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ADVANCED PERIODIC GALERKIN FINITE ELEMENT METHOD FOR SHALLOW WATER EQUATION

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The advanced periodic Galerkin method for the shallow water equation is presented using the exponential function for the constituents decomposition including the conjugate complex function. The steady flow which is caused due to the existance of periodic flow can be solved by the present method. To verify the effectiveness of the present method, the numerical calculation in one-dimensional problem of which the analytical solution is known is performed. The numerical results are well in agreement with the analytical solution. The residual flow analyses both in the model basin and in Tokyo Bay are carried out. The numerical solutions are well in agreement with the experimental data and measurements. Accordingly, the present method is shown to be adaptable for the analysis of the combined problem between the steady and harmonic flows.

Keywords: periodic method, shallow water equation, residual current induced tidal flow, finite element method

1. INTRODUCTION

Commonly, the shallow water equation is applied to coastal sea problems such as tidal current, tsunami propagation, storm surge, etc. The tidal flow is usually calculated by the shallow water equation with bottom friction, Coriolis force, etc. In the numerical analysis of the tidal flow, two approaches are usually used, one is the time stepping method and the other is the periodic method. The time stepping method[1),2)] can be easily applied to the calculation of non-linear problems. The residual flow due to the nonlinearlity can be expressed by averaging the flow over one period[3)]. But the calculation of this method should continue from zero to final quasi-steady periodic state. It is rather difficult to decide when the time stepping method can attain the periodic state. The stability condition of the time stepping method is relatively sensitive. In the case of tidal analysis in the nearshore flow, the determination of time increment is essential, because the number of iterations becomes large. Another method of tidal analysis is the periodic method. The periodic method is suitable for tidal analysis in the nearshore region because this method does not need to use time-wise calculation. The variables in the periodic method are separated to time and space. The flow is assumed to be quasi-steady.

Kawahara and Hasegawa[4)] applied the Galerkin method for the shallow water equation in both of the space and time domain. This periodic Galerkin method is able to solve the interaction of the various frequency components due to nonlinearlity. The tidal analysis of Katsurashima channel is carried out[5)] by this method. However the steady flow which is occurred by periodic flow due to nonlinearity such as tidal residual flow can not be calculated by this method. The several calculation methods of interaction between

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the steady and harmonic components are proposed by Provost *et al.*[6], Westerink *et al.*[7),8)], Walters[9] or Kouno *et al.*[10] and among others. Provost *et al.*[6] and Westerink *et al.*[7),8)] proposed an iterative method based on the periodic method employing the equation that the nonlinear term is transformed into the right side of the shallow water equation. Walters[9] proposed another iterative method. The velocity and water elevation are separated into the steady and harmonic components, in which the harmonic component has a pair of conjugate complex. A pair of the shallow water equation is calculated in an indirect way. The water elevation is calculated and then the velocity is calculated in a separate manner. The water elevation and velocity are calculated using the values of the previous step. This iterative procedure is also repeated until the convergence is obtained. The same shape function is chosen for velocity and water elevation. Kouno *et al.*[10] also tried the nonlinear analysis, in which the velocity and water elevation are separated into the steady and periodic components. The calculation of the steady flow is carried out by the time stepping method. The useful calculation method of interaction between the steady and harmonic components is not established yet.

In this paper the Galerkin method is applied in both the space and time domain like a conventional periodic Galerkin method[4]. The steady component induced by the harmonic component and also the interaction among the harmonic components can be calculated by the present method. The exponential function is adopted for the trial function in time and the expansion of harmonic components has a pair of conjugate complex. Quadratic and linear function are adopted for the shape function of velocity and water elevation respectively. The final simultaneous equation is a set of nonlinear equations. Newton-Raphson method is applied to solve the equation system is a direct manner.

To verify the effectiveness of the present method, one dimensional wave propagation considering the interaction between the steady and harmonic components is performed and is compared with the analytical solution. Both components of the steady and harmonic flow are well matched with the analytical solution. The residual flow induced by the tidal flow in the model basin is carried out. Both numerical and experimental results[12] show the counterclockwise flows. Both amplitudes of velocity are in good agreement. The residual flow in Tokyo Bay is calculated to verify the possibility of application. The numerical results also show good agreement with the measurements[14]. The numerical examples show that the present method is adaptable for practical computation.

2. GOVERNING EQUATION

The governing equation is the shallow water equation which is derived by integrating the Navier-Stokes equation vertically over the water depth and is expressed as :

$$\frac{\partial u_k}{\partial t}+u_j u_{k,j}+g\zeta_{,k}-\nu(u_{k,j}+u_{j,k})_{,j}=0 \quad (1)$$

$$\frac{\partial \zeta}{\partial t}+(hu_k)_{,k}=0 \quad (2)$$

where t, u_k, ζ, h, g and ν denote time, depth-averaged components of velocity in x, y coordinate directions, water elevation from mean sea level (MSL), water depth to MSL, acceleration due to gravity and eddy viscosity, and $(\ )_{,i}$ denotes the partial differentiation with respect to coordinate x_i. In the continuity equation (2) the amplitude of the water elevation can be assumed to be negligible compared with the water depth. The boundary conditions are prescribed on each boundary :

the velocity u_k is specified on boundary S_1

$$u_k=\hat{u}_k \text{ on } S_1 \quad (3)$$

the velocity gradient q_k is specified on boundary S_2

$$q_k=(u_{k,j}+u_{j,k})n_j=\hat{q}_k \text{ on } S_2 \quad (4)$$

and the water elevation ζ is specified on boundary S_3

$$\zeta=\hat{\zeta} \text{ on } S_3 \quad (5)$$

where the notation hat denotes given value on the boundary, n_j represents the unit normal vector.

