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SENSITIVITIES OF PARAMETERS DUE TO MODEL ERRORS AND MEASUREMENT NOISES IN STRUCTURAL IDENTIFICATION PROBLEMS

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This paper deals with the development of sensitivities of structural parameter estimates with respect to model errors and measurement noises.

Structural identification algorithm is normally developed based on the concept of least square method. Then the determination of parameters is treated as an optimization problem and the parameter estimates are an optimum solution of the problem. Hence the sensitivities of the parameters with respect to errors and noises can be formulated in the manner similar to sensitivities of optimum solutions.

Numerical examples are presented to demonstrate an effectiveness of the method proposed.

Keywords: *structural identification, sensitivity, model error, input noise, response noise*

1. INTRODUCTION

In order to verify the seismic safety of structures or to obtain informations for aseismic design, attempts have been made to measure the dynamic behavior of existing structures and to estimate their dynamic characteristics by observing microtremors and earthquakes continuously or by conducting strong vibration experiments[1)~5)].

Using the concept of the non-linear least square method, Hanada *et al.*[6)] identified the stiffness and the damping coefficient of a given structure on the condition that the residual of the characteristic equation of the system be minimized. In a study aimed at identifying system parameters, Distefano and Rath[7)] introduced a method of identification on the basis of the non-linear filter and invariant imbedding, while Simonian[8),9)] adopted dynamic programming for the same purpose. The methods of invariant imbedding and dynamic programming, although they differ in a conception, lead to identical governing equations[10)]. Shah and Udwadia[11)] identified the system constants based on the Gauss-Newton method together with the concept of point matching and further referred to the optimum sensor locations. Hoshiya and Saito[12),13)], and Maruyama *et al.*[14)], on the other hand, used the Kalman filter to estimate the system constants and dynamic characteristics. Applying the Gauss-Newton method, Matsui and Kurita[15)] derived a method for estimating the system constants using the data within an arbitrary time interval extracted from acceleration records while simultaneously estimating the initial displacement and velocity of the system at the same interval.

Although a number of methods for estimating the system constants and dynamic characteristics of

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structures have been reported on, as explained above, few studies have been made on the effects of the noise in the measured data and of the error in structural model on the estimation of unknown parameters.

This paper introduces a method of obtaining the sensitivities of parameters unknown or to be identified with respect to the noise and error. Since the estimated values in structural identification are a solution of optimization problem, the sensitivity of the parameters namely corresponds to the sensitivity of the optimum solution[16]. Calculated sensitivity values make it possible to determine the degree of accuracy required of prescribed constants in a structural model, as well as the accuracy required of the measured data. On the other hand, if the sensitivity of unknown parameters with respect to a noise is too large at a measuring point, it may be said that the point is inappropriate location to place a sensor on. Hereunder, the sensitivity formulas are theoretically developed and simple examples are presented for their verification.

2. EVALUATION FUNCTION IN THE IDENTIFICATION PROBLEM

When a structure consisting of N degrees of freedom is excited by dynamic external force, its equation of motion is given by

$$\boldsymbol{M}\ddot{\boldsymbol{z}}+\boldsymbol{C}\dot{\boldsymbol{z}}+\boldsymbol{K}\boldsymbol{z}=\boldsymbol{Q}(t) \quad \cdots\cdots (1)$$

where the symbols signify respectively,

$\boldsymbol{M}$: $N\times N$ mass matrix
$\boldsymbol{C}$: $N\times N$ damping matrix
$\boldsymbol{K}$: $N\times N$ stiffness matrix
$\boldsymbol{Q}(t)$: $N\times 1$ dynamic external vector
$\ddot{\boldsymbol{z}}$: $N\times 1$ acceleration response vector
$\dot{\boldsymbol{z}}$: $N\times 1$ velocity response vector
$\boldsymbol{z}$: $N\times 1$ displacement response vector

The initial conditions in Eq. (1) are given at $t=t_0$ as,

$$\boldsymbol{z}(t_0)=\boldsymbol{a}, \quad \dot{\boldsymbol{z}}(t_0)=\boldsymbol{b} \quad \cdots\cdots (2)$$

which represent the initial displacement and the initial velocity respectively. Let the unknown parameters in Eq. (1) be $\boldsymbol{X}=\{X_j\}$ $(j=1,\cdots,M)$ and the known constants be $\boldsymbol{Y}=\{Y_l\}$ $(l=1,\cdots,L)$. Then, the response vectors $\ddot{\boldsymbol{z}}$, $\dot{\boldsymbol{z}}$ and $\boldsymbol{z}$ are not only functions of time but are also dependent on $\boldsymbol{X}$ and $\boldsymbol{Y}$. When the measured acceleration record at an observation point i is given by $\ddot{u}_i$ and the corresponding analyzed acceleration by $\ddot{z}_i$, considering noise $\varepsilon_i(t)$ in $\ddot{u}_i$, the following relationship holds.

$$\ddot{u}_i(t)=\ddot{z}_i(t)+\varepsilon_i(t) \qquad i\in A \quad \cdots\cdots (3)$$

where A is a set of observation points. By employing the concept of the least square method, the evaluation function J may be defined as,

$$J(\boldsymbol{X},\boldsymbol{Y})=\frac{1}{2}\int_{t0}^{t1}\sum_{i\in A} w_i(\ddot{u}_i-\ddot{z}_i)^2 dt \quad \cdots\cdots (4)$$

where w_i is the weight coefficient. Since there is no appropriate means of selecting w_i at present, $w_i=1$ $(i\in A)$ is assumed in this paper.

