AN EXPLICIT FEM FORMULATION OF THE 2-D TRIANGULAR ELEMENT FOR FINITE STRAINS

Masahiro AI* and Fumio NISHINO**

Not by means of mathematical expansions, but on the basis of a physical decomposition of its total freedom into the parameters of position as a rigid and those of deformation, an explicit discretization is developed for the 2-D triangular element with large displacements. While the material is assumed elastic even for finite strains, any geometrically nonlinear effects are taken into account, systematically and rigorously.

Keywords : *geometrically nonlinear discretization, finite strains, isoparametric interpolation*

1. INTRODUCTION

Under the isoparametric interpolation applied to the finite displacements of solid continua, the Lagrangian expression for strain remains quadratic in terms of the nodal parameters. Depending upon this feature, the existing total-Lagrangian FEM formulations[3)~7)] classified into the B-notation and the N-notation methods are mathematically accomplished. However, those expansions are not to be understood physically. Further, to obtain the actual stiffness relations, an awful calculation is necessary in the numerical analysis.

In this paper, another formulation is presented for the 2-D triangular element, with a full physical explanation, which is developed explicitly in a complete accordance with an exisiting general procedure stated in Ref.1, 2) to separate the total freedom of an element into the parameters of position as a rigid and those of deformation. By the assumption of an elastic strain energy existing even for finite strains, the realistic material problems are disregarded, but the expansion classified into the total-Lagrangian is theoretical and rigorous as a geometrically nonlinear discretization.

2. DESCRIPTION OF GEOMETRY AND STRAIN

In the Cartesian $\{x, y\}$, let us consider a three-node triangular element (e). The spatial coordinates of its three nodes are employed as the **element position** :

$$\{x\}_{(e)} = \{(x, y)_i, (x, y)_j, (x, y)_k\} \cdots\cdots\cdots\cdots (1)$$

* Mem. of JSCE, Dr. Eng., Prof., Dept. of Civil Engng., Hosei Univ. (Koganei, Tokyo)

** Mem. of JSCE, Ph. D., Prof., Dept. of Civil Engng., Univ. of Tokyo

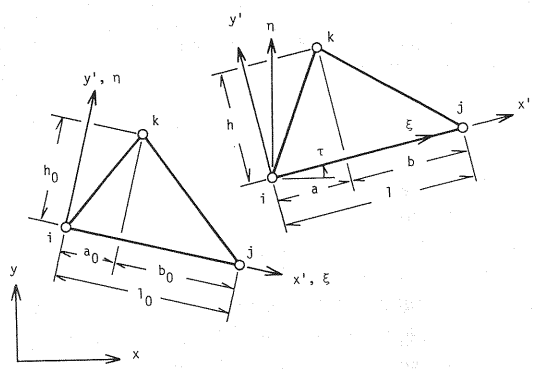

Fig.1 Geometry of Triangular Element

where $i<j<k$. As the **element coordinate system**, we take $\{x', y'\}$ in relation to the current configuration such that x' is directed from node i toward j with y' being perpendicular to x'. As well, in the initial (or stress-free) state, the $\{x', y'\}$ of material points are employed as Lagrangian coordinates $\{\xi, \eta\}$: $\xi = x'_0$, $\eta = y'_0$, where an initial quantity is denoted by subscript $(\ )_0$.

As shown in **Fig.1**, the triangle shape is characterized by $\{x', y'\}$ of the three nodes. Excepting the identical zeros, let **shape** $g_{(e)}$ be defined by collecting the remaining three into a set

$$g_{(e)} = \{l, a, h\}_{(e)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Those lengths are related to $\{x\}_{(e)}$ of (1) by

$$l = \sqrt{\bar{x}^2 + \bar{y}^2}, \quad a = \frac{1}{l}\{\bar{x}\hat{x} + \bar{y}\hat{y}\}, \quad h = \frac{1}{l}\{\bar{x}\hat{y} - \hat{x}\bar{y}\} \cdots\cdots\cdots (3. \text{a-c})$$

where $\{\bar{x}, \bar{y}\}$ and $\{\hat{x}, \hat{y}\}$ are position vectors of node j and k relative to i:

$$\{\bar{x}, \bar{y}\} = \{x_j - x_i, y_j - y_i\},$$

$$\{\hat{x}, \hat{y}\} = \{x_k - x_i,\ y_k - y_i\} \quad \cdots\cdots (4.\ \mathrm{a, b})$$