3. FINITE ELEMENT EQUATION

Following the conventional finite element Galerkin method, the weighted residual variational equation are formulated. Let u_k^* be the weighting function, the values of which are arbitrary except on the boundaries where they take zero. Multiplying both sides of equation (1) by u_k^*, integrating over the whole domain V and using Green's theorem lead to the variational equation as follows :

$$\int_v\left(u_k^*\frac{\partial u_k}{\partial t}\right)dV+\int_v(u_k^* u_j u_{k,j})dV+\nu\int_v(u_{k,j}^* u_{k,j})dV+\nu\int_v(u_{k,j}^* u_{j,k})dV+\int_v(u_k^* g\zeta_{,k})dV$$
$$=\int_s \nu\,(u_k^* q_k)dS \quad \cdots\cdots (6)$$

Multiplying both sides of equation (2) by the weighting function ζ^* and integrating over the whole domain V, the variational form of the continuity equation is derived as :

$$\int_v\left(\zeta^*\frac{\partial\zeta}{\partial t}\right)dV+\int_v\{\zeta^*(hu_k)_{,k}\}dV=0 \quad \cdots\cdots (7)$$

Both trial and weighting functions of velocity and water elevation are expressed in the form :

$$u_k=\Phi_\alpha u_{\alpha k},\quad u_k^*=\Phi_\alpha u_{\alpha k}^* \quad \cdots\cdots (8)$$
$$\zeta=\Psi_\lambda\zeta_\lambda,\quad \zeta^*=\Psi_\lambda\zeta_\lambda^* \quad \cdots\cdots (9)$$

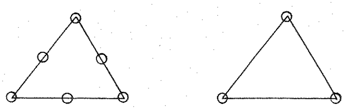

(a) Velocity (Φ_α) (b) Water elevation (Ψ_λ)

Fig. 1 Shape function.

where $u_{\alpha k}$ and ζ_λ are the nodal values of velocity at the αth node in the kth direction and water elevation at the λth node respectively. The interpolation function Ψ_λ for water elevation is linear function which is chosen for the reason that this is the lowest order function with three nodal unknown values. The interpolation function Φ_α for velocity is chosen as the quadratic function. This is based on the fact that the interpolation function for velocity should be heigher than that for water elevation in case of periodic shallow water analysis or imcompressible viscous flow analysis. This fact is already detected by author's group[11]. Substituting equations (8) and (9) into equations (6) and (7) and considering the arbitrariness of $u_{\alpha k}^*$ and ζ_λ^* yield a set of finite element equation :

$$\boldsymbol{F}_{\alpha k}\equiv\boldsymbol{M}_{\alpha k\beta j}\frac{\partial u_{\beta j}}{\partial t}+\boldsymbol{K}_{\alpha\beta\gamma j}u_{\beta j}u_{\gamma k}+\boldsymbol{H}_{\alpha k\mu}\zeta_\mu+\boldsymbol{S}_{\alpha k\beta j}u_{\beta j}-\hat{\boldsymbol{\Omega}}_{\alpha k}=0 \quad \cdots\cdots (10)$$

$$\boldsymbol{G}_\lambda\equiv\boldsymbol{A}_{\lambda\mu}\frac{\partial\zeta_\mu}{\partial t}+\boldsymbol{B}_{\lambda\mu\beta j}h_\mu u_{\beta j}=0 \quad \cdots\cdots (11)$$

where

$$\boldsymbol{M}_{\alpha k\beta j}=\delta_{kj}\int_v(\Phi_\alpha\Phi_\beta)dV,\quad \boldsymbol{K}_{\alpha\beta\gamma j}=\int_v(\Phi_\alpha\Phi_\beta\Phi_{\gamma,j})dV,\quad \boldsymbol{H}_{\alpha k\mu}=g\int_v(\Phi_\alpha\Psi_{\mu,k})dV$$

$$\boldsymbol{S}_{\alpha k\beta j}=\nu\delta_{kj}\int_v(\Phi_{\alpha,m}\Phi_{\beta,m})dV+\nu\int_v(\Phi_{\alpha,j}\Phi_{\beta,k})dV,\quad \hat{\boldsymbol{\Omega}}_{\alpha k}=\int_v\nu\,(\Phi_\alpha\hat{q}_k)dS$$

$$\boldsymbol{A}_{\lambda\mu}=\int_v(\Psi_\lambda\Psi_\mu)dV,\quad \boldsymbol{B}_{\lambda\mu\beta j}=\int_v(\Psi_\lambda\Psi_{\mu,j}\Phi_\beta)dV+\int_v(\Psi_\lambda\Psi_\mu\Phi_{\beta,j})dV$$