Let the true value of the constant vector of $\boldsymbol{Y}$ be $\overline{\boldsymbol{Y}}$. The unknown parameter $\boldsymbol{X}$ can be determined from $\partial J/\partial X_j=0$ $(j=1,\cdots,M)$, which are necessary conditions for minimizing the value of Eq. (4). Let values of $\boldsymbol{X}$ and J be $\overline{\boldsymbol{X}}$ and $J_{\min}$ respectively, when Eq. (4) takes a minimum value. Then, it follows,

$$J_{\min}=J(\overline{\boldsymbol{X}},\overline{\boldsymbol{Y}})=\frac{1}{2}\int_{t0}^{t1}\sum_{i\in A} w_i(\bar{\ddot{u}}_i-\ddot{z}_i)^2 dt \quad \cdots\cdots (5)$$

where $\bar{\ddot{u}}_i$ is the measured response from which a noise is completely removed by some means. When all these conditions are satisfied, $J_{\min}$ assumes zero. Also, to distinguish noise contaminated input wave from noiseless wave, the former is denoted by $\ddot{y}_0$ and the latter by $\bar{\ddot{y}}_0$.

In the evaluation of sensitivity, $\partial\ddot{z}/\partial X_j$, $\partial\dot{z}/\partial X_j$ and $\partial z/\partial X_j$ $(j=1,\cdots,M)$ are required, which are partial differentials of response vectors with respect to each unknown parameter X_j. After differentiating Eq. (1) by X_j, the following equation will be obtained[17].

$$\boldsymbol{M}\frac{\partial \ddot{z}}{\partial X_j}+\boldsymbol{C}\frac{\partial \dot{z}}{\partial X_j}+\boldsymbol{K}\frac{\partial z}{\partial X_j}=\frac{\partial \boldsymbol{Q}}{\partial X_j}-\frac{\partial \boldsymbol{M}}{\partial X_j}\ddot{z}-\frac{\partial \boldsymbol{C}}{\partial X_j}\dot{z}-\frac{\partial \boldsymbol{K}}{\partial X_j}z \quad (j=1,\cdots,M) \tag{6}$$

Initial conditions for the above equation are

i) when X_j is neither initial displacement nor initial velocity :

$$\frac{\partial z}{\partial X_j}(t_0)=\{0\},\quad \frac{\partial \dot{z}}{\partial X_j}(t_0)=\{0\} \tag{7}$$

ii) when X_j is initial displacement :

$$\frac{\partial z}{\partial X_j}(t_0)=\{1\},\quad \frac{\partial \dot{z}}{\partial X_j}(t_0)=\{0\} \tag{8}$$

iii) when X_j is initial velocity :

$$\frac{\partial z}{\partial X_j}(t_0)=\{0\},\quad \frac{\partial \dot{z}}{\partial X_j}(t_0)=\{1\} \tag{9}$$

Eqs. (8) and (9) are necessary to solve Eq. (6) when either the initial displacement or the initial velocity is unknown. Eqs. (7) denote the initial conditions when both initial conditions are prescribed. In this case, both initial displacement and velocity are considered to be included in $\boldsymbol{Y}$. $\{1\}$ refers to an $N\times 1$ vector in which one element corresponding to unknown parameter is 1 and the rests are zero.

3. EFFECTS OF MODEL ERRORS

(1) Derivation of formulas

In the structural identification problem, some parameters among masses and stiffnesses are treated as known and constant and all other unknown parameters are determined by minimizing the discrepancies between measured and calculated values. In case there are errors in the parameters provided as known, the value of $\boldsymbol{X}$, the result of identification, inevitably reflects the errors. If the sensitivity of $\boldsymbol{X}$ with respect to Y_l is obtained and the amount of error in Y_l is known, the effect of the error on $\boldsymbol{X}$ can be estimated. Conversely, if a certain degree of accuracy is required for the estimation of $\boldsymbol{X}$, the accuracy necessary for the known parameters can be evaluated. When an error ΔY_l exists in Y_l, for which $\overline{Y}_l$ is a true value, its effect on $\overline{X}_j$ may be written as $(\partial X_j/\partial Y_l)\Delta Y_l$. Namely

$$J=\frac{1}{2}\int_{t0}^{t1}\sum_{i\in A} w_i\left\{\ddot{\overline{u}}_i-\ddot{z}_i\left(\overline{\boldsymbol{X}}+\frac{\partial \boldsymbol{X}}{\partial Y_l}\Delta Y_l,\ \overline{Y}_l+\Delta Y_l\right)\right\}^2 dt \tag{10}$$

where $\ddot{\overline{u}}_i$ is the measured but noiseless data referred to in the previous section. If ΔY_l is small enough in magnitude, its effect on $\boldsymbol{X}$ will also be small. Hence, Eq. (10) will be written in the first order approximation as

$$J=\frac{1}{2}\int_{t0}^{t1}\sum_{i\in A} w_i\left\{\ddot{\overline{u}}_i-\ddot{z}_i(\overline{\boldsymbol{X}})-\sum_{j=1}^{M}\frac{\partial \ddot{z}_i}{\partial X_j}\frac{\partial X_j}{\partial Y_l}\Delta Y_l-\frac{\partial \ddot{z}_i}{\partial Y_l}\Delta Y_l\right\}^2 dt \tag{11}$$