And, the angular positon of the $\{x', y'\}$-system is given by

$$\tau = \arctan\{\hat{y}/\hat{x}\} \quad \cdots\cdots (5)$$

Then, $\boldsymbol{v}_{(e)} = \{x_i,\ y_i,\ \tau\}$ and $\boldsymbol{g}_{(e)} = \{l, a, h\}$ are a separation of the entire freedom into the **rigid position** and the shape, respectively.

By applying the isoparametric (or constant-strain) interpolation to the deformed (e) upon the $\{x', y'\}$-coordinates, we have

$$x'(\xi, \eta) = \frac{l}{l_0}\xi + \frac{1}{h_0}\left(a - \frac{a_0 l}{l_0}\right)\eta, \quad y'(\xi, \eta) = \frac{h}{h_0}\eta \quad \cdots\cdots (6.\ \mathrm{a, b})$$

Let **deformation** $\boldsymbol{\varepsilon}_{(e)}$ be defined by the resulting Green's strain components :

$$\boldsymbol{\varepsilon}_{(e)} = \{e_{\xi\xi},\ e_{\eta\eta},\ \gamma_{\xi\eta}\} \quad \cdots\cdots (7)$$

$$e_{\xi\xi} = \frac{1}{2}\left\{\left(\frac{l}{l_0}\right)^2 - 1\right\},$$

$$e_{\eta\eta} = \frac{1}{2}\left\{\left(\frac{a}{h_0} - \frac{a_0 l}{h_0 l_0}\right)^2 + \left(\frac{h}{h_0}\right)^2 - 1\right\},$$

$$\gamma_{\xi\eta}(=2e_{\xi\eta}) = \left(\frac{l}{l_0}\right)\left(\frac{a}{h_0} - \frac{a_0 l}{h_0 l_0}\right) \quad \cdots\cdots (8.\ \mathrm{a\text{-}c})$$

The constant strain state is determined for element position $\{\boldsymbol{x}\}_{(e)}$ through relations (3. a-c), (4. a, b) and (8. a-c). The associated tangential relations can be obtained by expanding their derivatives for infinitesimal variation $\delta\{\boldsymbol{x}\}_{(e)}$. But, we here develop those relations under the following physical decompositons : First, let the independent $\delta\{\boldsymbol{x}\}_{(e)}$ be re-decomposed into the $\{x', y'\}$:

$$\delta\{\boldsymbol{x}'\}_{(e)} = [T(\tau)]_{(e)}\delta\{\boldsymbol{x}\}_{(e)},$$

$$[T]_{(e)} = \begin{bmatrix} \begin{bmatrix} \cos\tau, & \sin\tau \\ -\sin\tau, & \cos\tau \end{bmatrix} & & \\ & [\ \ ] & \\ & & [\ \ ] \end{bmatrix} \quad \cdot (9.\ \mathrm{a, b})$$

The vectors $\vec{ij}$ and $\vec{ik}$ changed by $\delta\{\boldsymbol{x}'\}_{(e)}$ can be represented in the preceding $\{x', y'\}$ as

$$\begin{Bmatrix} \vec{ij} \\ \vec{ik} \end{Bmatrix} = \begin{bmatrix} l + \delta x'_j - \delta x'_i, & \delta y'_j - \delta y'_i \\ a + \delta x'_k - \delta x'_i, & h + \delta y'_k - \delta y'_i \end{bmatrix} \begin{Bmatrix} \boldsymbol{i}_{x'} \\ \boldsymbol{i}_{y'} \end{Bmatrix} \quad \cdots\cdots (10)$$