4. ADVANCED PERIODIC GALERKIN METHOD

To solve equations (10) and (11), the periodic Galerkin method is applied in time based on the fact that the long wave such as tidal flow is periodic motion. Kawahara and Hasegawa[4] succeeded in analyzing the non-linear shallow water equation to solve the interaction with the various harmonic component by the periodic Galerkin method. But the steady flow induced by the harmonic flow can not be considered. The advanced periodic Galerkin method is capable of solving the steady component induced by the harmonic component and the interaction between the steady and harmonic components. The dependent variables, velocity and water elevation, are represented by the Fourier series expansion in terms of integer values of a base frequency ω.

$$u_\beta = u_\beta^{(0)} + \frac{1}{2}\sum_{\tau=1}^{\infty}(u_\beta^{(\tau)}e^{i\omega\tau t} + u_\beta^{(-\tau)}e^{-i\omega\tau t}), \quad \zeta = \zeta^{(0)} + \frac{1}{2}\sum_{\tau=1}^{\infty}(\zeta^{(\tau)}e^{i\omega\tau t} + \zeta^{(-\tau)}e^{-i\omega\tau t}) \quad \cdots\cdots (12)$$

where τ is the index for harmonic components, superscript 0 refers to the steady component, ω is angular frequency and i is the imaginary unit. The harmonic components have a pair of conjugate complex which are superscripted τ and $-\tau$. This is the main point in the advanced periodic Galerkin method. The weighting function in time is expressed in the following form:

$$u_\beta = u_\beta^{*(0)} + \frac{1}{2}\sum_{\tau=1}^{\infty}(u_\beta^{*(\tau)}e^{i\omega\tau t} + u_\beta^{*(-\tau)}e^{-i\omega\tau t}), \quad \zeta = \zeta^{*(0)} + \frac{1}{2}\sum_{\tau=1}^{\infty}(\zeta^{*(\tau)}e^{i\omega\tau t} + \zeta^{*(-\tau)}e^{-i\omega\tau t}) \quad \cdots\cdots (13)$$

Where, $u_\beta^{*(0)}$, $u_\beta^{*(\tau)}$, $u_\beta^{*(-\tau)}$, $\zeta^{*(0)}$, $\zeta^{*(\tau)}$ and $\zeta^{*(-\tau)}$ are the arbitrary weighting values. Multiplying equation (10) and (11) by weighting function (13), integrating over one period $[0, T=2\pi/\omega]$, rearrenging the terms using the arbitrariness of the weighting values, a set of weighted residual equations can be obtained.

$$\int_0^T \boldsymbol{F}_{\alpha k}dt=0, \quad \int_0^T e^{-i\omega\tau t}\boldsymbol{F}_{\alpha k}dt=0, \quad \int_0^T e^{i\omega\tau t}\boldsymbol{F}_{\alpha k}dt=0 \quad (\tau=1,2,3,\cdots\infty) \quad \cdots\cdots (14)$$

$$\int_0^T \boldsymbol{G}_{\lambda}dt=0, \quad \int_0^T e^{-i\omega\tau t}\boldsymbol{G}_{\lambda}dt=0, \quad \int_0^T e^{i\omega\tau t}\boldsymbol{G}_{\lambda}dt=0 \quad (\tau=1,2,3,\cdots\infty) \quad \cdots\cdots (15)$$

Substituting equation (12) into equation (14) and (15) and manipulating using orthogonality condition of exponential function, the discrete nonlinear simultaneous equation system can be derived in the following form, steady state from equations (14)-1 and (15)-1:

$$\boldsymbol{K}_{\alpha\beta\gamma j}\left[u_{\beta j}^{(0)}u_{\gamma k}^{(0)} + \frac{1}{4}\sum_{v=1}^{\infty}\sum_{\chi=1}^{\infty}\{u_{\beta j}^{(v)}u_{\gamma k}^{(-\chi)} + u_{\beta j}^{(-v)}u_{\gamma k}^{(\chi)}\}\delta(v,\chi)\right] + \boldsymbol{H}_{\alpha k\mu}\zeta_\mu^{(0)} + \boldsymbol{S}_{\alpha k\beta j}u_{\beta j}^{(0)} = \hat{\boldsymbol{\Omega}}_{\alpha k}^{(0)} \quad \cdots\cdots (16)$$

$$\boldsymbol{B}_{\lambda\mu\beta j}h_\mu u_{\beta j}^{(0)} = 0 \quad \cdots\cdots (17)$$

harmonic component from equation (14)-2 and (15)-2 ($\tau=1,2,3,\cdots\infty$):