Since $\ddot{\overline{u}}_i$ agrees with $\ddot{z}_i(\overline{\boldsymbol{X}})$, the first two terms in { } disappears. $\partial X_j/\partial Y_l$ is the sensitivity of unknown parameters with respect to the model error, which implies that the estimated values of the unknown parameters are larger than their true values by $\partial X_j/\partial Y_l$ when there is a unit error in Y_l. Letting $\lambda_{jl}=\partial X_j/\partial Y_l$, Eq. (11) can be rewritten as

$$J=\frac{1}{2}\int_{t0}^{t1}\sum_{i\in A} w_i\left\{-\sum_{j=1}^{M}\frac{\partial \ddot{z}_i}{\partial X_j}\lambda_{jl}-\frac{\partial \ddot{z}_i}{\partial Y_l}\right\}^2(\Delta Y_l)^2 dt \tag{12}$$

Since $(\Delta Y_l)^2$, the square of the error, in Eq. (12) is arbitrary, the necessary condition to minimize Eq. (12) irrespective of the value of $(\Delta Y_l)^2$ is $\partial J/\partial \lambda_{kl}=0$; that is

$$\frac{\partial J}{\partial \lambda_{kl}}=(\Delta Y_l)^2\int_{t0}^{t1}\sum_{i\in A} w_i\left\{-\sum_{j=1}^{M}\frac{\partial \ddot{z}_i}{\partial X_j}\lambda_{jl}-\frac{\partial \ddot{z}_i}{\partial Y_l}\right\}\left(-\frac{\partial \ddot{z}_i}{\partial X_k}\right)dt=0 \tag{13}$$

After some manipulations, Eq. (13) reduces to

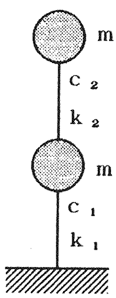
$$\sum_{j=1}^{M}\left\{\int_{t0}^{t1}\sum_{i\in A} w_i \frac{\partial \ddot{z}_i}{\partial X_j}\frac{\partial \ddot{z}_i}{\partial X_k}dt\right\}\lambda_{jl}=-\int_{t0}^{t1}\sum_{i\in A} w_i \frac{\partial \ddot{z}_i}{\partial Y_l}\frac{\partial \ddot{z}_i}{\partial X_k}dt \quad (k=1,\cdots,M) \cdots\cdots \quad (14)$$

The above formula gives a set of linear simultaneous equations for λ_{jl}, which can be solved easily.

(2) An example

To verify the theory introduced in the foregoing section, a two-degrees-of-freedom system, as shown in Fig. 1, is adopted, to which the El Centro wave (El Centro 1940 NS Imperial Valley Earthquake), $\ddot{y}_0(t)$, was applied as the input wave with a time increment $\Delta t=0.02$ seconds. The dynamic responses of the masses 1 and 2 are calculated by Newmark's β method for a time period from 0 to 20 seconds, the effect of the errors in these masses on damping coefficients c_1 and c_2, and stiffnesses k_1 and k_2, is examined. The results are given in Table 1, where $\partial c_i/\partial m_1=1.8453$, for example, implies that the estimated value of c_1 becomes 1.8453 tf·sec/m larger if the value of m_1 used in the identification is 1 tf·sec^2/m larger. The table shows that an increase of m_1 will make the estimates of c_1, k_1 and k_2 greater than the respective true values, while it makes the estimate of c_2 smaller than its true value. The table also shows that an increase of m_2, on the other hand, makes all the estimates of c_1, c_2, k_1 and k_2 larger than their respective true values. The effect on c_1 is very small, however. To study the accuracy of sensitivities in Table 1, the results from the sensitivity analyses are compared with those obtained by the method of identification in reference 15), assuming that the latter is correct. Also assuming that either of m_1 and m_2 contains 5 %, 10 % and 20 % error, the values of c_1, c_2, k_1 and k_2 are computed by the method of sensitivity which is developed in the previous section and the method described in reference 15). The results are presented in Table 2. 'Estimated' in Table 2 implies that the values are obtained using the sensitivities in Table 1 and 'Identified' means the results calculated by the method in reference 15). Since the difference between the values from two methods is relatively small, it can be stated that the method based on sensitivity analysis provide good estimates to the effect of error involved.

