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COLLOCATION METHOD FOR DETERMINING THE NATURAL VIBRATION CHARACTERISTICS OF CYLINDRICAL SHELLS WITH EITHER INTERNAL OR EXTERNAL FLUIDS

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This paper proposes a simple and effective method for determining the vibration characteristics of cylindrical shells with either internal or external fluids. The fluid region is treated analytically, and the collocation method is used to solve the integro-differential equations which govern the motion of the shell. The reliability and accuracy of the proposed method is illustrated through several numerical examples. Furthermore, a numerical study of the frequency characteristics of the shells is carried out. Finally, a simple equation which accurately predicts the fundamental natural periods of a wide range of shells in the swaying mode is presented.

Keywords: *cylindrical shells, vibration analysis, shell-fluid interaction, integro-differential equations, collocation method*

1. INTRODUCTION

Vibration problems of cylindrical shells in contact with a fluid are of great importance in civil engineering. In Fig. 1 we show two typical shells investigated: (1) the fluid is contained within the shell (this type will be referred to herein as an internal problem), such as in the case of storage tanks; and (2) when the shell is submerged in the fluid (this type will be referred to herein as an external problem), such as in the case of offshore structures. For the design of such structures subject to dynamic forces, it is of fundamental importance to clarify the vibration characteristics of the cylindrical shells. Such a knowledge therefore needs to be determined as accurately as possible.

It is well known that the effect of the fluid can alter the vibration characteristics of a structure. A number of studies[1)~9)] have been reported on this subject. However, most of these studies were primarily concerned with the internal problem, with a few exceptions[8),9)], and many of them were devoted to the study of the vibration characteristics of cylindrical shells with small values of n (n is the circumferential wavenumber). Moreover, comparative studies on both the internal problem and the external problem have not been reported, although it seems to be essential for a better understanding of the vibration characteristics of shells. For the shell-fluid interaction problems, it is extremely difficult to obtain a closed-form solution, primarily because of the different characteristics of the two continua. Various methods have been developed to describe the behaviour of the shell as well as the fluid. The shell has been most commonly modelled by using the finite element method (FEM), although there exist a variety of methods such as the matrix progression method[7),8)], the finite strip method[9)], and the Rayleigh-Ritz method[6),8)]. While the fluid has been treated by three approaches: (1) use of the FEM[1),3),4)]; (2) use of

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the boundary element method (BEM)[5] ; and (3) analytical treatments[2),6)~9)], using a series of generalized functions. Of the methods mentioned above, the FEM and the BEM have recently become the major means of solving the fluid-shell system. However, in using these methods, we encounter some difficulties. First, the use of the BEM* requires the existence of appropriate fundamental solutions. Second, the use of the FEM leads to a large system of equations resulting from refined discretization, especially for the infinite fluid. Therefore, alternative methods are clearly required, serving as a check for the major methods or as an efficient means of solving the problems under consideration. In the seismic design of shells, evaluation of the fundamental natural period of the shells is an important step. For this reason, several practical formulas[3),13),14)] have been proposed for the prediction of the fundamental natural period of the shells in the swaying mode ($n=1$). However, they are valid only for the internal problem and are applicable only for relatively short shells. Moreover, some of the existing formulas may produce a significant error in the fundamental natural period except for very thin shells, because in their formulas, the contribution of the shell mass to the period has been neglected.

Under the circumstances described above, this paper presents an efficient solution procedure, based on the collocation method, for the vibration analysis of cylindrical shells with either internal or external fluids. Although the collocation method[10),11)] has been used for the vibration and transient analysis of shells without the fluid, its application to the subject considered herein has been rather limited[12)]. The reason for employing the method in this study lies firstly, in the simplicity of the theory and the brevity of the associated computer code. In addition, the method yields very good results even with a reasonably small number of discrete points (collocation points), if the roots of the orthogonal polynominal are used as collocation points. In this paper, the fluid is treated analytically, and the equations of motion of the shells are reduced to the integro-differential equations in terms of the displacements of the shell. The resulting equations are then solved by using the collocation method. The present study is an extension of a previous paper by the authors[12)]. Specifically, the objectives of this paper are : (1) To investigate the effect of the fluid with emphasis on the frequency characteristics of both of the shells mentioned above ; and (2) to present a simple equation for estimating the fundamental natural period (corresponding to $n=1$) of both shells under a wide range of the geometric parameters.

2. SHELL-FLUID SYSTEM AND EQUATIONS GOVERNING FLUID MOTION

As shown in **Fig. 1** the shell is of uniform thickness h, radius a and height L, made of homogeneous, isotropic material with an elasticity modulus E, Poisson's ratio ν and shell density ρ_S. The shell is in contact with the fluid to a height H. The locations of points in the fluid and shell are specified by the cylindrical coordinate system (r, θ, z), where θ is the circumferential coordinate. The axial coordinate of the shell is denoted by x.

The fluid is considered to be incompressible and inviscid. According to linear potential flow theory, the velocity potential, Φ, must satisfy the Laplace equation

$$\Phi_{,rr}+\Phi_{,r}/r+\Phi_{,\theta\theta}/r^2+\Phi_{,zz}=0 \quad (1)$$

where the subscripts following a comma indicate differentiation. The boundary conditions are given by

$$\Phi_{,z}|_{z=H}=0 \quad (2\cdot a)$$

$$\Phi_{,t}|_{z=0}=0 \quad (2\cdot b)$$

$$\Phi_{,r}|_{r=a}=W_{,t} \quad (2\cdot c)$$

$$\Phi|_{r\to\infty}=0 \quad (2\cdot d)$$

where t=time ; W=the radial displacement of the

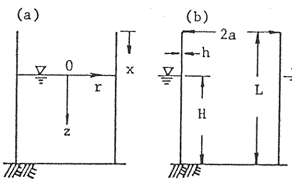

Fig. 1 Shell-fluid interaction : (a) Internal problem ; (b) External problem.