$$\frac{1}{2}i\omega\tau\boldsymbol{M}_{\alpha k\beta j}u_{\beta j}^{(\tau)} + \frac{1}{2}\boldsymbol{S}_{\alpha k\beta j}u_{\beta j}^{(\tau)} + \frac{1}{2}\boldsymbol{H}_{\alpha k\mu}\zeta_\mu^{(\tau)} + \boldsymbol{K}_{\alpha\beta\gamma j}\left[\frac{1}{2}u_{\beta j}^{(0)}u_{\gamma k}^{(\tau)} + \frac{1}{2}u_{\beta j}^{(\tau)}u_{\gamma k}^{(0)}\right.$$
$$\left. + \sum_{v=1}^{\infty}\sum_{\chi=1}^{\infty}\left\{\frac{1}{4}u_{\beta j}^{(v)}u_{\gamma k}^{(\chi)}\delta(\tau, v+\chi) + \frac{1}{4}u_{\beta j}^{(v)}u_{\gamma k}^{(-\chi)}\delta(\tau, v-\chi) + \frac{1}{4}u_{\beta j}^{(-v)}u_{\gamma k}^{(\chi)}\delta(\tau, -v+\chi)\right\}\right] = \hat{\boldsymbol{\Omega}}_{\alpha k}^{(\tau)} \quad \cdots\cdots (18)$$

$$\frac{1}{2}i\omega\tau\boldsymbol{A}_{\lambda\mu}\zeta_\mu^{(\tau)} + \frac{1}{2}\boldsymbol{B}_{\lambda\mu\beta j}h_\mu u_{\beta j}^{(\tau)} = 0 \quad \cdots\cdots (19)$$

and conjugate harmonic component from equation (14)-3 and (15)-3 ($\tau=1,2,3,\cdots\infty$):

$$\frac{-1}{2}i\omega\tau\boldsymbol{M}_{\alpha k\beta j}u_{\beta j}^{(-\tau)} + \frac{1}{2}\boldsymbol{S}_{\alpha k\beta j}u_{\beta j}^{(-\tau)} + \frac{1}{2}\boldsymbol{H}_{\alpha k\mu}\zeta_\mu^{(-\tau)} + \boldsymbol{K}_{\alpha\beta\gamma j}\left[\frac{1}{2}u_{\beta j}^{(0)}u_{\gamma k}^{(-\tau)} + \frac{1}{2}u_{\beta j}^{(-\tau)}u_{\gamma k}^{(0)}\right.$$
$$+ \sum_{v=1}^{\infty}\sum_{\chi=1}^{\infty}\left\{\frac{1}{4}u_{\beta j}^{(v)}u_{\gamma k}^{(-\chi)}\delta(-\tau, v-\chi) + \frac{1}{4}u_{\beta j}^{(-v)}u_{\gamma k}^{(\chi)}\delta(-\tau, -v+\chi)\right.$$
$$\left.\left. + \frac{1}{4}u_{\beta j}^{(-v)}u_{\gamma k}^{(-\chi)}\delta(-\tau, -v-\chi)\right\}\right] = \hat{\boldsymbol{\Omega}}_{\alpha k}^{(-\tau)} \quad \cdots\cdots (20)$$

$$\frac{-1}{2}i\omega\tau\boldsymbol{A}_{\lambda\mu}\zeta_\mu^{(-\tau)} + \frac{1}{2}\boldsymbol{B}_{\lambda\mu\beta j}h_\mu u_{\beta j}^{(-\tau)} = 0 \quad \cdots\cdots (21)$$

where

$$\delta(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{otherwise} \end{cases}$$

Compared with the conventional periodic method and the present method, the steady formulation (16) and (17) are quite different. The steady equation of the present method has the harmonic term which consists of conjugate harmonic components. Therefore, the steady component induced by the harmonic component can be calculated by the present method. These sets of non-linear equation systems can be solved by the Newton Raphson method.

5. NUMERICAL CALCULATION

The steady component which is occurred by the harmonic component can not be calculated by the conventional periodic Galerkin method. This steady component can be calculated by the present method.

Considering the steady component induced by the harmonic component, it is convenient to assume that index τ in expansion of the series in equation (12) is chosen as $\tau=1$. Thus for practical computation, the following three equation systems are derived.
The steady equation is obtained from equation (16) and (17) as :

$$\boldsymbol{K}_{\alpha\beta\gamma j}\left(u^{(0)}_{\beta j}u^{(0)}_{\gamma k}+\frac{1}{4}u^{(-1)}_{\beta j}u^{(1)}_{\gamma k}+\frac{1}{4}u^{(1)}_{\beta j}u^{(-1)}_{\gamma k}\right)+\boldsymbol{H}_{\alpha k\mu}\zeta^{(0)}_{\mu}+\boldsymbol{S}_{\alpha k\beta j}u^{(0)}_{\beta j}=\hat{\boldsymbol{\Omega}}^{(0)}_{\alpha k} \qquad (22)$$

$$\boldsymbol{B}_{\lambda\mu\beta j}h_{u}u^{(0)}_{\beta j}=0 \qquad (23)$$

harmonic component from equation (18) and (19) as :

