of the open boundary of a structure has been studied for some time[5),6),20),31)~41)]. However, design problems with a closed boundary, pre-defined shape, and locations of the design boundaries have not yet been extensively treated.

During the past years, a few methods have been used to parameterize structural boundaries for optimal shape design : boundary shape described by coordinates of boundary nodes[20),31)~33)], coefficients of polynomials[6),34)~36)], control points of splines or spline blending functions[5),14),37)~41),53)~56)], and parameters of generic primitive models[23),24),28),29),42),43)]. A selection of a particular parameterization means a restriction on the set of feasible designs that may be different from original intention.

**(1) Coordinates of element boundary nodes**

The use of coordinates for boundary nodes in the finite element or boundary element model as shape variables is the earliest method[13),20),31)~33),44)~51)]. It is simple and easy. However, severe drawbacks have been reported[3)] : (1) the number of design parameters tends to become very large, which may lead to high computational cost and difficulty; (2) smoothness of the design boundary is not retained across boundary nodes, which may lead to an unacceptable or impractical design; and (3) analysis error due to a selected discretization can be amplified and thus the optimized shape based on this wrong information is meaningless when looked at from the design problem initially set. Such problems can be seen in the typical fillet or hole problems[3),4),40),227)].

**(2) Polynomials representation of boundaries**

Polynomial representation is a natural choice for describing boundaries. Some early references are[34)~36)]. The total number of shape design parameters can be reduced by using polynomials for shape representation. However, as reported by Ding[3)], using high order polynomials to represent the boundary shape may result in oscillatory boundaries.

A more general approach is to define the boundary as a linear combination of certain shape functions with the coefficients being the design variables. Thus, Kristensen and Madsen[6)] defined the boundary using linear combination of orthogonal functions added to the initial design by treating the coefficients of the functions as design parameters. Dems[52)] also used a set of prescribed shape functions and applied it to the simple case of piecewise linear boundaries.

**(3) Spline representation of boundaries**

The use of high-order polynomials to describe the boundary can result in an oscillatory boundary shape as mentioned already. Splines eliminate this problem, since they are composed of low-order polynomial segments that are combined to maximize smoothness of the design boundaries. Furthermore, the spline representation has been shown to yield better sensitivity accuracy than a piecewise linear representation of the boundary[37)].

The cubic spline function, which has two continuous derivatives everywhere and possesses minimum mean curvature, is a natural choice for defining the boundary[38),39)]. Braibant et al.[40),41),53),54)] used Bezier and B-spline blending functions to describe design element boundaries. The blending functions provide great flexibility for the geometric description. With the spline formulation[14),50),55),56)], boundary regularity requirements are automatically taken into account and also an analytical formulation of the sensitivity derivatives can be established.

**(4) Generic primitive models**

Many CAGD (Computer Aided Geometric Design) programs or pre-processors in FEM packages have modules which can generate geometric models defined by parameters of generic primitive model. Some examples of primitives are unit sphere, unit cube, unit cylinder and so on. Sometimes the primitives are defined by B-Rep (Boundary Representation), such as NURB (Non-uniform Rational B-spline).

These primitives are used to define the shape of the solid model or boundary shape. In this case, the design variables are parameters of the primitives which define the model. Some efforts are seen[24),28),42)] to connect the shape optimization technique with those models defined by generic primitives.

**(5) Design element concept**

One way to achieve an adequate finite element model is to use the design element concept that was first introduced by Imam[33)], and is used by several researchers[20),22),40),41),53),54),58)]. In this approach, the structure is divided into a few regions. Each of these regions corresponds to a design element that is described by a set of master nodes that control the geometry of the design element[15)]. Associated with the design element is a set of suitably chosen design variables that specify the locations of the master nodes only. The design element boundary is described by using two- or three-dimensional isoparametric finite element interpolation functions[33),58)], or spline blending functions[40),41),53)].