The effect of errors existing in the initial values will be examined next. Dynamic identification, in general, is made for an arbitrary time interval chosen from the measured response acceleration, but neither the value of the initial displacement nor the value of the initial velocity is known. Table 3 gives the sensitivities of identified parameters with respect to errors in the initial values, which are calculated by the proposed method. It shows that the effect of the error in the initial velocity is very small, while the error in the initial displacement produces comparatively large effects on the estimated values of parameters. Assuming three different magnitudes of errors on initial conditions, which are 5 %, 10 % and 20 % of respective absolute maximums of relative velocity and displacement, the parameters are estimated



$m_1=50/9.8$ (tf·sec^2/m)
$m_2=50/9.8$ (tf·sec^2/m)
$c_1=10$ (tf·sec/m)
$c_2=10$ (tf·sec/m)
$k_1=3000$ (tf/m)
$k_2=3000$ (tf/m)

Fig. 1 Two degrees-of-freedom model.

Table 1 Sensitivities with respect to error in masses.

Y_φ	$\frac{\partial c_1}{\partial Y_\varphi}$	$\frac{\partial c_2}{\partial Y_\varphi}$	$\frac{\partial k_1}{\partial Y_\varphi}$	$\frac{\partial k_2}{\partial Y_\varphi}$
m_1	1.8453	-1.6455	75.895	390.08
m_2	0.11466	3.6055	512.10	197.92

units: $\frac{\partial c_i}{\partial m_j}$ (1/sec), $\frac{\partial k_i}{\partial m_j}$ (1/sec^2)

Table 2 Estimates based on sensitivities and identified results.

		5%		10%		20%	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in m_1	c_1	10.471	10.403	10.941	10.716	11.883	11.213
	c_2	9.5802	9.6928	9.1605	9.5401	8.3209	9.4666
	k_1	3019.4	3020.8	3038.7	3045.0	3077.4	3103.7
	k_2	3099.5	3100.9	3199.0	3202.2	3398.0	3398.7
Error in m_2	c_1	9.9708	9.9316	9.9415	9.5996	9.8830	7.7830
	c_2	10.920	11.081	11.840	12.595	13.679	17.628
	k_1	3130.6	3131.3	3261.3	3261.5	3522.6	3501.4
	k_2	3050.5	3053.6	3101.0	3117.1	3202.0	3296.1

units: c_i (tf·sec/m), k_i (tf/m)

Table 3 Sensitivities with respect to error in initial values.

Y_ℓ	$\frac{\partial c_1}{\partial Y_\ell}$	$\frac{\partial c_2}{\partial Y_\ell}$	$\frac{\partial k_1}{\partial Y_\ell}$	$\frac{\partial k_2}{\partial Y_\ell}$
a_1	-24.416	124.30	9056.7	-24141
a_2	100.06	-107.14	-5207.2	12671
b_1	-2.4514	9.7793	113.80	-218.12
b_2	4.3591	-5.1787	25.568	64.596

a_1 : initial displacement of mass 1
a_2 : initial displacement of mass 2
b_1 : initial velocity of mass 1
b_2 : initial velocity of mass 2

units: $\frac{\partial c_i}{\partial a_j}$(tf·sec/m²), $\frac{\partial k_i}{\partial a_j}$(tf/m²), $\frac{\partial c_i}{\partial b_j}$(tf·sec²/m²), $\frac{\partial k_i}{\partial b_j}$(tf·sec/m²)

Table 4 Estimates from the sensitivities and identified results.

		5%		10%		20%	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in a_1	c_1	9.9753	9.6311	9.9505	8.5217	9.9011	4.3799
	c_2	10.126	11.003	10.252	13.773	10.504	23.771
	k_1	3009.2	3010.8	3018.3	3022.9	3036.7	3029.5
	k_2	2975.6	2971.3	2951.1	2938.3	2902.2	2895.0
Error in a_2	c_1	10.160	9.8433	10.321	9.0184	10.642	5.1636
	c_2	9.8282	10.687	9.6565	13.169	9.3130	23.695
	k_1	2991.7	2995.4	2983.3	2997.1	2966.6	3004.3
	k_2	3020.3	3010.5	3040.6	3003.1	3081.3	2958.9
Error in b_1	c_1	9.9649	9.9170	9.9298	9.7390	9.8596	9.0923
	c_2	10.140	10.267	10.280	10.785	10.560	12.570
	k_1	3001.6	3001.8	3003.2	3003.9	3006.5	3008.4
	k_2	2996.9	2996.4	2993.8	2992.0	2987.5	2981.6
Error in b_2	c_1	10.098	10.065	10.196	10.063	10.393	9.8360
	c_2	9.8833	9.9867	9.7667	10.187	9.5334	11.265
	k_1	3000.6	3000.8	3001.2	3001.9	3002.3	3004.9
	k_2	3001.5	3000.9	3002.9	3000.7	3005.8	2998.3

units: c_i(tf·sec/m), k_i(tf/m)

by the sensitivity approach and the method of identification. Both results are tarbulated in Table 4. The table shows that the approach presented here is applicable when the errors in the initial displacement and velocity are comparatively small, but the accuracy worsens as the errors become larger, presumably because the effects of the initial displacement and velocity attenuate quickly during the time period of 20 seconds and thus have little influence on the evaluation function. In reference 15), the initial displacement and the initial velocity are also identified more precisely when the shorter duration of response records is used. This also endorses the discussion presented above.