$$\frac{1}{2}i\omega\boldsymbol{M}_{\alpha k\beta j}u^{(1)}_{\beta j}+\frac{1}{2}\boldsymbol{K}_{\alpha\beta\gamma j}(u^{(0)}_{\beta j}u^{(1)}_{\gamma k}+u^{(1)}_{\beta j}u^{(0)}_{\gamma k})+\frac{1}{2}\boldsymbol{S}_{\alpha k\beta j}u^{(1)}_{\beta j}+\frac{1}{2}\boldsymbol{H}_{\alpha k\mu}\zeta^{(1)}_{\mu}=\hat{\boldsymbol{\Omega}}^{(1)}_{\alpha k} \qquad (24)$$

$$\frac{1}{2}i\omega\boldsymbol{A}_{\lambda\mu}\zeta^{(1)}_{\mu}+\frac{1}{2}\boldsymbol{B}_{\lambda\mu\beta j}h_{\mu}u^{(1)}_{\beta j}=0 \qquad (25)$$

and conjugate harmonic component is obtained from equation (20) and (21) as :

$$\frac{-1}{2}i\omega\boldsymbol{M}_{\alpha k\beta j}u^{(-1)}_{\beta j}+\frac{1}{2}\boldsymbol{K}_{\alpha\beta\gamma j}(u^{(0)}_{\beta j}u^{(-1)}_{\gamma k}+u^{(-1)}_{\beta j}u^{(0)}_{\gamma k})+\frac{1}{2}\boldsymbol{S}_{\alpha k\beta j}u^{(-1)}_{\beta j}+\frac{1}{2}\boldsymbol{H}_{\alpha k\mu}\zeta^{(-1)}_{\mu}=\hat{\boldsymbol{\Omega}}^{(-1)}_{\alpha k} \qquad (26)$$

$$\frac{-1}{2}i\omega\boldsymbol{A}_{\lambda\mu}\zeta^{(-1)}_{\mu}+\frac{1}{2}\boldsymbol{B}_{\lambda\mu\beta j}h_{\mu}u^{(-1)}_{\beta j}=0 \qquad (27)$$

(1) One-dimensional numerical calculation

The numerical calculation in one-dimensional flow is carried out. These results are compared with the analytical solution which are introduced in appendix.

a) Periodic flow coupled with the steady flow

The numerical calculations are carried out using the mesh idealization as shown in Fig. 2 and the boundary condition in Table 1. The length of analytical domain, gravity acceleration and water depth are given 32 km, 10 m/sec^2 and 10 m. The numerical calculation is carried out for various values of steady flow U which is 0.25, 0.5, 0.75 and 1.0 m/sec from the top respectively in Fig. 3. The amplification factor is compared with the numerical and the analytical solutions in Fig. 3.

Both solutions show reasonable agreement as shown in Fig. 3. The distribution of water elevation is

Table 1 The condition for one-dimensional numerical calculation.

Steady component	Normal velocity = 0	on $\overline{AB}$ and $\overline{CD}$
	Normal velocity = Um/sec	on $\overline{AD}$ and $\overline{BC}$
	Water elevation = 0	on $\overline{AD}$
Harmonic component	Normal velocity = 0	on $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$
	periodic velocity=1$e^{i\omega t}$m/sec	on AD

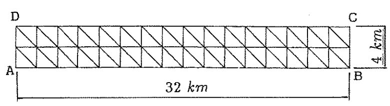


Fig. 2 Mesh idealization for one-dimensional shallow water flow.

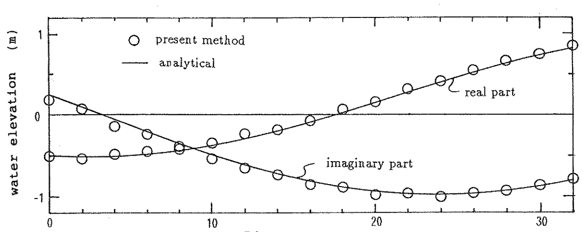


Fig. 4 Comparison of the water elevation in harmonic component.

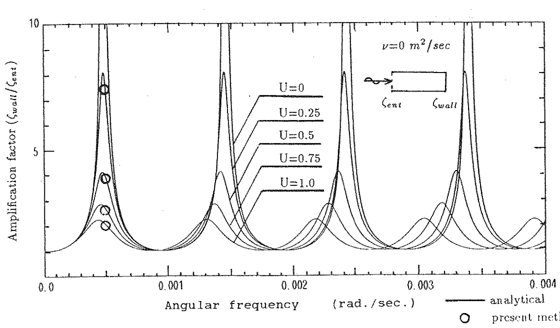


Fig. 3 Variation of the amplification factor due to steady flow.

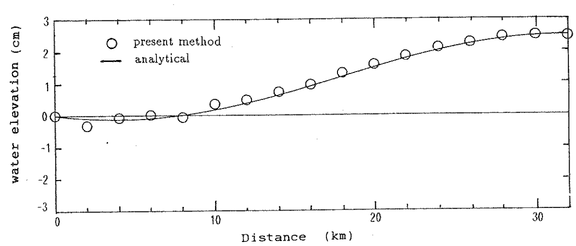


Fig. 5 Comparison of the water elevation in steady component.

compared and is shown in Fig. 4. For this calculation, gravity acceleration, water depth, eddy viscosity ν, steady flow U and angular frequency ω are used 10 m/sec^2, 10 m, 100 m^2/sec, 5 m/sec and 0.0005/sec, respectively.

b) One-dimensional steady flow analysis

The numerical calculation of the steady flow is performed. The computed condition is the same as in section a). The numerical result is well in agreement with the analytical solution as shown in Fig. 5. The water elevation of steady flow is smaller than that of harmonic flow for the order of 2.