* The BEM using particular integrals, recently proposed by Wang and Banerjee[19)] for the free vibration analysis of axisymmetric solids, may be applied to the subject considered herein.

shell ; and Eq. (2·d) is applicable only to the external problem. The hydrodynamic pressure, p, acting on the inner or outer surface of the shell can be determined from the Bernoulli equation and is given by

$$p=-\rho_f \Phi_{,t} \quad (3)$$

where ρ_f=the mass density of the fluid.

3. EQUATIONS GOVERNING SHELL MOTION

The analytical formulation is based on an improved shell theory[15] with the effects of transverse shear deformation and rotary inertia. This results in the same equations as those given in Ref. 15), except for the term representing the hydrodynamic pressure due to the fluid. Therefore, the material presented in this section will be discussed briefly.

The generalized displacement field consists of the displacement components (U, V, W) in the (x, θ, r) coordinate directions and the rotation components (β_x, β_θ). The generalized displacements, the potential function Φ and the hydrodynamic pressure p are expanded in a Fourier series in the circumferential coordinate θ. The following expressions are used :

$$\left.\begin{aligned}(U, V, W)&=\frac{\sigma a^2}{Eh}\sum_n[u\cos n\theta,\quad v\sin n\theta,\quad w\cos n\theta]\,e^{i\omega t}\\ (\beta_x, \beta_\theta)&=\frac{\sigma a}{Eh}\sum_n[\bar{\beta}_x\cos n\theta,\quad \bar{\beta}_\theta\sin n\theta]\,e^{i\omega t}\\ \Phi&=\frac{i\omega\sigma a^3}{Eh}\sum_n\phi\cos n\theta e^{i\omega t},\qquad p=\rho_f\omega^2\frac{\sigma a^3}{Eh}\sum_n\bar{p}\cos n\theta e^{i\omega t}\end{aligned}\right\} \quad (4\cdot\text{a}\sim\text{d})$$

where $i=\sqrt{-1}$; σ=a reference stress ; n=the number of circumferential waves ; and ω=the natural frequency of the shell-fluid system.

Eqs. (1) and (2) are solved to obtain the expression for ϕ, and the expression for $\bar{p}$ is obtained with the use of Eq. (3). The resulting expressions are given by

$$\phi_{IN}=\bar{p}_{IN}=\sum_{r=1}^{\infty}\frac{2\,I_n(\lambda_r)}{\lambda_r I'_n(\lambda_r)}\sin(\lambda_r H\eta/a)\int_0^1 w\sin(\lambda_r H\eta/a)\,d\eta \quad (5\cdot\text{a})$$

$$\phi_{EX}=\bar{p}_{EX}=-\sum_{r=1}^{\infty}\frac{2\,K_n(\lambda_r)}{\lambda_r K'_n(\lambda_r)}\sin(\lambda_r H\eta/a)\int_0^1 w\sin(\lambda_r H\eta/a)\,d\eta \quad (5\cdot\text{b})$$

where the subscripts IN and EX hold for the internal and external problems, respectively ; $\lambda_r=(2r-1)\pi a/2H$; $I_n(\lambda_r)$ and $K_n(\lambda_r)$ =the modified Bessel functions of the order n of the first and second kind, respectively ; $I'_n(\lambda_r)$ and $K'_n(\lambda_r)$ =the derivatives of the modified Bessel functions with respect to λ_r ; and η $(=z/H)$=the nondimensional z coordinate in the fluid domain. In the evaluation of Eq. (5) only a finite number of terms, say N_r, in the truncated series are taken into account. The effect of N_r on the solution will be discussed in next section.

With the aid of Eq. (5), the equations of motion of the shell for each circumferential number can be written in the matrix form :

$$[C]\{X''\}+[D]\{X'\}+[E]\{X\}=\Omega^2([F]\{X\}+\{p\}) \quad (6)$$

in which $(\ )'=d(\ )/d\xi$, where ξ $(=x/L)$=the nondimensional axial coordinate ; $\Omega^2=\rho_s(1-\nu^2)a^2\omega^2/E$=a frequency parameter ; $[C]$, $[D]$, $[E]$ and $[F]$=the 5×5 coefficient matrices whose elements have been given in Ref. 12) ; and $\{X\}$ and $\{p\}$=the displacement and hydrodynamic pressure vectors given by

$$\{X\}^T=\{u, v, w, \bar{\beta}_x, \bar{\beta}_\theta\},\quad \{p\}^T=(0, 0, p_w, 0, 0) \quad (7\cdot\text{a, b})$$

where p_w is identical to an expression of the added mass of shell-fluid problems, and using Eq. (5) it can be written as

$$p_w=(\rho_f/\rho_s)\cdot(a/h)\cdot\bar{p}_i,\quad (i=IN, EX) \quad (8)$$

It is seen from Eq. (8) that p_w depends on the fluid-shell density ratio and the radius-to-thickness ratio. Due to the integral introduced by Eq. (8) [see Eq. (5)], Eq. (6) is the so-called integro-differential equations which govern the motion of shell.