4. EFFECTS OF THE NOISE IN THE MEASURED RESPONSE

(1) Derivation of formulas

It is assumed that there are no errors in the structural model $\overline{Y}$, and no noises in the measured input value $\ddot{\overline{y}}_0$ and in measured response values $\ddot{\overline{u}}_i(t)$ $(i \in A,\ i \neq l)$ except the response at the measuring point l. Though measured response accelerations are, in general, the absolute accelerations, the relative accelerations, subtracting the input acceleration from the absolute response accelerations, is used for theoretical development, since the measured input values are considered to be the true values without noise here. When the measured response at measuring point l is denoted by $\ddot{u}_l$, the response $\ddot{u}_l$ can be expressed by the sum of true response $\ddot{\overline{u}}_l$ and noise $\Delta\ddot{u}_l=\varepsilon_l\eta_l(t)$; that is $\ddot{u}_l=\ddot{\overline{u}}_l+\Delta\ddot{u}_l$.

$\eta_l(t)$ is a normalized noise having a maximum (absolute) value of 1, and ε_l is a scalar which represents the magnitude of the maximum value. When the noise $\Delta\ddot{u}_l$ in measurement exists only at the point l, the identified parameter may be written as $\overline{X}+\Delta X^{(l)}$, where $\Delta X^{(l)}$ is defined as the subtraction of the true value $\overline{X}$ from the identified result X which reflects the noise $\Delta\ddot{u}_l$. Then, Eq. (4) can be prescribed as,

$$J=\int_{t0}^{t1}\left[\frac{1}{2}\sum_{\substack{i\in A\\ i\neq l}} w_i\{\ddot{u}_i-\ddot{z}_i(\overline{X}+\Delta X^{(l)})\}^2+\frac{1}{2}w_l\{\ddot{u}_l+\Delta\ddot{u}_l-\ddot{z}_l(\overline{X}+\Delta X^{(l)})\}^2\right]dt \quad\cdots\cdots (15)$$

It is reasonable to assume that X varies continuously with ε_l in $\Delta\ddot{u}_l$ and that the magnitude of $\Delta X^{(l)}$ is small when ε_l is small. Therefore,

$$\begin{aligned}\ddot{z}_i(\overline{X}+\Delta X^{(l)})&=\ddot{z}_i(\overline{X})+\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\frac{\partial X_j}{\partial\ddot{u}_l}\Delta\ddot{u}_l\\ &=\ddot{z}_i(\overline{X})+\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\frac{\partial X_j}{\partial\varepsilon_l}\frac{\partial\varepsilon_l}{\partial\ddot{u}_l}\Delta\ddot{u}_l\\ &=\ddot{z}_i(\overline{X})+\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\Gamma_{jl}\varepsilon_l \quad\cdots\cdots (16)\end{aligned}$$

where $\Gamma_{jl}=\partial X_j/\partial\varepsilon_l$. Substituting Eq. (16) into Eq. (15) and using the relationship $\ddot{z}_i(\overline{X})=\overline{\ddot{u}}_i$, the following equation is obtained after rearranging terms.

$$J=\frac{\varepsilon_l^2}{2}\int_{t0}^{t1}\sum_{\substack{i\in A\\ i\neq l}} w_i\left\{-\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\Gamma_{jl}\right\}^2 dt+\frac{\varepsilon_l^2}{2}\int_{t0}^{t1} w_e\left\{-\sum_{j=1}^{M}\frac{\partial\ddot{z}_l}{\partial X_j}\Gamma_{jl}+\eta_l(t)\right\}^2 dt \quad\cdots\cdots (17)$$

From the necessary condition for Eq. (17) to be minimum irrespective of the value of ε_l^2, with respect to the unknown variable Γ_{jl}, one obtains

$$\frac{\partial J}{\partial\Gamma_{kl}}=\varepsilon_l^2\int_{t0}^{t1}\sum_{j=1}^{M}\left(\sum_{i\in A} w_i\frac{\partial\ddot{z}_i}{\partial X_j}\frac{\partial\ddot{z}_i}{\partial X_k}\right)\Gamma_{jl}dt-\varepsilon_l^2\int_{t0}^{t1} w_e\eta_l(t)\frac{\partial\ddot{z}_l}{\partial X_k}dt=0$$

$$\therefore\quad \sum_{j=1}^{M}\left[\int_{t0}^{t1}\left(\sum_{j\in A} w_i\frac{\partial\ddot{z}_i}{\partial X_j}\frac{\partial\ddot{z}_i}{\partial X_k}\right)dt\right]\Gamma_{je}=\int_{t0}^{t1} w_e\eta_l(t)\frac{\partial\ddot{z}_l}{\partial X_k}dt \quad (k=1,\cdots,M) \quad\cdots\cdots (18)$$

where Γ_{jl} is the sensitivity with respect to the noise in measured response $\ddot{u}_l$. The above equation gives a set of linear simultaneous equations with respect to the unknown variable Γ_{jl}, which can be solved easily.