## 3. SHAPE OPTIMIZATION

Finding the optimum shape of a structure under some constraints has been one of the most attractive concerns of many great mathematicians for a long time from Galileo Galilei, who presented the problem of finding the shape of a cantilever beam in order to obtain a uniform stress distribution in the 17th century, to many researchers in the 20th century. Euler and Lagrange had set the necessary conditions for optimization problems. Since then there are many analytical tools that contribute to the study of optimum design of structural components. The work of Michell (1904)[59)] was a prevailing basic theory of these analytical optimization works. The conference book edited by Haug and Cea (1981)[60)] provides an extensive review of analytical works in this field. Unfortunately, these analytical approaches have limitations in solving practical problems. However they are very valuable and important because they can provide insight into the design process as well as lay down a foundation for numerical methods.

Numerical optimization methods have seen extensive development over the past thirty years. It was Schmit[61)] who proposed a general approach to structural optimization in 1960, which indicated the feasibility of coupling finite element analysis and non-linear mathematical programming for the optimum structural design. Following Schmit's work, major advances in structural optimization have been made by Kicher, Gallagher, Gellatly *et al.*[62),63)]. However the major interest of their effort was the member sizing problem where design variables are the cross-sectional area or thickness.

**(1) Shape Optimization using FEM**

One of the first treatments of shape optimization was presented by Zienkiewicz and Campbell[31)]. They formulated the shape optimization problem using an FE model and treated the boundary nodes of the FE model as design variables. They obtained derivatives by directly differentiating the discretized equations and employed a sequential linear programming method for numerical solution. Francavilla *et al*[12)], Schnack[64)], and Oda[65)] employed the finite element method for a fillet optimization to minimize stress concentration. Similar but more extensive methods to minimize stress concentrations are presented by Tvergaard[66)], Kristensen and Madsen[6)] with a sequential linear programming method.

Bhavikatti and Ramakrishnan[34)] presented a refinement of the formulation of[31)] and Ramakrishnan and Francavilla[67)] employed a similar finite element formulation, but they used a penalty function method for numerical optimization. A function space gradient projection method for two-dimensional elastic bodies was presented by Chun and Haug[68)], using the design sensitivity analysis methods similar to those presented by Rousselet and Haug[69)].

Optimality criteria have been developed for selected classes of shape optimal design problems. Banichuk[70)] formulated a general problem of selecting the optimum shape of the cross section of a nonhomogeneous shaft, to maximize torsional stiffness, with a given amount of material available. He used variations of functional with respect to both the warping function and boundary variation, using material derivatives, and obtained a necessary condition for optimum location of the boundary. Dems and Mroz[17),71)~76)] presented a quite general approach of shape optimal design using both the optimality criterion method and the variational calculus, and the optimality conditions were generated for both the conservative and the nonconservative load system.

Three dimensional shape optimization problems were proposed by Kodiyalam and Vanderplaats[77)] using the forced approximation technique, in which the approximate stresses were obtained through linearization of nodal forces instead of direct linearization of the stresses. Applications of three-dimensional shape optimization are found in[5),23),28),33),78),79)].

Bendsoe and Kikuchi[225)] and coworkers[81)~83)] proposed a topological shape optimization method, called homogenization method that utilizes infinitely many microscale holes in a design domain. For a skeleton topology optimization readers are referred to the two papers by Topping[1)] and Levy[2)].

**( 2 ) Shape optimization using BEM**

Because of convenience in remeshing compared to FEM as well as relatively good accuracy of the solutions at the boundary, the BEM has become an attractive analysis method in shape optimization.

The use of the BEM in shape optimization started from the 1980's. One of the first formulation of shape optimal design was presented by Futagami[84),85)]. He coupled the BEM with the linear programming to optimize systems governed by partial differential equations Barone and Caulk[30)] optimized position, size and surface temperature of circular holes inside a two-dimensional heat conductor to produce the minimum variation in surface temperature. They employed a special boundary integral method and a conjugate gradient method. Mota Soares *et al.*[44),45)] presented a model for optimization of the geometry of solid and hollow shafts in terms of boundary elements and the nonlinear programming. They extended the formulation to two-dimensional elasticity problems[13),46)]. The determination of the optimum shape under displacements and geometrical constraints was presented by Zochowski and Mizukami[86)] with the objective of minimum area.