The stress resultants that appear in the statement of the boundary conditions are N_x, $N_{x\theta}$, Q_x, M_x and $M_{x\theta}$. As before, these resultants for each Fourier harmonic are taken as

$$(N_x, N_{x\theta}, Q_x)=\sigma a \sum_n [n_x \cos n\theta, n_{x\theta} \sin n\theta, q_x \cos n\theta]\, e^{i\omega t} \quad (9\cdot a)$$

$$(M_x, M_{x\theta})=\sigma a^2 \sum_n [m_x \cos n\theta, m_{x\theta} \sin n\theta]\, e^{i\omega t} \quad (9\cdot b)$$

The Fourier coefficients in Eq. (9) can be expressed in terms of the displacements (Eq. (7·a)), i. e.,

$$\{T\}^T=[G]\{X'\}+[H]\{X\} \quad (10)$$

where $[G]$ and $[H]$=the 5×5 coefficient matrices (see Ref. 12)); and $\{T\}$ is the stress resultant vector given by

$$\{T\}^T=(n_x, n_{x\theta}, q_x, m_x, m_{x\theta}) \quad (11)$$

Finally, the boundary conditions at each edge of the shell are specified as a set of five conditions, one from each of the following five pairs:

$$(u, n_x),\ (v, n_{x\theta}),\ (w, q_x),\ (\bar{\beta}_x, m_x),\ (\bar{\beta}_\theta, m_{x\theta}) \quad (12)$$

4. METHOD OF SOLUTION

For the present study the shell is assumed to consist of dry and wet portions. As shown in Fig. 2(a), the dry and wet portions are divided into N_d and N_w elements, respectively; i. e., the total number of elements is $N=N_d+N_w$. A local nondimensional independent variable is denoted by ξ, where ξ takes the values 0 to 1 in each element. Denote the point along the axial coordinate by i, where i varies from 0 to N. Of these points, the points from 1 to $N-1$ are identified at the common boundries of different elements, and these points will be called "dividing points". The remaining points 0 and N of the ends of the shell will be called "boundary points". Let us construct a set of N_w+1 points $0=\eta_0<\eta_1<\cdots<\eta_{N_w}=1$ in the fluid domain $0\leqq\eta\leqq1$, so that location of these points coincides with that of the dividing points.

The proposed method is to approximate a derivative and an integral as a linear sum of the displacement values at discrete points so that the integro-differential equations can be reduced to a set of algebraic equations. To this end, over each element we place a set of $M+2$ discrete points (Fig. 2(b)) which are composed of the end-points $\xi_0=0$, $\xi_{M+1}=1$ and the interior collocation points ξ_j $(j=1\sim M)$, such that $0=\xi_0<\xi_1<\cdots<\xi_{M+1}=1$. In this paper, the interior collocation points are selected to be zeros of the M th shifted Legendre polynomial $P^*_M(\xi)$ since these zeros are distributed near the two end-points and are therefore optimal for the boundary values problems.

The displacement functions (Eq. (7·a)) for the k th element are interpolated by

$$X_j^{(k)}=\sum_{i=1}^{M+2} N_i(\xi)\, X_{j,i}^{(k)} \qquad (i=1\sim5) \quad (13)$$

where the notation $(\ \ )^{(k)}$ will designate quantities associated with the k th element; $X_1\sim X_5$ correspond to u, v, w, $\bar{\beta}_x$, $\bar{\beta}_\theta$, respectively; $N_i(\xi)$=the $(M+2)$ th interpolation functions[17] (often called the Legendre polynomials) associated with the i th descrete points; and $X_{j,i}$=the values of the displacements X_j at the i th discrete points.

Before describing the details of the proposed method, the following comments seem to be in order:

(a) To decrease the computational effort required, the following two matrices $[A]$ and $[B]$ are used to approximate the first and second derivatives that appear in Eqs. (6) and (10):

$$\{X_j^{(k)\prime}\}=[A]\{X_j^{(k)}\},\quad \{X_j^{(k)\prime\prime}\}=[B]\{X_j^{(k)}\} \qquad (i=1\sim5) \quad (14\cdot a, b)$$

where $[A]$ and $[B]$ are the $(M+2)\times(M+2)$

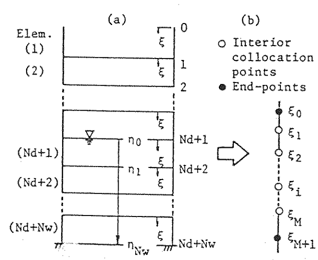

Fig. 2 Shell element: (a) Division of shell into N $(=N_d+N_w)$ elements; (b) Location of discrete points in an element.

matrices and are obtained by differentiation of the interpolation functions; and the displacement vector $\{X_j^{(k)}\}$, etc. are $\{X_j^{(k)}\}^T=(X_j^{(k)}(\xi_0),\ X_j^{(k)}(\xi_1),\ \cdots,\ X_j^{(k)}(\xi_{M+1}))$, etc.

(b) The complete set of displacement vector $\{X_j^{(\,)}\}$ is partitioned into two groups $\{X_{j,c}^{(\,)}\}$ and $\{X_{j,e}^{(\,)}\}$, the first one is associated with the interior collocation points while the second one is associated with the end-points; thus

$$\left.\begin{aligned}\{X_{j,c}^{(k)}\}^T&=(X_j^{(k)}(\xi_1),\ X_j^{(k)}(\xi_2),\cdots,\ X_j^{(k)}(\xi_M))\\ \{X_{j,e}^{(k)}\}^T&=(X_j^{(k)}(\xi_0),\ X_j^{(k)}(\xi_{M+1}))\end{aligned}\right\}\quad (j=1\sim5) \tag{15}$$

Henceforth, the subscripts c and e appearing in Eq. (15) are used to designate quantities associated with the interior collocation points and end-points, respectively.