(2) An example

To verify the method introduced in the above section, the model in Fig. 1 is used together with a normalized noise shown in Fig. 2, which is a band-limited white noise in the range of 0.1 Hz to 25 Hz and is adjusted to have an absolute maximum value 1. Fig. 3 shows the response accelerations of masses 1 and 2 due to $\ddot{y}_0$ (El Centro wave treated as true excitation) when the noise in Fig. 2 is superimposed on these responses, the identified parameters are expected to differ from their true values. Sensitivities of the parameters with respect to the noise are tarbulated in Table 5. Table 6 compares damping coefficients and

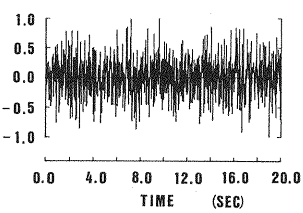
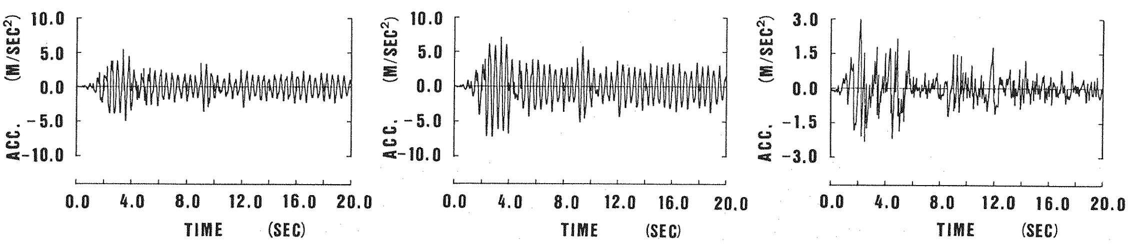


Fig. 2 Band-limited white noise.

Table 5 Sensitivities with respect to noise in measured response.

ε_ℓ	$\frac{\partial c_1}{\partial \varepsilon_\ell}$	$\frac{\partial c_2}{\partial \varepsilon_\ell}$	$\frac{\partial k_1}{\partial \varepsilon_\ell}$	$\frac{\partial k_2}{\partial \varepsilon_\ell}$
ε_1	-0.00824	0.13871	4.8032	-11.393
ε_2	-0.13882	0.05514	7.0382	-16.385

units: $\frac{\partial c}{\partial \varepsilon_\ell}$; (tf·sec³/m²), $\frac{\partial k}{\partial \varepsilon_\ell}$; (tf·sec²/m²)



a) Response acceleration of mass 1 b) Response acceleration of mass 2 c) Input acceleration

Fig. 3 Response and input acceleration waves.

Table 6 Estimates from the sensitivities and the identified results.

		5%		10%		20%	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in $\ddot{u}_1$	c_1	9.9977	9.9904	9.9955	9.9808	9.9910	9.9615
	c_2	10.038	10.048	10.076	10.097	10.152	10.193
	k_1	3001.3	3001.0	3002.6	3001.9	3005.3	3003.9
	k_2	2996.9	2997.9	2993.8	2995.9	2987.5	2991.5
Error in $\ddot{u}_2$	c_1	9.9503	9.9060	9.9006	9.8117	9.8013	9.6219
	c_2	10.020	10.128	10.039	10.255	10.079	10.508
	k_1	3002.5	3001.8	3005.0	3003.6	3010.1	3007.5
	k_2	2994.1	2995.5	2988.3	2990.9	2976.5	2981.5

units: c_i(tf·sec/m), k_i(tf/m)

stiffnesses by using the sensitivities and those by the identification method in reference 15) for three different magnitudes of errors which are chosen to be 5 %, 10 % and 20 % of maximum of corresponding responses. It may be said that the effect of noise is very small, as long as it is given by white noise.

5. EFFECT OF INPUT NOISE

(1) Derivation of formulas

An earthquake acceleration record with $\Delta t=0.02$ sec is adopted as the input. The true value of the earthquake acceleration $\ddot{y}_0(t)$ is designated by $\bar{\ddot{y}}_0$, and the measurement noise by $\Delta\ddot{y}_0(t)=\varepsilon_0\eta_0(t)$, a noise contaminated input acceleration is expressed as $\ddot{y}_0(t)=\bar{\ddot{y}}_0+\Delta\ddot{y}_0(t)$. $\eta_0(t)$ is a normalized noise whose absolute maximum value is 1. It is assumed that no error exists in the model or no noise is involved in the measured responses. Since analysis is performed by using the input acceleration, computed response reflects this noise and differs from the true response. Therefore, the identified values of unknown parameters X cannot be free from errors either. Thus, X can be written as $X=\bar{X}+\Delta X^{(0)}$, and the analyzed acceleration is denoted as $\ddot{z}_i(\bar{\ddot{y}}_0+\Delta\ddot{y}_0, \bar{X}+\Delta X^{(0)})$. Thus, the evaluation function, Eq. (4) becomes

$$J=\int_{t0}^{t1}\left[\frac{1}{2}\sum_{i\in A} w_i\{\bar{\ddot{u}}_i-\ddot{z}_i(\bar{\ddot{y}}_0+\Delta\ddot{y}_0, \bar{X}+\Delta X^{(0)})\}^2\right]dt \quad\cdots\cdots(19)$$