(2) Numerical simulation of tidal circulation

The two numerical examples of tidal circulation are performed. One is the simulation of experimental tidal circulation in the model basin and another is the tidal circulation in Tokyo Bay.

a) Tidal circulation in model basin

The experiment of residual current induced by the tidal flow with the use of a hydraulic model was carried out by Yanagi[12]. The right half of the basin is a model bay, which is a square of 5 m sides with a one sided mouth of 1 m and the water depth of 0.1 m. In the experiment, the period of tide was 6 minutes and the tidal range 1 cm at the plunger. The steady residual flow was obtained in 6 tidal periods after the start of tide generation. Water elevation and velocity were measured during the periods from the 6 th period to the 10th period. The numerical computation is performed to simulate the phenomena of the experimental results. The calculation conditions and mesh idealization are listed in Table 2 and shown in Fig. 6. The residual flow is compared with the experimental data in Fig. 7. The two cases of numerical calculation are carried out. Eddy viscosity is assumed as 10 cm^2/sec and 100 cm^2/sec. The former is in good agreement with the experimental result, but in the case of eddy viscosity=100 cm^2/sec the velocity of the circulation is very smaller than the experimental result. The periodic flow in both cases is obtained as in almost the same pattern. The computational aspects of the harmonic flow is significantly influenced by the frequency. The steady flow is sensitive to the viscosity. The viscosity effect is more significant for the steady flow analysis.

Table 2 The calculation condition of tidal circulation in model basin.

Steady component	Normal velocity = 0	on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$ and $\overline{HA}$
	Water elevation = 0	on $\overline{HA}$
Harmonic component	Normal velocity = 0	on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$
	Periodic water elevation = 0.5 $e^{i\omega t}$ cm	on $\overline{HA}$

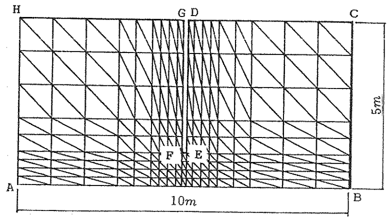
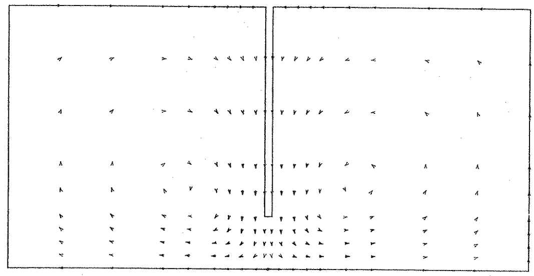
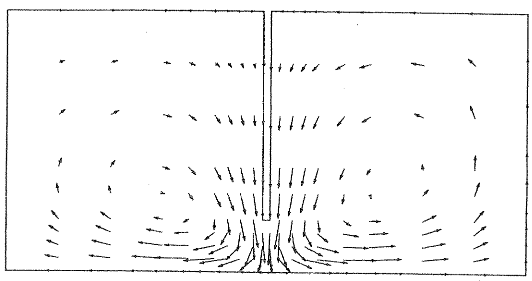


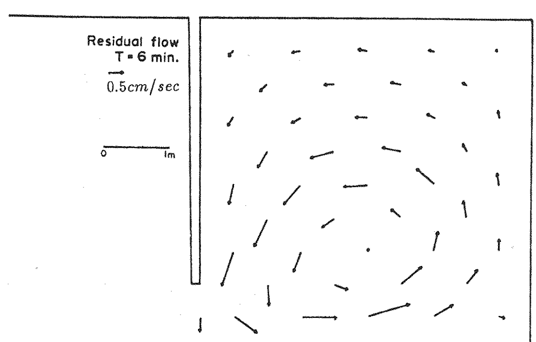
Fig. 6 Mesh idealization for tidal circulation in model basin.



(a) Numerical residual flow 0.5cm/sec (ν = 100cm^2/sec)



(b) Numerical residual flow 0.5cm/sec (ν = 10cm^2/sec)



(c) Experimental residual flow[12]

Fig. 7 Comparison of the residual flow in the model basin.

b) The simulation of residual circulation in Tokyo Bay

The residual circulation analysis in Tokyo Bay is performed to verify the possibility of application. The configulation of Tokyo Bay is modeled with reference to the Maritime chart of Tokyo datum[13] (see in Fig. 9). The finite element idealization is shown in Fig. 8. Total numbers of nodes and elements are 1 360 and 637 respectively. For the boundary condition, the amplitude of the water elevation on the entrance of the bay is given 0. 42 cm which is obtained by combined the amplitude of M_2 and S_2 tide near the entrance[2] and the normal velocity is set at zero on the other boundary in harmonic component. With the boundary condition in steady component the normal velocity is zero on all the boundaries and the water elevation is given as zero on the entrance of the bay. The period of tide is set 12 hours and eddy viscosity is assumed 50 m²/sec. For periodic flow the numerical results are compared with the measurements in the Chart of Tidal Streams in Maritime Safety Agency[14] which is shown in Fig. 10. Both data shows to be well in agreement. In numerical results near Futtsu, the relatively large velocities are obtained. These are computed due to the characteristics of the land configulation, namely, the region is straitened by the Cape