(c) The integral involved in the evaluation of Eq. (8) is carried out by an appropriate numerical integration rule[18]. Let us recall that we select the interior collocation points as the zeros of the shifted Legendre polynomials. It is natural, therefore, to choose the Gauss-Legendre quadrature formula with this set of points as the sampling points. Before the application of the quadrature formula, the integral variable η in Eq. (5) is related to the local axial coordinate ξ of each element in the wet portion by the following linear transformation:

$$\eta^{(j)}(\xi)=\eta_{j-1}+\Delta\eta\cdot\xi \qquad (\Delta\eta=\eta_j-\eta_{j-1},\quad j=1\sim N_w) \tag{16}$$

With the aid of Eq. (16), the integral that appears in Eq. (5) is approximated by

$$\int_0^1 w\sin(\lambda_r H\eta/a)\,d\eta=\sum_{j=1}^{N_w}[Y^{(j,r)}]\{w_c^{(Na+j)}\} \tag{17}$$

where $\{w_c^{(Na+j)}\}$ is the vector composed of the radial displacements at the interior collocation points (i. e., the sampling points); $[Y^{(j,r)}]$ is the $M\times1$ row matrix whose elements are defined as

$$Y^{(j,r)}(i)=\Delta\eta\cdot W_i\cdot\sin(\lambda_r H\eta^{(j)}(\xi_i)/a) \qquad (i=1\sim M) \tag{18}$$

in which W_i=the quadrature weights.

From Eq. (13), the number of unknowns per element is $5(M+2)$. That is, the total number of unknowns for the shell having N elements is $5(M+2)N$. The application of the present method to Eq. (6) yields $5MN$ linear algebraic equations. In addition to this, there are 10 boundary conditions at the boundary points and $10(N-1)$ continuity conditions of the displacements and stress resultants at the dividing points. Since $5MN+10+10(N-1)=5(M+2)N$ we have the same number of equations as unknowns. These equations will be explained further in the following subsections.

(1) $5MN$ equations

By using Eq. (14) to approximate the derivatives and by using Eq. (17) to compute the integral, Eq. (6) for the k th element leads to $5M$ linear equations. After dividing all the unknowns into two groups as discussed previously, these equations can be expressed in the matrix form as

$$[\alpha_c^{(k)}]\{\delta_c^{(k)}\}+[\alpha_e^{(k)}]\{\delta_e^{(k)}\}=\Omega^2([MS_c^{(k)}]\{\delta_c^{(k)}\}+\sum_{j=1}^{Nw}[MF_c^{(l,j)}]\{w_c^{(Na+j)}\})$$
$$(k=1\sim N,\quad l=1\sim N_w) \tag{19}$$

For the element in the dry portion, the second term on the right hand-side of Eq. (19) vanishes. $[\alpha_c^{(k)}]$ and $[\alpha_e^{(k)}]$ are the $5M\times5M$ and $5M\times10$ matrices, which depend on the elements of $[A]$, $[B]$ (given by Eq. (14)) and $[C]$, $[D]$, $[E]$ (appearing in Eq. (6)). $[MS_c^{(k)}]$ is the $5M\times5M$ matrix, which is dependent only on the elements of $[F]$ in Eq. (6). Moreover, by making use of the expression Eq. (15) of the displacement vector, the vectors $\{\delta_c^{(k)}\}$ and $\{\delta_e^{(k)}\}$ are as follows:

$$\left.\begin{aligned}\{\delta_c^{(k)}\}^T&=(\{u_c^{(k)}\}^T,\ \{v_c^{(k)}\}^T,\ \{w_c^{(k)}\}^T,\ \{\beta_{xc}^{(k)}\}^T,\ \{\beta_{\theta c}^{(k)}\}^T)\\ \{\delta_e^{(k)}\}^T&=(\{u_e^{(k)}\}^T,\ \{v_e^{(k)}\}^T,\ \{w_e^{(k)}\}^T,\ \{\beta_{xe}^{(k)}\}^T,\ \{\beta_{\theta e}^{(k)}\}^T)\end{aligned}\right\} \tag{20}$$

In Eq. (19), $[MF_c^{(l,j)}]$ is the $M\times M$ matrix computed as

$$[MF_c^{(l,j)}]=\sum_{r=1}^{\infty}[YY^{(l,j)}][Y^{(j,r)}] \tag{21}$$

where r=the number of terms in the potential function; $[Y^{(j,r)}]$ is given by Eq. (18); and $[YY^{(l,j)}]$ is the

$1\times M$ column matrix with the following elements :

$$\left.\begin{aligned} YY^{(l,j)}_{IN}(i)&=\frac{\rho_f}{\rho_s}\frac{a}{h}\frac{2\,I_n(\lambda_r)}{\lambda_r I'_n(\lambda_r)}\sin[\lambda_r\frac{H}{a}\eta^{(j)}(\xi_i)], \quad (i=1\sim M)\\ YY^{(l,j)}_{EX}(i)&=-\frac{\rho_f}{\rho_s}\frac{a}{h}\frac{2\,K_n(\lambda_r)}{\lambda_r K'_n(\lambda_r)}\sin[\lambda_r\frac{H}{a}\eta^{(j)}(\xi_i)], \quad (i=1\sim M)\end{aligned}\right\} \quad \cdots\cdots (22)$$