Since $\Delta\ddot{y}_0$ is noise, its effect will be presumed small. Hence Taylor series expansion of $\ddot{z}_i$ in Eq. (19) leads to

$$\begin{aligned}\ddot{z}_i(\bar{\ddot{y}}_0+\Delta\ddot{y}_0, \bar{X}+\Delta X^{(0)}) &\doteqdot \ddot{z}_i(\bar{\ddot{y}}_0, \bar{X})+\frac{\partial\ddot{z}_i}{\partial\ddot{y}_0}\Delta\ddot{y}_0+\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\frac{\partial X_j}{\partial\ddot{y}_0}\Delta\ddot{y}_0\\ &=\ddot{z}_i(\bar{\ddot{y}}_0, \bar{X})+\frac{\partial\ddot{z}_i}{\partial\varepsilon_0}\frac{\partial\varepsilon_0}{\partial\ddot{y}_0}\varepsilon_0\eta_0+\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\frac{\partial X_j}{\partial\varepsilon_0}\frac{\partial\varepsilon_0}{\partial\ddot{y}_0}\varepsilon_0\eta_0\\ &=\ddot{z}_i(\bar{\ddot{y}}_0, \bar{X})+\frac{\partial\ddot{z}_i}{\partial\varepsilon_0}\varepsilon_0+\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\Gamma_{j0}\varepsilon_0 \quad\cdots\cdots(20)\end{aligned}$$

where $\Gamma_{j0}=\partial X_j/\partial\varepsilon_0$ is the sensitivity with respect to the input noise.

Substitution of Eq. (20) into Eq. (19) results in

$$\begin{aligned}J&=\int_{t0}^{t1}\left[\frac{1}{2}\sum_{i\in A} w_i\left\{\bar{\ddot{u}}_i-\ddot{z}_i(\bar{\ddot{y}}_0, \bar{X})-\frac{\partial\ddot{z}_i}{\partial\varepsilon_0}\varepsilon_0-\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\Gamma_{j0}\varepsilon_0\right\}^2\right]dt\\ &=\int_{t0}^{t1}\left[\frac{1}{2}\sum_{i\in A} w_i\left\{-\frac{\partial\ddot{z}_i}{\partial\varepsilon_0}\varepsilon_0-\sum_{j=1}^{M}\frac{\partial\ddot{z}_i}{\partial X_j}\Gamma_{j0}\varepsilon_0\right\}^2\right]dt \quad\cdots\cdots(21)\end{aligned}$$

In Eq. (21), ε_0^2, the square of the maximum value of the noise, is positive and arbitrary. Therefore, as minimizing Eq. (21) is equivalent to minimizing Eq. (19), Γ_{k0} can be determined from the necessary condition for Eq. (21) to be minimum. By taking differentiation of Eq. (21) with respect to Γ_{k0}, the necessary condition is given as

Table 7 Sensitivities with respect to noise in input acceleration.

$\frac{\partial c_1}{\partial \varepsilon_0}$	$\frac{\partial c_2}{\partial \varepsilon_0}$	$\frac{\partial k_1}{\partial \varepsilon_0}$	$\frac{\partial k_2}{\partial \varepsilon_0}$
-1.7749	0.59407	-66.060	64.423

units: $\frac{\partial c_i}{\partial \varepsilon_0}$(tf·sec³/m²), $\frac{\partial k_i}{\partial \varepsilon_0}$(tf·sec²/m²)

Table 8 Estimates from the sensitivities and the identified results.

		5 %		10 %		20 %	
		Estimated	Identified	Estimated	Identified	Estimated	Identified
Error in $\ddot{y}_0$	c_1	9.7338	9.7453	9.4675	9.5195	8.9351	9.1720
	c_2	10.089	10.134	10.178	10.351	10.356	10.997
	k_1	2990.1	2989.9	2980.2	2979.5	2960.4	2957.7
	k_2	3009.7	3009.7	3019.3	3019.4	3038.7	3038.0

units: c_i(tf·sec/m), k_i(tf/m)

$$\frac{\partial J}{\partial \Gamma_{k0}}=\varepsilon_0^2\int_{t0}^{t1}\sum_{i\in A}w_i\left\{-\frac{\partial \ddot{z}_i}{\partial \varepsilon_0}-\sum_{j=1}^{M}\frac{\partial \ddot{z}_i}{\partial X_j}\Gamma_{j0}\right\}\left(-\frac{\partial \ddot{z}_i}{\partial X_k}\right)dt=0 \quad (22)$$

Rearrangement of the above equation yields

$$\sum_{j=1}^{M}\left[\int_{t0}^{t1}\left(\sum_{i\in A}w_i\frac{\partial \ddot{z}_i}{\partial X_j}\frac{\partial \ddot{z}_i}{\partial X_k}\right)dt\right]\Gamma_{j0}=-\int_{t0}^{t1}\left(\sum_{i\in A}w_i\frac{\partial \ddot{z}_i}{\partial \varepsilon_0}\frac{\partial \ddot{z}_i}{\partial X_k}\right)dt \quad (k=1,\cdots,M) \quad (23)$$