Meric[87)~89)] applied the BEM to solve the necessary conditions for optimality of performance index derived by the calculus of variations using a Lagrange multiplier technique. He extended his method for non-linear anisotropic heat conduction problems and applied his method to obtain an optimal outer boundary profile of an orthotropic solid body[90)]. Optimal cross-sectional shapes for minimum viscous drag for fully developed magnetohydrodynamic channel flow are investigated by using a similar method[51)]. Kobelev[91)] also used the BEM for the best shape of an elastic bar in torsion. On the other hand, Burczynski and Adamczk[92),93)] started with integral optimality conditions and used boundary element method. The resulting nonlinear system was solved by the Newton-Raphson method.

Sandgren and Wu[55)] obtained the optimal shape of ladle hook with substructuring method. They have shown that the subregion approach can reduce the computing time significantly. Carter *et al*[47)]. described an iterative numerical optimization procedure for generating the cryosurgical probe tip geometry to produce the desired lethal temperature envelope for a steady state axisymmetric system. They used the Kirchhoff transformation to include the nonlinear effect of variable thermal conductivity at cryogenic temperatures. Gracia and Doblare[49)] obtained the solution of the shape optimization problem for simply and multiply connected orthotropic sections under Saint Venant torsion. Espiga *et al.*[94)] used the BEM for two-dimensional elastic orthotropic solids.

Saigal and Chandra[48)] adopted the implicit differentiation of discretized boundary integral equations for the shape optimization of heat conduction problem. A boundary element formulation for acoustic shape sensitivity analysis is formulated by Kane *et al.*[160)]. Shape optimization of structures to minimize stress concentration is formulated as a sequential linear programming problem with an adaptive move limit by Xu and Yu[95)]. A method for automated grid refinement and grid adaptation of boundary elements is introduced by Hajela and Jih[15)] to interface with the optimum shape design problem. They used a predefined control function in a variational formulation and master node concept to obtain an optimal node distribution.

Kwak and Choi[96)] developed a general procedure and formulas for the SDSA based on the BIE formulation for a potential problem and applied it to a seepage problem. They extended the formulation to plane elasticity problems and studied a fillet and an elastic ring design problem[50)]. Lee and Kwak[97),98)] extended the adjoint method of Choi and Kwak to two dimensional thermoelasticity problems and considered a shape optimal design to minimize the weight of a turbine disc under stress constraints. Lee and Kwak[56)] also extended the approach to transient diffusion problem and applied to a shape optimization problem of a glass forming plunger to minimize the variation of temperature along the cavity surface.

An optimal design technique for magnetostatic fields is described by Ishiyama *et al.*[99),100)]. They have shown two application examples ; a 1-Tesla superconducting magnet system with a magnetic shielding for magnetic resonance imaging (MRI) and a magnetic levitation system.

Chaudouet *et al.*[101)] applied the BEM to three dimensional optimum design with a growing-reforming technique. A modular approach for shape optimization used in the finite element[28)] is employed for the boundary elements by Yang[16)].

Stochastic shape optimal design problems are investigated by Nakagiri[102)], Tada *et al.*[103)] and Burczynski[104)]. The boundary element SDSA and the optimal design of vibrating structures for the criterion of maximizing the lowest natural frequency are considered by Fedelinski and Burczynski[105)].

The coupling of FEM and BEM is employed by

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Kamiya and Kita[106] that takes the advantages of easier remeshing of BEM and the sparse matrix of FEM. They also proposed a numerical approach to search for the design synthesis of the optimal shape of a spring wire under a minimum weight restriction and stress criterion[57].

## 4. SHAPE DESIGN SENSITIVITY ANALYSIS (SDSA)

There are many special methods for solving the shape design problems, for example, optimality criterion methods, intuitive method (pattern transformation method)[65], experimental techniques employing photoelastic models and so on. Optimality criterion methods for shape optimization consist of the following two steps : 1) the derivation of a set of necessary conditions that must be satisfied at the optimum design ; and 2) the development of an iterative redesign procedure that drives the initial trial design toward a design which satisfies the previously established set of necessary conditions (see, for example, 57), 92), 93)). Pattern transformation method is a technique of scaling up or down the shape of the boundary based on their stress ratio or strain energy ratio. The detail and other methods are explained well in[3),4)].