where $\{\boldsymbol{i}_{x'}, \boldsymbol{i}_{y'}\}$ are the unit vectors into $\{x', y'\}$. At the same time, those unit vectors are also rotated by $\delta\{\boldsymbol{x}'\}_{(e)}$:

$$\begin{Bmatrix} \boldsymbol{i}_{x'} + \delta\boldsymbol{i}_{x'} \\ \boldsymbol{i}_{y'} + \delta\boldsymbol{i}_{y'} \end{Bmatrix} = \begin{bmatrix} 1, & \delta\tau \\ -\delta\tau, & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{i}_{x'} \\ \boldsymbol{i}_{y'} \end{Bmatrix},$$

$$\delta\tau = \frac{1}{l}(\delta y'_j - \delta y'_i) \quad \cdots\cdots (11.\ \mathrm{a, b})$$

The above matrix containing 1 and differential $\delta\tau$ can be inverted by the transposition. Introducing the inverse of (11) into (10), we have the vectors $\vec{ij}$ and $\vec{ik}$ represented upon the current $\{\boldsymbol{i}_{x'} + \delta\boldsymbol{i}_{x'},\ \boldsymbol{i}_{y'} + \delta\boldsymbol{i}_{y'}\}$. The nonzero elements of that 2×2 matrix relating $\{\vec{ij}, \vec{ik}\}$ to $\{\boldsymbol{i}_{x'} + \delta\boldsymbol{i}_{x'},\ \boldsymbol{i}_{y'} + \delta\boldsymbol{i}_{y'}\}$ are to be $l + \delta l$, $a + \delta a$ and $h + \delta h$. This result is written in the matrix form

$$\delta\boldsymbol{g}_{(e)} = [Q^I_X(\boldsymbol{g})]_{(e)}\delta\{\boldsymbol{x}'\}_{(e)},$$

$$[Q^I_X(\boldsymbol{g})]_{(e)} = \begin{bmatrix} -1, & 0, & 1, & 0, & 0, & 0 \\ -1, & -\frac{h}{l}, & 0, & \frac{h}{l}, & 1, & 0 \\ 0, & -\frac{b}{l}, & 0, & -\frac{a}{l}, & 0, & 1 \end{bmatrix} \quad \cdots\cdots (12.\ \mathrm{a, b})$$

where $b = l - a$. Succeedingly, by the mathematical differentiation of (8. a-c), we have

$$\delta\boldsymbol{\varepsilon}_{(e)} = [Q^{II}_X(\boldsymbol{g})]_{(e)}\delta\boldsymbol{g}_{(e)},$$

$$[Q^{II}_X]_{(e)} = \begin{bmatrix} \frac{l}{l_0^2}, & 0, & 0 \\ \frac{-a_0}{h_0^2 l_0}\left(a - \frac{a_0 l}{l_0}\right), & \frac{1}{h_0^2}\left(a - \frac{a_0 l}{l_0}\right), & \frac{h}{h_0^2} \\ \frac{1}{h_0 l_0}\left(a - \frac{2a_0 l}{l_0}\right), & \frac{l}{h_0 l_0}, & 0 \end{bmatrix} \quad \cdots\cdots (13.\ \mathrm{a, b})$$

Then, collecting (9), (12) and (13) into a unified matrix form, we have

$$\delta\boldsymbol{\varepsilon}_{(e)} = [Q_X(\{\boldsymbol{x}\})]_{(e)}\delta\{\boldsymbol{x}\}_{(e)} \quad \cdots\cdots (14.\ \mathrm{a})$$