Eq. (19) can be determined for each element separately, and for the overall shell, these equations yield a system of $5\,MN$ algebraic equations and can be expressed in the matrix form as

$$[\alpha_c]\{\delta_c\}+[\alpha_e]\{\delta_e\}=\Omega^2([MS_c]+[MF_c])\{\delta_c\} \quad \cdots\cdots (23)$$

where $[\alpha_c]$, $[\alpha_e]$ and $[MS_c]$ are the global matrices with submatrices only on the diagonal position ; i. e.,

$$[\alpha_i]=\lceil[\alpha_i^{(1)}],\ [\alpha_i^{(2)}],\ \cdots,\ [\alpha_i^{(N)}]\rfloor, \quad (i=c,\ e) \quad \cdots\cdots (24\cdot a)$$

$$[MS_c]=\lceil[M\,S_c^{(1)}],\ [M\,S_c^{(2)}],\ \cdots,\ [M\,S_c^{(N)}]\rfloor \quad \cdots\cdots (24\cdot b)$$

and $\{\delta_c\}$ and $\{\delta_e\}$ are the global vectors denoted by

$$[\delta_i]^T=(\{\delta_i^{(1)}\}^T,\ \{\delta_i^{(2)}\}^T,\ \cdots,\ \{\delta_i^{(N)}\}^T), \quad (i=c,\ e) \quad \cdots\cdots (25)$$

In Eq. (23), $[MS_c]$ and $[MF_c]$ are the structual mass matrix and the added mass matrix, respectively. $[MF_c]$ is obtained by adding the submatrix $[M\,F_c^{(l,j)}]$ (Eq. (19)) into the appropriate positions related to the radial displacement vectors $\{w_c^{(Na+j)}\}$ $(j=1\sim N_w)$ in the global vector $\{\delta_c\}$.

(2) 10 and 10 $(N-1)$ equations

From any given set of boundary conditions at the boundary points 0 and N, we have 10 equation. Using Eqs. (10), (12) and (14・a), and repeating the similar procedure which is used to obtain Eq. (19), these equations can be written as

$$\left.\begin{aligned}[\gamma_{c,0}^{(1)}]\{\delta_c^{(1)}\}+[\gamma_{e,0}^{(1)}]\{\delta_e^{(1)}\}&=\{0\}\\ [\gamma_{c,N}^{(N)}]\{\delta_c^{(N)}\}+[\gamma_{e,N}^{(N)}]\{\delta_e^{(N)}\}&=\{0\}\end{aligned}\right\} \quad \cdots\cdots (26)$$

where the subscripts i $(=0,\ N)$ following a comma represent the boundary points ; $[\gamma_{c,0}^{(1)}]$ and $[\gamma_{c,N}^{(N)}]$=the $5\times5\,M$ matrices ; and $[\gamma_{e,0}^{(1)}]$ and $[\gamma_{e,N}^{(N)}]$=the 5×5 matrices.

The remaining 10 $(N-1)$ equations are obtained from the compatibility and equilibrium conditions at the dividing points. These conditions can be expressed as

$$\{X^{(i)}\}_1=\{X^{(i+1)}\}_0, \quad \{T^{(j)}\}_1=\{T^{(i+1)}\}_0, \quad (i=1\sim N-1) \quad \cdots\cdots (27)$$

where $\{X\}$ and $\{T\}$=the displacement and stress resultant vectors given by Eqs. (7・a) and (11), respectively ; and the subscripts 0 and 1 which appear outside the braces, refer to the values at $\xi=0$ and $\xi=1$, respectively. Utilizing Eqs. (10) and (14・a) for Eq. (27), we obtain an expression similar to Eq. (26) as follows :

$$[\gamma_{c,i}^{(i)}]\{\delta_c^{(i)}\}+[\gamma_{e,i}^{(i)}]\{\delta_e^{(i)}\}+[\gamma_{c,i}^{(i+1)}]\{\delta_c^{(i+1)}\}+[\gamma_{e,i}^{(i+1)}]\{\delta_e^{(i+1)}\}=\{0\} \quad (i=1\sim N-1) \quad \cdots\cdots (28)$$

where the subscript i refers to the dividing points ; and $[\gamma_{c,i}^{(\ )}]$ and $[\gamma_{e,i}^{(\ )}]$=the $10\times5\,M$ and 10×10 matrices, respectively.

Eqs. (26) and (28) can be combined into a single matrix equation of the form

$$[\gamma_c]\{\delta_c\}+[\gamma_e]\{\delta_e\}=\{0\} \quad \cdots\cdots (29)$$

where $[\gamma_c]$ and $[\gamma_e]$=the $10\,N\times5\,MN$ and $10\,N\times10\,N$ matrices, respectively.

(3) Eigenvalue problem

When Eq. (29) is solved for $\{\delta_e\}$ and the result is substituted into Eq. (23), we obtain

$$([\alpha_c]-[\alpha_e][\gamma_e]^{-1}[\gamma_c])\{\delta_c\}=\Omega^2([MS_c]+[MF_c])\{\delta_c\} \quad \cdots\cdots (30)$$

Eq. (30) represents the generalized eigenvalue problem, and is the condensed form that contains only the unknowns associated with the interior collocation points. The solution of Eq. (30) yields the estimate for the $5\,MN$ eigenvalues and the corresponding eigenvectors.