Most of the work, however, is based on employing mathematical programming methods coupled with finite element method or boundary element method. Most methods in nonlinear programming require gradient information at each iteration.

Design sensitivity analysis, that is, the calculation of quantitative information on how the response of a structure is affected with respect to changes in the variables that define its shape, plays a key role in shape optimization. The first partial derivative of structural response quantities with respect to shape design variables provides the essential information required to couple mathematical optimization and structural analysis procedures. This problem of shape design sensitivity analysis has been addressed over about the past 20 years. There are two popular baseds : variational formulation and boundary integral equation (BIE) formulation.

### (1) SDSA based on Variational Formulation

The dominantly used analysis method based on variational formulation is the Finite Element Method (FEM). There are two main approaches to calculate the shape design sensitivities in this context ; the discretized approach and the continuum approach.

**a)** Discretized Approach

The discretized method uses a discretized model to carry out the sensitivity analysis, which includes three methods : Finite Differentiation Method (FDM), analytical method and semi-analytical method.

The simplest method for obtaining the partial derivatives is actually calculating the increments using FDM[22),58)]. The FDM is to disturb the design variables one by one, and using finite difference formula to approximate the derivatives. FDM has the advantage of being simple in concept, and easy in implementation, but it has two disadvantages. First, changes in shape can lead to a distortion of the finite elements[22),58)], and so the accuracy often depends upon the size of perturbation step. Second, the computation cost is comparatively high especially when the number of design variables is larger than that of constraints.

An analytical method is to differentiate the system equation directly with respect to the design variables[20),107)~109)]. Analytical shape sensitivity may be obtained through the implicit differentiation approach[110)], which is quite straightforward in terms of mathematical derivation and programming. From the initial efforts of Zienkiewicz and Campbell[31)] and Ramakrishnan and Francavilla[12)] to the more recent work of Braibant and Fleury[40)], and Wang *et al.*[20)], the theory of the implicit differentiation approach has been established. But, unfortunately the stiffness matrix is usually nonlinear with the shape design variables, therefore it is difficult to obtain the derivative of the stiffness matrix analytically.

The semi-analytical method differentiates the system equation first as in the analytical method, then employs the finite difference method to calculate the derivative of the stiffness matrix[111),112)]. This is one of the attractive method in practical problems because of its generality and easy implementation. However many researchers indicated that semi-analytical method could have serious accuracy problem for beam. plate and solid problem[40),58),113)~117)].

**b)** Continuum Approach

In the continuum approach, the sensitivity formulas are derived for the system before discretization, so there is no approximation involved in the formulation. There are two standard methods for describing the variation of a functional over a varying domain : Material Derivative Method (MDM) and Domain Parameterization Method (DPM).

The first approach, MDM involves the introduction of time-like parameters to describe the evolution of the undeformed geometry into neighboring shapes. The material derivative approach is based on the calculus of variation[118),119)].

The variation is obtained by determining the first order changes in the functional on the moving domain as the time-like parameters are varied[120]. Such an approach forms the basis of the MDM for shape sensitivity analysis[110),121)].

The material derivative approach of structural design sensitivity analysis has been developed over the last ten years from several different points of view[8),9),17),76),110),115),118),122)~127)]. This approach was first proposed by Cea, Zolesio and Rousselet[115),126),127)], and further developed by Haug and Choi *et al.*[110),118)]. The general formulation for elasticity problems was very well summarized in Haug *et al.*[110)].

An efficient approach, referred as the Direct Differentiation Method (DDM), involves implicit differentiation of the elasticity equation to obtain the partial derivatives[110)]. It is expensive for problems with a large number of design variables. Some recent articles on the DDM discusse a rigorous treatment of shape variations[114),128)]. On the other hand, the so called Adjoint Variable Method (AVM) has been derived by direct application of the weak governing equations, often in the form of virtual work, without introducing mutual energy principles. Haug, Choi and their co-workers[110),112),121),134),135)] considered both discrete and continuous systems. Dems and Mroz[76)] presented a similar approach based on the variational method, which included more general boundary conditions. They also identified the physical interpretation of the adjoint field variables as extended set of design variable to include shape, loading and support parameters. Belegundu and Arora[133)] showed that the adjoint variables represent the sensitivity of the cost function and constraint functions with respect to the loading or forcing functions in the design problem. Comparisons of the variational and implicit differentiation approaches have been investigated[37),47),55),74),112)]. Other researchers[69),76),114),118),122)~124),128)~133)] have presented formulations for SDSA of linear structures by introducing adjoint structures from a physical consideration.