$$[Q_X(\{\boldsymbol{x}\})]_{(e)}(=[Q^{II}_X]_{(e)}[Q^I_X]_{(e)}[T]_{(e)}) =$$

$$\begin{bmatrix} \frac{-1}{l_0^2}\bar{x}, & \frac{-1}{l_0^2}\bar{y}, & \frac{1}{l_0^2}\bar{x}, & \frac{1}{l_0^2}\bar{y}, & 0, & 0 \\ \frac{a_0 b_0}{h_0^2 l_0^2}\bar{x} - \frac{b_0}{h_0^2 l_0}\hat{x}, & \frac{a_0 b_0}{h_0^2 l_0^2}\bar{y} - \frac{b_0}{h_0^2 l_0}\hat{y}, & \frac{a_0^2}{h_0^2 l_0^2}\bar{x} - \frac{a_0}{h_0^2 l_0}\hat{x}, & \frac{a_0^2}{h_0^2 l_0^2}\bar{y} - \frac{a_0}{h_0^2 l_0}\hat{y}, & \frac{-a_0}{h_0^2 l_0}\bar{x} + \frac{1}{h_0^2}\hat{x}, & \frac{-a_0}{h_0^2 l_0}\bar{y} + \frac{1}{h_0^2}\hat{y} \\ \frac{(a_0 - b_0)}{h_0 l_0^2}\bar{x} - \frac{1}{h_0 l_0}\hat{x}, & \frac{(a_0 - b_0)}{h_0 l_0^2}\bar{y} - \frac{1}{h_0 l_0}\hat{y}, & \frac{-2a_0}{h_0 l_0^2}\bar{x} + \frac{1}{h_0 l_0}\hat{x}, & \frac{-2a_0}{h_0 l_0^2}\bar{y} + \frac{1}{h_0 l_0}\hat{y}, & \frac{1}{h_0 l_0}\bar{x}, & \frac{1}{h_0 l_0}\bar{y} \end{bmatrix} \quad \cdots\cdots (14.\ \mathrm{b})$$

3. DEFORMATION FORCE

For simplicity, let us consider an elastic finite-strain problem, in which a strain-energy-density function, $A(\boldsymbol{e})$, is prescribed in terms of the Green's strain components. As a set of force components conjugate to $\boldsymbol{\varepsilon}_{(e)}$, we here define **deformation force** $\boldsymbol{f}_{(e)}$. Since the strain energy in our constant-strain element is given by $U(\boldsymbol{\varepsilon})_{(e)} = V_0 A(\boldsymbol{e})$, where V_0 is the initial volume, the present

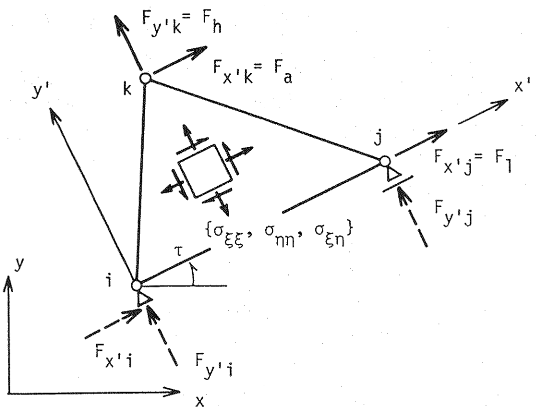

Fig.2 Nodal Forces in Simple Support

$f_{(e)} - \varepsilon_{(e)}$ relation is given by

$$f_{\xi\xi}(=V_0\sigma_{\xi\xi}) = V_0\frac{\partial A(e)}{\partial e_{\xi\xi}},$$

$$f_{\eta\eta}(=V_0\sigma_{\eta\eta}) = V_0\frac{\partial A(e)}{\partial e_{\eta\eta}},$$

$$f_{\xi\eta}(=V_0\sigma_{\xi\eta}) = V_0\frac{\partial A(e)}{\partial \gamma_{\xi\eta}} \cdots\cdots\cdots (15.\text{ a-c})$$

where $\{\sigma_{\xi\xi}, \sigma_{\eta\eta}, \sigma_{\xi\eta}\}$ denote the second Piola-Kirchhoff stress components.