5. FREE VIBRATION ANALYSIS

In this section, to sharpen the focus of the study, discussion is limited to cylindrical shells which are clamped at the base and free at the top. Based on the authors' past work[10),11)], the dry portion of the shell was modelled by one element and the number of collocation points was taken as $M=11$. In all the computations, the following shell and fluid properties were used : $E=2.1\times10^4$ kgf/mm^2 (206 GPa) ; $\rho_s=8\times10^{-6}$ kgf·sec^2/cm^4 (7.84×10^3 kg/m^3) ; $\rho_f=1.02\times10^{-6}$ kgf · sec^2/cm^4 (10^3 kg/m^3) ; and $\nu=0.3$. Also in all the following tables and figures, m denotes the number of half waves in the axial direction, n the number of circumferential waves. The numerical computations were carried out a HITAC M-682 H computer. The execution time required to obtain the solution (natural frequencies, mode shapes and hydrodynamic pressure distribution) for a shell-fluid system is less than 0.5 seconds.

(1) Convergence

For a fixed M, the convergence of the proposed method depends on the number of elements in the wet portion of the shell N_w, as well as, the number of terms N_r in the series expansion of the velocity potential. To examine the convergence characteristics of the method, two shells were considered. For convenience these are referred to as follows : (i) shell (A), its dimensions are $L=H=21.96$ m, $a=7.32$ m, and $h=1.09$ cm ; and (ii) shell (B), its dimensions are $L=H=12.2$ m, $a=18.3$ m, and $h=2.54$ cm. Calculations were performed by using $N_w=1$ and $N_w=2$, and by varying N_r from 5 to 15.

For the external problem, frequencies for the first three modes are presented in Fig. 3 for $n=1$. The convergence of the solutions is reasonable even with $N_w=1$, and is insensitive to choices of N_r. The convergence characteristics of the external problem are the same as those of the internal problem, which are not shown here. It is also of interest to check the convergence of the hydrodynamic pressure distribution. For the internal problem, results obtained using $N_w=2$ are presented in Fig. 4. In the figure, the results on the left are shown for shell (B) and the results on the right are shown for shell (A). As can be seen, the results indicate that the convergence with respect to the number of terms used is somewhat slower than for the natural frequencies, and that the 12-term solution should be reasonablly accurate for practical engineering purposes.

It appears from the foregoing numerical results that to produce the converged solution, it is sufficient to use $N_r=12$. All the following results have therefore been calculated with $N_w=2$ and $N_r=12$.

(2) Comparisons with existing results

In order to check the accuracy of the frequencies obtained, some comparative studies were performed. Firstly comparisons were made with the finite element solutions of Ref. 2), for the internal problem of shell (A) and shell (B) mentioned previously. The results for various modes are given in Table 1 together with those of Ref. 2). There are no appreciable differences between both the results. A second set of comparisons was made with the Rayleigh-Ritz and matrix progression solutions of Ref. 8), for the external problems. The shell considered was : $L=80$ m ; $a=40$ m ; $H=64$ m ; and $h=0.4$ m. The present solutions are in good agreement with those of Ref. 8) as summarized in Table 2.

(3) Frequency characteristic of internal and external problems

To illustrate the frequency characteristic of the internal and external problems, the relationships

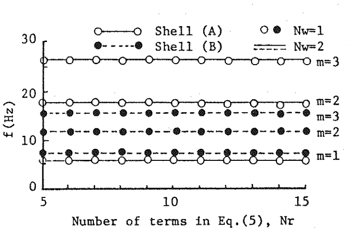

Fig. 3 Convergence of frequencies f (Hz) for external problems.

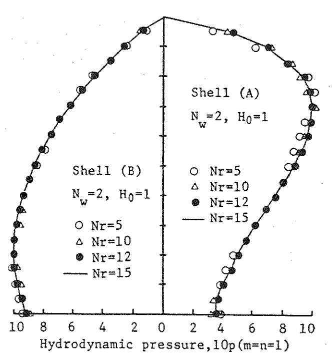

Fig. 4 Effect of number of terms in Eq. (5) on hydrodynamic pressures.

Table 1 Comparison of frequencies f (Hz) for internal problems.

(a) shell (A)

Solution procedure	m	n=1	n=2	n=3	n=4
Present(Nw=1)	1	3.55	1.64	0.93	0.63
Present(Nw=2)		3.55	1.64	0.93	0.63
FEM[Ref.2)]		3.56	1.65	0.95	0.65
Present(Nw=1)	2	10.34	6.55	4.40	3.16
Present(Nw=2)		10.33	6.58	4.43	3.19
FEM[Ref.2)]		10.45	6.66	4.52	3.28

(b) shell (B)

Solution procedure	m	n=1	n=2	n=3	n=4
Present(Nw=1)	1	6.18	5.19	4.14	3.31
Present(Nw=2)		6.18	5.19	4.14	3.31
FEM[Ref.2)]		6.18	5.19	4.14	3.31
Present(Nw=1)	2	11.25	10.52	9.94	9.16
Present(Nw=2)		11.25	10.52	9.93	9.18
FEM[Ref.2)]		11.28	10.60	9.98	9.22

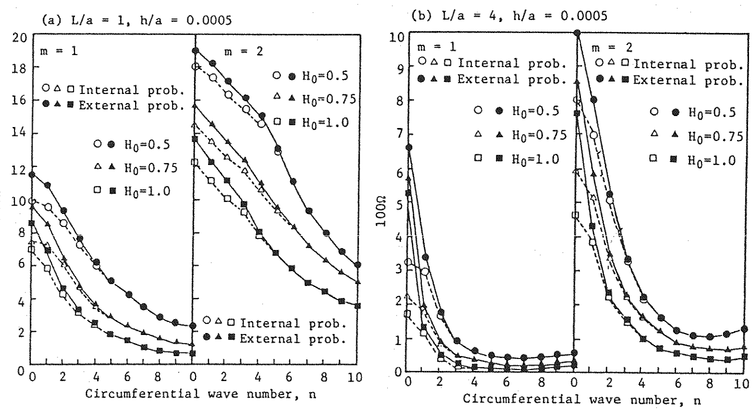

Fig. 5 Relationship between frequency Ω and wave number n.