Yang and Botkin[112)] demonstrated equivalence of variational and implicit differentiation method for linear problems. This equivalence can also be shown for nonlinear problems when finite element formulations are used[8)]. Several authors have proposed formulations based on boundary integrals and the adjoint method[69),118),123),129),132)]. But there are considerable numerical difficulties with the evaluation of boundary integrals[110),136)]. These problems were avoided by using domain integrals instead of boundary ones[131),137)], but it becomes expensive to calculate the full domain integration, so a boundary layer approach was suggested[10)]. Hou *et al.*[138)] pointed out some discontinuity problems that can rise at the interface between finite elements in the domain method. A problem of accuracy occurs when the adjoint system under a singular load must be solved, especially for a stress sensitivity. Even though a local averaging may be used to smooth the singularity[115)], the problem still exists and remains as an open problem.

The second approach, DPM uses a variable mapping to transform the problem to one with a nonvarying domain[139)~147)]. The geometric coordinates and the usual set of dependent variables are written as functions of parametric coordinates defined on a fixed domain. Functionals are rewritten on the fixed domain using the parametric coordinates as the independent variables, and functional variations are then determined in the usual way. This method forms the basis of the approach for shape sensitivity analysis found in the reference[148)], in which the shape variations are described in terms of a mapping from an independent, fixed reference geometry.

The DPM can be considered as an extension of the isoparametric concept of finite element analysis to the design and optimization problems. Comparison between MDM and DPM is done by Tortorelli *et al.*[143)], and Arora and Cardoso[146)]. Recently, the two approaches have been shown to be theoretically equivalent[147),149)]. However, their numerical implementation can be quite different.

**(2) SDSA based on BIE**

In the past decade the Boundary Element Method (BEM) has been recognized as an alternative numerical method for engineering problems, especially in the area of shape optimization. The BEM can reduce the two major drawbacks of the FEM : the remeshing problems during iterations and inaccuracy of boundary value evaluation. The principal advantage is that there is no need to discretize the interior of the body. There is also a large reduction in the number of unknowns. These can be seen from the researches in early 1980's such as Futagami[84),85)], Barone and Caulk[30)], Meric[87)~89)], Mota Soares *et al.*[13),44)~46)] and Zochowski and Mizukami[86)].

As in the variational formulation, there are two approaches to perform the SDSA based on boundary integral equations. One is a discretized approach, and the other a continuum approach.

**a)** Discretized Approach

This approach uses a discretized model to obtain the shape design sensitivity, and could be divided by three categories: Finite Difference Method (FDM), analytical method, semi-analytical method.

The FDM is straightforward and easy to

implement, and many authors[47),55),80),150)] have used it as a reference for comparison or a tool for sensitivity calculation. However, the FDM have a few shortcomings. It can not be exact unless the system is linear in the design variables. Thus, the result is highly dependent upon the size of perturbation. Also the computational cost can be high. Zhao and Adey[151)] presented a different SDSA scheme, which is based on FDM but independent of the perturbation step.

As an analytical approach, Kane and Saigal[152),153)] proposed an implicit direct differentiation method, in which the system matrix discretized from the boundary integral equation is differentiated analytically. Their formulation involves the products of shape functions, fundamental solutions and their derivatives. They introduced a rigid body motion technique of Barone and Yang[154)] to treat the singularity that exists in the derivative of fundamental solutions. While those approachs provide an easy and straightforward process, they have shown some problems such as the computational burden of performing singular integration of new kernels. They and their coworkers have extended the implicit differentiation method to various problems[14),48),155)~165)]. Similar method is used in[49),94),95),102)].