4. RELATIONS FROM $f_{(e)}$ TO NODAL FORCES

The **element force**, $\{\boldsymbol{F}\}_{(e)}$, is defined as the nodal forces conjugate to $\{\boldsymbol{x}\}_{(e)}$. Those force components into the spatial $\{x, y\}$ can be derived in accordance with the former geometrical decompositions.

With our $\boldsymbol{v}_{(e)} = \{x_i, y_i, \tau\}$ and $\boldsymbol{g}_{(e)} = \{l, a, h\}$, the simple support shown in **Fig.2** is associated : if that support is fixed in the space, the rigid displacements into $\{x_i, y_i, \tau\}$ are constrained, but any deformations are possible by the variety of $\{l, a, h\}$.

Upon that support fixed, the force components into $\boldsymbol{g}_{(e)} = \{l, a, h\}$ are now denoted by $\boldsymbol{G}_{(e)} = \{F_l, F_a, F_h\}$. By substituting (13) into the virtual work equation, $\boldsymbol{f}_{(e)} \cdot \delta\boldsymbol{\varepsilon}_{(e)} = \boldsymbol{G}_{(e)} \cdot \delta\boldsymbol{g}_{(e)}$, we have

$$\boldsymbol{G}_{(e)} = [Q_F^{II}(\boldsymbol{g})]_{(e)}\boldsymbol{f}_{(e)}, \quad [Q_F^{II}]_{(e)} = [Q_X^{II}]_{(e)}^T \cdots (16)$$

Next, we consider the entire nodal forces, $\{\boldsymbol{F}'\}_{(e)}$, resolved into the element coordinates. Apparently, $F_{x'j} = F_l$, $F_{x'k} = F_a$ and $F_{y'k} = F_h$. By the equilibrium condition as a rigid upon the current shape, or as the reactive forces in the simple support, the remaining components in $\{\boldsymbol{F}'\}_{(e)}$ are determined for $\boldsymbol{G}_{(e)}$. This result is written in the form :

$$\{\boldsymbol{F}'\}_{(e)} = [Q_F^I(\boldsymbol{g})]_{(e)}\boldsymbol{G}_{(e)} \cdots\cdots\cdots (17.\text{ a})$$

where matrix $[Q_F^I(\boldsymbol{g})]_{(e)}$ is found to be in the contragredience with (12) :

$$[Q_F^I(\boldsymbol{g})]_{(e)} = [Q_X^I(\boldsymbol{g})]_{(e)}^T \cdots\cdots\cdots (17.\text{ b})$$

Finally, element force $\{\boldsymbol{F}\}_{(e)}$ is obtained from $\{\boldsymbol{F}'\}_{(e)}$ by the inverse rotation to (9) :

$$\{\boldsymbol{F}\}_{(e)} = [T(\tau)]_{(e)}^T\{\boldsymbol{F}'\}_{(e)} \cdots\cdots\cdots (18)$$

Collecting (16), (17) and (18) into a unified matrix form, we have

$$\{\boldsymbol{F}\}_{(e)} = [Q_F(\{\boldsymbol{x}\})]_{(e)}\boldsymbol{f}_{(e)},$$

$$[Q_F(\{\boldsymbol{x}\})]_{(e)} = [T(\tau)]_{(e)}^T[Q_F^I(\boldsymbol{g})]_{(e)}[Q_F^{II}(\boldsymbol{g})]_{(e)} \cdots\cdots\cdots (19.\text{ a, b})$$

Apparently,

$$[Q_F(\{\boldsymbol{x}\})]_{(e)} = [Q_X(\{\boldsymbol{x}\})]_{(e)}^T \cdots\cdots\cdots (20)$$

5. TANGENT STIFFNESS

By the use of (3), (4), (7), (15) and (19), element force $\{\boldsymbol{F}\}_{(e)}$ is obtained for element position $\{\boldsymbol{x}\}_{(e)}$. Let us consider the tangent stiffness matrix upon freedom $\{\boldsymbol{x}\}_{(e)}$.