$\partial \ddot{z}_i/\partial \varepsilon_0$ in the right hand side of Eq. (23) can be computed in the following manner. The response of the structure due to the noise contaminated input $\ddot{y}_0=\ddot{\bar{y}}_0+\varepsilon_0\eta_0(t)$ is

$$\boldsymbol{M}\ddot{\boldsymbol{z}}+\boldsymbol{C}\dot{\boldsymbol{z}}+\boldsymbol{K}\boldsymbol{z}=\boldsymbol{f}(\ddot{\bar{y}}_0+\varepsilon_0\eta_0) \quad (24)$$

where $\boldsymbol{f}$ is an $N\times 1$ vector. When compared with Eq. (1), $\boldsymbol{Q}(t)=\boldsymbol{f}(\ddot{\bar{y}}_0+\varepsilon_0\eta_0)$. Partial differentiation of Eq. (24) with respect to ε_0 gives

$$\boldsymbol{M}\frac{\partial \ddot{\boldsymbol{z}}}{\partial \varepsilon_0}+\boldsymbol{C}\frac{\partial \dot{\boldsymbol{z}}}{\partial \varepsilon_0}+\boldsymbol{K}\frac{\partial \boldsymbol{z}}{\partial \varepsilon_0}=\boldsymbol{f}\eta_0 \quad (25)$$

By solving the above equation, $\partial \ddot{z}_i/\partial \varepsilon_0$ on the right side of Eq. (23) can be obtained. Eq. (23) is a set of linear simultaneous equations with respect to the unknown variable Γ_{j0}, which can be solved without so much difficulty.

(2) An example

To verify the theory induced in the preceding section, the model in Fig. 1 is used again together with the normalized band-limited white noise $\eta_0(t)$ in Fig. 2. Table 7 gives the sensitivity of the identified parameters with respect to the noise in the input wave. The El Centro wave is superimposed by three different levels of white noise to generate noise contaminated data. The magnitude of noise ε_0 are 5 %, 10 %, and 20 % of maximum of the El Centro wave. Table 8 compares the estimated values of $\boldsymbol{X}$ obtained by the sensitivity analysis and those by the identification method in reference 15). Comparatively good agreements between these data suggest that the sensitivity of unknown parameters with respect to the input noise is highly precise. Identified values which are very close to the true values also imply that the effect of the white noise on the identification is very small.

6. CONCLUSIONS

The formulas have been developed, which determine the effects (sensitivities) of model error and measurement noise on the parameter estimation in structural identification problems. This problem is a sort of optimization, in which the sensitivity of a parameter with respect to error or noise is fundamentally the same as sensitivity of the optimum solution.

From the example problems, one can conclude the following.

(1) Compared with noise in measurements, model error has larger effect on the estimated value of a parameter. Using the sensitivity, the effect of an error on the estimated value of the parameter can be calculated with considerable accuracy.

(2) Error in the initial velocity has little effect on the estimated value of a parameter, while the effect of an error in the initial displacement is not necessarily small. Though the same calculation is made on different models, the same tendency is also observed. Given the fact that both the initial displacement and the initial velocity attenuate in a short period of time due to the damping characteristic of the system in the

response analysis, the reason for the difference between the effects of the displacement and velocity is not clear.

(3) Noise involved in the response and input measurements has little effect on the estimated value of a parameter. It may be due to the reason that the white noise does not have dominant frequency characteristics.

(4) In all examples, the existence of an error or noise is assumed at only one point. If there are model errors and noises at more than one place, the estimated values may deviate more from the true values due to the interaction of these errors. There are also a few contradictory cases where on estimated value from sensitivity analysis is larger than the true value while the corresponding identified result is smaller than it. This contradiction occurs, when the sensitivity is extremely small, presumably as a result of calculation error including the effect of trapezoidal integration. The reason that an estimate of the damping coefficient is less accurate, is among others, due to the substantially smaller order of the parameter compared with that of the stiffness. This is expected to improve, however, if magnitudes of parameters are balanced or the technique of variable transformation is introduced, as is considered in structural optimization[16].

It is also confirmed that the accuracy of estimation improves when an identification is performed with the measured and calculated displacement in stead of using measured and calculated accelerations. Only one example of white noise is used in this paper, but there are no consipiquous differences even if the sensitivity is calculated using other forms of white noise as long as the system has two degrees of freedom. Though no discussion is made on noise with dominant frequency characteristics, the methods proposed here are applicable whether the noise has a particular feature or not. It has been further confirmed that the effect of an white noise on the identification is small enough from the engineering point of view, even if it has an SN ratio of 100 %. The methods developed above can provide a good estimate of the effect with considerable accuracy.

In this study, the response $\ddot{z}_i$ is expanded around, the true value of the unknown parameter, to obtain its sensitivity with respect to noise or error. Though the true value is usually unknown, all the theoretical formulas introduced above can be still utilized by adopting the estimated value $\overline{X}$ based on data containing error or noise as the true value.

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