In the semianalytical method, the discretized system matrix is differentiated analytically, but the derivative of the stiffness matrix is calculated by employing the finite difference method. This method has economical and practical advantages but also the disadvantages of the FDM, and reliability problems[15),157),166)].

**b)** Continuum Approach

The continuum approach uses the material derivative concept of continuum mechanics to represent the variation of responses with respect to a shape change. The first step of this approach is to differentiate the boundary integral equation. No approximation is involved in the expression of the sensitivity, until the derived equations are discretized by boundary elements.

There are basically two methods to perform the SDSA. One is the direct differentiation method and the other the adjoint variable method. In the first method the state boundary integral is directly differentiated with respct to design variables and then a boundary integral equation similar to the original BIE is obtained in terms of derivatives of state variables. In the adjoint variable method, the constraint functional is first differentiated and the state variable dervatives are then replaced by terms calculable from adjoint systems.

Kwak and Choi[50),96),178)] and their followers[56),97),98)] have developed a general method for SDSA using the formal boundary integral equation. They used the material derivative concept and adjoint variable method utilizing a boundary integral identity to obtain an explicit expression for the variation of the performance functional in terms of boundary shape change, and the formulations were applied to thermal and elastostatic problems. Although this approach had aleady proven successful through numerical examples, there are some difficulties to determine the approximate adjoint tractions uniquely. An improved formulation of Kwak and Choi was developed by Zhao and Adey[226)], in which a singularity subtraction technique was employed to model the adjoint problem. Park and Yoo[179),180)] proposed a method employing the material derivative idea and an adjoint variable method in variational form for heat transfer system and axisymmetric elastic problem. They transformed the variational adjoint equation into an equivalent partial differential equation, and solved it by boundary integral equation method. Meric[51),90),181),182)] used the material derivative and adjoint variable method by means of an augmented functional method using the Lagrange multiplier. While his work throughout their derivation was independent of the boundary element formulation, he proposed the BEM for the solution of the original and the adjoint system. The same procedure is investigated by Aithal *et al.*[183)] and Kobelev[91)].

Barone and Yang[154),167)] and Yang[16)] developed a direct differentiation method, that is based on a direct application of the material derivative concept to the conventional boundary integral equations for displacements and stresses in an elastic solid. They employed a rigid body motion to remove high-order singularities that arise when taking derivatives of the basic Kelvin kernels in the displacement sensitivity. Zhang and Mukherjee[168)~171)] and Mukherjee and Chandra[172),173)] used the same concept of Barone and Yang but they used another boundary integral equation, derivative boundary integral equation, in which the basic unknowns are the tractions and the tangential derivatives of the displacements. Chandra and Chan[174),175)] utilized these for a steady state conduction-convection problem. Choi and Choi[176)] obtained the design derivatives directly by solving a new BIE, which is obtained by taking the material derivative of the boundary integral identity. The Authors[177),229)] presented formulas that consider changes in boundary conditions.

In the direct differentiation method we need one SDSA for each design variable, whereas we need one for each active constraint in the adjoint method. Therefore one can be computationally more efficient that the other depending on the number of design variables versus the number of active constraints. Ignoring the computational burden of complexities of singular kernels, the direct method may be more advantageous than the adjoint method, since the concentrated adjoint loads occurring in some cases are not suitable for the usual boundary element anlysis. Neither the adjoint variable nor direct method, however, can provide the most efficient computation if used alone. There may be some efficient hybrid methods. Choi and Kwak[184] proposed a unified approach for SDSA in the BIE formulation, which covers both the adjoint variable and direct method.