By differentiating (19.a), we have

$$\delta\{\boldsymbol{F}\}_{(e)} = [Q_F]_{(e)}\delta\boldsymbol{f}_{(e)} + (\delta[Q_F]_{(e)})\boldsymbol{f}_{(e)} \cdots\cdots\cdots (21)$$

The form for $\delta\{\boldsymbol{F}\}_{(e)}$ to be developed is

$$\delta\{\boldsymbol{F}\}_{(e)} = [k(\{\boldsymbol{x}\})]_{(e)}\delta\{\boldsymbol{x}\}_{(e)},$$

$$[k]_{(e)} = [k_M(\{\boldsymbol{x}\})]_{(e)} + [k_G(\boldsymbol{f}, \{\boldsymbol{x}\})]_{(e)} \quad (22.\text{ a, b})$$

where $[k_M]_{(e)}$ and $[k_G]_{(e)}$ are the so-called deformation and geometrical stiffnesses, associated with the first and second terms of (21), respectively.

By the use of (14) and (15), the deformation stiffness is written as

$$[k_M(\{\boldsymbol{x}\})]_{(e)} = [Q_F(\{\boldsymbol{x}\})]_{(e)}[\kappa(\varepsilon)]_{(e)}[Q_X(\{\boldsymbol{x}\})]_{(e)},$$

$$[\kappa]_{(e)} = V_0\left[\frac{\partial}{\partial\varepsilon}<\frac{\partial A(\varepsilon)}{\partial\varepsilon}>^T\right]_{(e)} \cdots\cdots\cdots (23.\text{ a, b})$$

From (14) and (19), matrix $[Q_F(\{\boldsymbol{x}\})]_{(e)}$ itself can be rewritten as

$$[Q_F(\{\boldsymbol{x}\})]_{(e)} = [Q_X(\{\boldsymbol{x}\})]_{(e)}^T = \left[\frac{\partial\varepsilon}{\partial\{\boldsymbol{x}\}}\right]_{(e)}^T$$

Hence, the second term of (21) is developed as

$$(\delta[Q_F]_{(e)})\boldsymbol{f}_{(e)} = \left[\left[\!\left[\frac{\partial}{\partial\{\boldsymbol{x}\}}\left[\frac{\partial\varepsilon}{\partial\{\boldsymbol{x}\}}\right]^T\right]\!\right]\delta\{\boldsymbol{x}\}\right]_{(e)}\boldsymbol{f}_{(e)}$$

$$= \left[\frac{\partial}{\partial\{\boldsymbol{x}\}}<\frac{\partial(\varepsilon\cdot\boldsymbol{f})}{\partial\{\boldsymbol{x}\}}>^T\right]_{(e)}\Bigg|_{f(e)=\text{const.}}\delta\{\boldsymbol{x}\}_{(e)} \cdot (24)$$

where notation ⟦ ⟧ denotes a three-dimensional matrix ; and subscript $|_{f(e)=\text{const.}}$ means to regard $\boldsymbol{f}_{(e)}$ not subject to the differentiation. As the actual expansion, first, we have the three derivative matrices by differentiating each transposed row of $[Q_X(\{\boldsymbol{x}\})]_{(e)}$ of (14. b) with respect to $\{\boldsymbol{x}\}_{(e)}$. And, as the sum of those matrices multiplied by respective $f_{\xi\xi}$, $f_{\eta\eta}$ and $f_{\xi\eta}$, we have the geometrical stiffness matrix :

$$[k_G(\boldsymbol{f},\ \{x\})]_{(e)}=$$

$$V_0\sigma_{\xi\xi}\begin{bmatrix} \frac{1}{l_0^2}, & 0, & -\frac{1}{l_0^2}, & 0, & 0, & 0 \\ & \frac{1}{l_0^2}, & 0, & -\frac{1}{l_0^2}, & 0, & 0 \\ & & \frac{1}{l_0^2}, & 0, & 0, & 0 \\ & & & \frac{1}{l_0^2}, & 0, & 0 \\ & Sym. & & & 0, & 0 \\ & & & & & 0 \end{bmatrix}$$