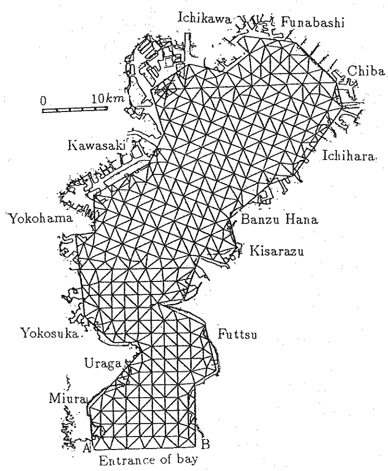


Fig. 8 Mesh idealization in Tokyo Bay.

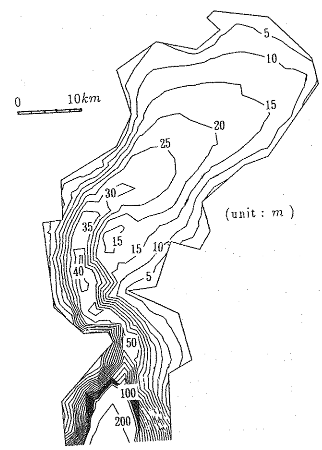
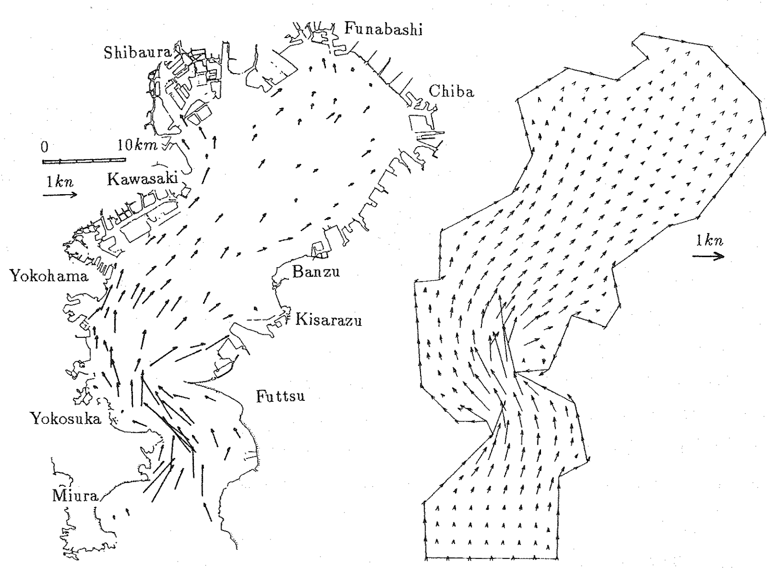


Fig. 9 Configulation of water depth.



(a) Measurements[14] (b) Present method

Fig. 10 Comparison of maximum North West stream in Tokyo Bay.

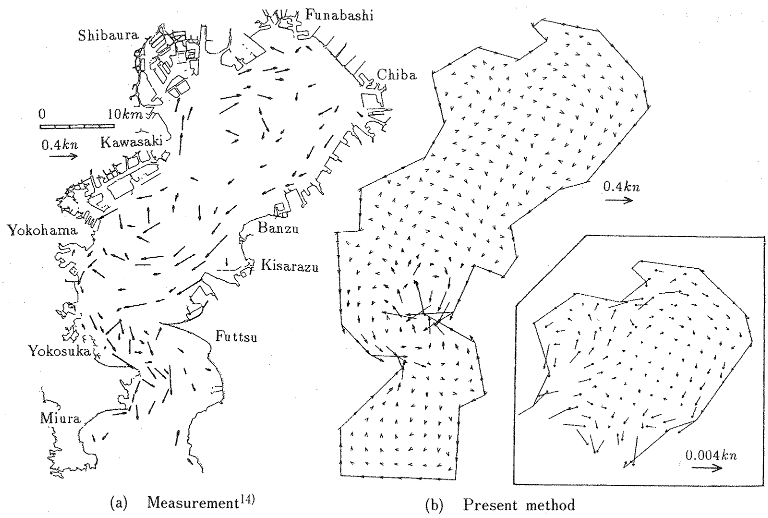


Fig. 11 Comparison of the residual current in Tokyo Bay.

of Futtsu, the water depth in the area is significantly shallow compared with other area and the gradient of water depth is quite steep, etc. To overcome these discrepancies the finer finite element mesh idealization should be used.