**Table 1** Sensitivity Application Fields

| Application Fields | References |
|---|---|
| Linear elastic | 9, 11, 13, 14, 15, 32, 50, 55, 75, 76, 86, 92, 94, 95, 118, 122, 130, 132, 139, 152～157, 163, 167～170, 176～178, 180, 183, 186, 187 |
| Nonlinear elastic | 8, 140, 144, 145, 171～173, 188～191 |
| Unilateral plane elasticity | 191 |
| Plate/Shell | 58, 71～73, 108, 188, 190, 192～197 |
| Nonsmooth boundary | 18, 71～73, 76 |
| Thermal | 30, 47, 48, 56, 85, 87~90, 96, 141, 159, 161, 162, 165, 174, 175, 179, 182, 188, 198～200 |
| Thermoelasticity | 79, 97, 98, 143, 158, 181, 201～204 |
| Thermoviscoelasticity | 205 |
| Elastodynamic | 142 |
| Frequency/Eigenvalue | 105, 135, 189, 195, 197, 206, 207 |
| Dynamic | 84, 123, 137, 206, 207 |
| Acoustics | 160 |
| Magnetostatics | 51, 99, 208 |
| Stochastic structure | 102～104 |

**Table 2** Optimization literature by specific problem

| Application Problems | References |
|---|---|
| Elastic bar/Beam | 28, 33, 44, 45, 49, 52, 70, 91, 93, 106, 124, 138, 201, 209, 210 |
| Disks | 32, 34, 38, 40, 41, 52, 72 |
| Plate/Shell with a hole | 8, 11, 24, 50, 53, 54, 58, 64, 97, 119, 192～194, 211～214 |
| Plane arch/Arch dam | 17～21 |
| Fillet/Weld surface | 5～16, 46, 50, 54, 97, 215, 216 |
| Torque arm | 7, 53, 58, 107, 217 |
| Control arm | 23, 43, 209, 216 |
| Engine connecting rod | 12, 22, 27, 28, 35, 42, 79, 94, 209 |
| Engine bearing cap | 5, 28, 79 |
| Steering knuckle | 23 |
| Chain link | 26, 119 |
| Cable crimping device | 107 |
| Culvert | 26, 217 |
| Bracket | 53, 214 |
| Pressure vessel | 218 |
| Penstock stiffener plate | 219 |
| Tire | 25 |
| Hook | 55 |
| Helical spring | 57 |
| Piezoelectric structure | 220 |
| Magnetic pole/electrode | 99, 100, 208, 221 |
| MHD channel section | 51 |
| Cryosurgical probe tip | 47 |
| Die/mold | 30, 56, 222 |
| Photo cell | 223 |
| Speciman for shear test | 224 |
| Furnace hearth | 48 |
| Obstacle in an Eulerian flow | 225 |

## 5. APPLICATIONS OF SDSA AND OPTIMIZATION

Test probelms and application cases appearing in the literature may be grouped in terms of SDSA formulation and optimization. It is seen that this area of shape optimal design is still in the growing stage, studying various formulations and testing rather simple problems. Although some of the applications are implemented on commerical softwares in conjunction with the FEM, it may take some more years to see any routine practical applications.

### (1) SDSA Formulation

The information on areas of SDSA formulation is summarized in **Table 1**. Refer to Adelman and Hafka[185] for more detailed information about sensitivity analysis application fields.

### (2) Specific Problem Applications

References are listed in **Table 2** by area of applications. Refer to Ding[3] and Haftka[4] for more structural shape optimization literature.

## 6. DISCUSSIONS AND CONCLUSION

An approach for the rapid creation of design and analysis model which is based on the integration of parameterized surface models, called 3-D shape design primitives and fully automatic mesh generation is now under developments. An integrated system for shape optimal design consists of geometric modeling, mesh generation, analysis, and design sensisivity analysis modules. Current trends in structural shape parameterization and optimization use the concept of generating the velocity field for the material derivative method. Many commerical packages have SDSA modules, but their approaches are based on finite difference, or semi-analytic method. The semi-anlytical approach is attractive for its generality and numerical efficiency. However, severe error in sensitivity may cause numerical difficulty and bring in divergence in optimization. In general, analytical sensitivities such as the material derivative approach give relatively good results and better convergence in shape optimal design problems. This suggests that anlytical sensivities should be used whenever available.

It is recognized that the essential and most influential content of a design is the shape, but its determination most difficult. Theories and algorithms for shape optimal design are found available in the literature, although their reliability, efficiency and accuracy remain to be studied more. It is, however, the reviewer's wish to see more realistic and practical application cases. Otherwise the many researchers in this area may spend too much time on trivial improvements in the methods with no attention from potential industry users.

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