$$+V_0\sigma_{\eta\eta}\begin{bmatrix} \left(\frac{b_0}{h_0l_0}\right)^2, & 0, & \frac{a_0b_0}{(h_0l_0)^2}, & 0, & \frac{-b_0}{h_0^2l_0}, & 0 \\ & \left(\frac{b_0}{h_0l_0}\right)^2, & 0, & \frac{a_0b_0}{(h_0l_0)^2}, & 0, & \frac{-b_0}{h_0^2l_0} \\ & & \left(\frac{a_0}{h_0l_0}\right)^2, & 0, & \frac{-a_0}{h_0^2l_0}, & 0 \\ & & & \left(\frac{a_0}{h_0l_0}\right)^2, & 0, & \frac{-a_0}{h_0^2l_0} \\ & Sym. & & & \frac{1}{h_0^2}, & 0 \\ & & & & & \frac{1}{h_0^2} \end{bmatrix}$$

$$+V_0\sigma_{\xi\eta}\begin{bmatrix} \frac{2b_0}{h_0l_0^2}, & 0, & \frac{(a_0-b_0)}{h_0l_0^2}, & 0, & \frac{-1}{h_0l_0}, & 0 \\ & \frac{2b_0}{h_0l_0^2}, & 0, & \frac{(a_0-b_0)}{h_0l_0^2}, & 0, & \frac{-1}{h_0l_0} \\ & & \frac{-2a_0}{h_0l_0^2}, & 0, & \frac{1}{h_0l_0}, & 0 \\ & Sym. & & \frac{-2a_0}{h_0l_0^2}, & 0, & \frac{1}{h_0l_0} \\ & & & & 0, & 0 \\ & & & & & 0 \end{bmatrix}$$

………………………………………(25)

6. CONCLUDING REMARKS

Guided by the stated-in-general-terms procedure[1),2)], an explicit discretization is developed for the triangular FEM element, in which as intermediate parameters toward $\varepsilon_{(e)}$ and $\boldsymbol{f}_{(e)}$, respectively, shape $\boldsymbol{g}_{(e)}$ and its associated force $\boldsymbol{G}_{(e)}$ are defined exceptionally here in this paper. The present formulation is based on a physical decomposition of the geometrically nonlinear problem. The geometrical relations and force transformations are related into each pair by the contragredience.

REFERENCES

1) Ai, M. and Nishino, F. : Mechanics in Geometrically Nonlinear Problem of Discrete System and Application to Plane Frame-Works, Proc. of JSCE, No.304, pp.17～32, Dec., 1980 (in Japanese).

2) Ai, M. and Nishino, F. : Geometrically Nonlinear Mechanics in Discrete Systems and Application to Plane Frame-Works, J. of faculty of Engng. (B), The Univ. of Tokyo, Vol. XXXVI, No.4, pp.745～775, 1982.

3) Felippa, C.A. : Discussion on "Incremental Finite Element Matrices, " by S. Rajasekaran and D.W. Murray, J. Struct. Div., ASCE, Vol.100, pp.2521～2523, Dec., 1974.

4) Mallett, R.H. and Marcal, P.V. : Finite Element Analysis of Nonlinear Structures, J. Struct. Div., ASCE, Vol.94, pp.2081～2105, Sep., 1968.

5) Rajasekaran, S. and Murray, D.W. : Incremental Finite Element Matrices, J. Struct Div., ASCE, Vol.99, pp.2423～2438, Dec., 1973.

6) Wood, R.D. and Schrefler, B. : Geometrically Nonlinear, Analysis — A Correlation of Finite Element Notations, Int. J. Num. Meth. Engng., Vol. 12, pp.635～642, 1978.

7) Zienkiewicz, O.C. and Nayak, G.C. : A General Approach to Problems of Plasticity and Large Deformation using Isoparametric Elements, Proc. Conf. Matrix Meth. Struct. Mech., Wright-Patterson A.F.B. Ohio (1971), (Pub. 1973).

(Received August 29, 1991)