Table 2 Comparison of frequencies f (Hz) for external problems.

Solution procedure	m	n=0	n=1	n=2	n=3
Present(Nw=1)	1	6.63	3.60	1.90	1.17
Present(Nw=2)		6.63	3.60	1.90	1.17
Present(Nw=1)	2	9.96	7.87	5.61	3.94
Present(Nw=2)		10.00	7.87	5.62	3.94
MPM[Ref.8)]					3.90
RRM[Ref.8)]					4.10

MPM = Matrix progression method
RRM = Rayleigh-Ritz method

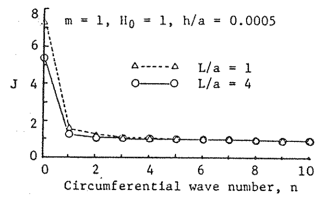

Fig. 6 Relationship between factor J and wave number n.

between the first two frequency Ω and the circumferential wave number n are presented in Fig. 5, for shells having L/a of 1 and 4 and the same h/a of 0.0005. Three different values of the fluid height-to-radius ratio, H_0 $(=H/a)$, were considered.

Some points are worthy of note in these results. First, it can be seen that as H_0 increases, the frequency decreases. This is the obvious result since the added mass of the fluid increases with H_0, while the structure stiffness properties remain unchanged. Furthermore, we can see that the frequencies of the external problem are larger than the corresponding ones of the internal problem. It should be pointed out, however, that for higher values of n (say $n \geqq 5$), the frequencies of both internal and external problems are nearly equal. This is due entirely to the added mass of the fluid and can be explained by a factor, J, defined as the ratio of the generalized virtual mass of the internal problem to that of the external problem, i.e.,

$$J=\int_0^1 \phi_{IN} w_{IN} d\eta \Big/ \int_0^1 \phi_{EX} w_{EX} d\eta \cdots\cdots\cdots\cdots (31)$$

Table 3 Effect of h/a on Ω_{EX}/Ω_{IN} (L/a=4, H/a=1).

h/a	m	Ω_{EX}/Ω_{IN} n=1	n=5
0.0005	1	1.18	1.02
	2	1.11	1.02
0.001	1	1.18	1.02
	2	1.11	1.02
0.005	1	1.16	1.02
	2	1.09	1.02
0.01	1	1.14	1.01
	2	1.08	1.01

Table 4 Effect of L/a on Ω_{EX}/Ω_{IN} (h/a=0.0005, H/a=1).

L/a	m	Ω_{EX}/Ω_{IN} n=1	n=5
0.5	1	1.14	1.03
	2	1.05	1.03
1.0	1	1.18	1.02
	2	1.11	1.02
2.0	1	1.15	1.02
	2	1.16	1.02
4.0	1	1.13	1.01
	2	1.13	1.01
8.0	1	1.10	1.00
	2	1.10	1.00

where, again, the subscripts IN and EX are used to indicate quantities related to the internal and external problems, respectively. In Fig. 6 the values of J for the first mode are plotted as a function of n, for shells having L/a of 1 and 4. It is evident that the factor J approaches rapidly unity as n increases. This is an expression of the essentially frequency character of the two problems.

The dependence of the frequency ratio Ω_{EX}/Ω_{EI} on the thickness-to-radius ratio h/a and shell height-to-radius ratio L/a can be illustrated through Tables 3 and 4. The computations were performed for both $n=1$ and $n=5$, in the ranges of $h/a=0.0005$ to 0.01 and $L/a=0.5$ to 8. It can be observed from the tables that when $n=5$, Ω_{EX}/Ω_{IN} is equal to 1 regardless of the values of both L/a and h/a.

It is concluded from a series of results that within the range of the parameters considered, it is possible for the two problems to have identical natural frequencies, which can occur for $n \geqq 5$.

(4) Simple equation for determination of fundamental natural period

A simple equation for the determination of the fundamental natural periods (corresponding to $n=1$) is presented. It is assumed for this end that the shell is in contact with the fluid to a height L (i.e., $H/L=1$), and its proportion is in the ranges of $L/a=0.5$ to 8 and $h/a=0.0005$ to 0.01.

The approximate fundamental natural period of the shells, T, may be determined from Dunkerley's approximation[16], i.e.,

$$T=\sqrt{T_s^2+T_f^2} \quad (32)$$

where T_s is the fundamental natural period of the empty shell and can be evaluated from Eq. (30) with $[MF_c]=0$, while T_f is the fundamental natural period of the same shell of negligible mass with fluid mass and can similarly be determined from Eq. (30) with $[MS_c]=0$. If T_s and T_f are considered to be functions of L/a, Eq. (32) can be rewritten as

$$T=2\pi\sqrt{\rho_s a^2/E}\ \sqrt{F_s^2+(\rho_f/\rho_s)(a/h)F_f^2} \quad (33)$$

where the functions F_s and F_f are obtained through a least square fitting technique as

$$F_s=0.631+0.8174x+0.2868x^2+0.5738\times10^{-2}x^3 \quad (34)$$

$$F_f=0.3413+0.5131x+0.1197x^2+0.1602\times10^{-1}x^3-0.6161\times10^{-3}x^4$$

(for the internal problem) (35·a)

$$F_f=0.2432+0.4747x+0.9966\times10^{-1}x^2+0.1544\times10^{-1}x^3-0.5266\times10^{-3}x^4$$

(for the external problem) (35·b)

in which $x=L/a$; and the functions F_s and F_f are applicable for the range of $L/a=0.5$ to 8. In the special case of $F_f=0$, Eq. (33) reduces to the formula of empty shells.