For the residual flow, the numerical results are compared with the measurements in Fig. 11. The numerical results and measurements have a clockwise circulation which is located from Kawasaki to Banzu. The area off Kawasaki through Miura has current toward the south. The flow pattern of the computed and measured results show quite good agreement, but the absolute value shows a slight discrepancy. This is because the present computation include several approximations, i. e., 1) the computation does not consider the wind effect, 2) the discharge from the river is not considered, 3) the effects of periodic flow longer than a one-day period are not included, 4) the eddy viscosity is taken as a constant over the flow domain, 5) the mesh idealization used is rather coarse, etc. Taking those effects into consideration, it is seen that a reasonably good computation can be performed.

6. CONCLUSION

This paper has presented the finite element method of discretizing space and time functions using the Galerkin method. To discretize the space function the conventional finite element method has been used. The quadratic interpolation function is used for velocity and the linear is used for water elevation. For the time integration, the conventional periodic Galerkin method[4)] has been improved to be able to calculate the steady flow which is occurred by the harmonic flow. The interpolation function is taken as a pair of conjugate complex exponential functions of the periodic motion. The numerical solutions in one-dimensional shallow water equation show quite good correspondence with the analytical Galerkin solutions. The residual flow can also be computed by the present method. The counterclockwise circulation in right side of the model basin can be computed which is reasonably well suited to the experimental consideration. In the residual circulation in Tokyo Bay, the clockwise circulation is obtained but the absolute value of velocity in the residual flow is not matched by the measurement data. This is due to the several other origins. The advanced periodic Galerkin method has shown to enlarge the application field of the periodic method.

APPENDIX ONE-DIMENSIONAL ANALYSIS

One-dimensional analysis is suitable to clarify the adaptability of the present method. The analytical solution of one-dimensional shallow water equation is introduced.

(1) One-dimensional periodic flow

The one-dimensional shallow water equation is :

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \zeta}{\partial x}-2\nu\frac{\partial^2 u}{\partial x^2}=0,\quad \frac{\partial \zeta}{\partial t}+h\frac{\partial u}{\partial x}=0 \qquad (28)$$

Eliminating the water elevation ζ from equations (28), the non-linear wave equation with u variable can be derived as :

$$\frac{\partial^2 u}{\partial t^2}-gh\frac{\partial^2 u}{\partial x^2}-2\nu\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)+u\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}=0 \qquad (29)$$

The velocity can be assumed as :

$$u=U+\frac{1}{2}u^{(1)}e^{i\omega t}+\frac{1}{2}u^{(-1)}e^{-i\omega t} \qquad (30)$$

where U is constant steady flow and $u^{(1)}$ and $u^{(-1)}$ are a pair of conjugate harmonic components. Substituting equation (30) into equation (29) multiplying $e^{-i\omega t}$ both side of the resulting equation and integrating over the one period, the analytical solution can be obtained as follows :

$$u=Ae^{\frac{i\omega}{2\kappa}(U+\sqrt{U^2+4\kappa})x}+Be^{\frac{i\omega}{2\kappa}(U-\sqrt{U^2+4\kappa})x} \qquad (31)$$

Substituting (31) into equation (28)-2, the water elevation ζ is derived.

$$\zeta=\frac{-h}{i\omega}\left\{A\frac{i\omega}{2\kappa}(U+\sqrt{U^2+4\kappa})e^{\frac{i\omega}{2\kappa}(U+\sqrt{U^2+4\kappa})x}+B\frac{i\omega}{2\kappa}(U-\sqrt{U^2+4\kappa})e^{\frac{i\omega}{2\kappa}(U-\sqrt{U^2+4\kappa})x}\right\} \qquad (32)$$

where $\kappa=gh+2i\omega$, A and B are determined by the boundary condition. The variation of response which is the ratio of the water elevation at the wall to that at the entrance versus frequency is shown in Fig. 3. The ratio of water elevation has several peak values corresponding to the frequency of resonance. The larger the steady flow, the lower the peak values. At high frequency domain the steady flow gives strong significance to the characteristics of periodic flow.

(2) One-dimensional steady flow

One-dimensional shallow water equation is described as in equation (28). The velocity and water elevation are assumed to be expressed as :

$$u=U+\frac{1}{2}u^{(1)}e^{i\omega t}+\frac{1}{2}u^{(-1)}e^{-i\omega t},\quad \zeta=\zeta^{(0)}+\frac{1}{2}\zeta^{(1)}e^{i\omega t}+\frac{1}{2}\zeta^{(-1)}e^{-i\omega t} \qquad (33)$$

Substituting equation (33) into (28), the Galerkin method is applied in time domain. The steady component is obtained by integration over one period as follows :

$$\frac{\partial U}{\partial x}=0,\quad \frac{\partial \zeta^{(0)}}{\partial x}=\frac{-1}{4g}\left\{u^{(1)}\frac{\partial u^{(-1)}}{\partial x}+u^{(-1)}\frac{\partial u^{(1)}}{\partial x}\right\} \qquad (34)$$

The steady velocity gradient is zero. That means the velocity of steady flow U is constant in the whole domain. The water elevation of steady flow can be calculated by numerical integration starting from $\zeta=0$ at $x=0$.

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