To test the accuracy of Eq. (33), numerical examples were considered. A graphical comparison between the periods calculated using the proposed method and those calculated from Eq. (33) is shown in Fig. 7(a) for the internal problem and in Fig. 7(b) for the external problem, where the results are plotted as a function of L/a for three different values of h/a. Also included in these figures are the results obtained by dropping the shell mass term F_s in Eq. (33). The results obtained from Eq. (33) with $F_s \neq 0$ are in good agreement with those obtained from Eq. (30), and error is usually less than 5 percent. Moreover, Fig. 7 indicates that the shell mass term F_s does make a measurable difference in the period; the maximum error in the value of period determined from Eq. (33) with $F_s=0$ is 13 percent, and corresponds to the larger values of h/a.

A similar numerical comparison, based on the different values of L/a and h/a, is given in Table 5. In the case of internal problem (Table 5(a)), the results obtained from the other approximate formulas, which were given by Sakai, *et al.*[13] and by Shimizu, *et al.*[3], and are applicable for the ranges of $0.3 \leqq L/a \leqq 4$ and $0.4 \leqq L/a \leqq 4$, respectively, are also presented. Note that the present results obtained by the simple equation (Eq. (33)) agree favorably with those obtained by the other approximate formulas and by using the proposed method (Eq. (30)).

In the case of shells with both internal and external fluids (not shown here), the approximate fundamental natural period may be determined from the following equation similar to Eq. (33):

$$T=2\pi\sqrt{\rho_s a^2/E}\ \sqrt{F_s^2+(\rho_{IN}/\rho_s)(a/h)F_{IN}^2+(\rho_{EX}/\rho_s)(a/h)F_{EX}^2} \quad (36)$$

where ρ_{IN}=the mass density of internal fluid; ρ_{EX}=the mass density of external fluid; and the functions F_{IN} and F_{EX} are defined by Eqs. (35·a) and (35·b), respectively. The period may be estimated with good

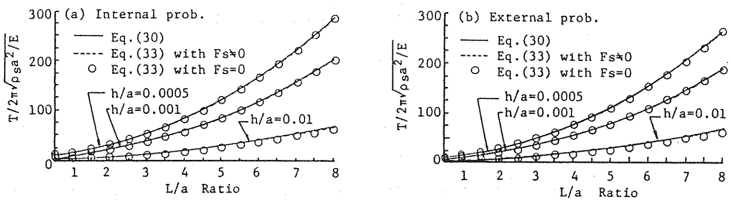

Fig. 7 Accuracy of fundamental natural periods obtained from a simple equation.

Table 5 Numerical comparison of fundamental natural periods obtained from a simple equation.

(a) Internal prob.

Source	L/a	h/a		
		0.0005	0.001	0.01
Present(1)	0.5	9.94	6.99	2.40
Present(2)		10.12	7.20	2.51
Ref.13)		9.25	6.53	
Ref.3)		9.93	7.03	
Present(1)	1.0	16.30	11.55	3.86
Present(2)		15.90	11.31	3.94
Ref.13)		15.55	11.00	
Ref.3)		16.34	11.56	
Present(1)	4.0	83.08	59.01	20.43
Present(2)		83.14	59.12	20.50
Ref.13)		79.42	56.16	
Ref.3)		83.68	59.17	
Present(1)	8.0	284.99	202.52	69.47
Present(2)		285.44	202.83	69.57

Present(1) = Eq.(30), Present(2) = Eq.(33)

(b) External prob.

Source	L/a	h/a		
		0.0005	0.001	0.01
Present(1)	0.5	8.50	6.03	2.07
Present(2)		8.18	5.84	2.12
Present(1)	1.0	13.79	9.79	3.39
Present(2)		13.40	9.56	3.44
Present(1)	4.0	73.78	52.53	18.56
Present(2)		73.83	52.58	18.63
Present(1)	6.0	149.84	106.63	37.38
Present(2)		149.99	106.81	37.45
Present(1)	8.0	259.34	184.99	64.24
Present(2)		259.74	184.76	64.36

Present(1) = Eq.(30), Present(2) = Eq.(33)

accuracy, as indicated in Ref. 20).

6. CONCLUSIONS

The principal conclusions of this study are summarized as follows:

(1) An efficient computational procedure, based on the collocation method, has been proposed for the analysis of free vibration of cylindrical shells with internal or external fluids. The comparisons with other numerical solutions show that the method yields very good results.

(2) Within the range of geometric parameters considered ($0.5 \leqq L/a \leqq 8$ and $0.0005 \leqq h/a \leqq 0.01$), the frequency of the external problem is larger than that of the internal problem. When the circumferential wave number is greater than or equal to five ($n \geqq 5$), however, the frequencies for the two problems become almost the same. Thus, in the case of $n \geqq 5$, the frequency of the internal problem can be used to estimate that of the external problem, and the opposit is also true.

(3) By using Dunkerley's approximation in combination with the method developed in the paper and by considering both the shell mass and the fluid mass, an approximate equation for determining the fundamental natural periods has been presented for shells in contact with the fluid to the height of shell. The equation is applicable for a wide range of shells ($0.5 \leqq L/a \leqq 8$ and $0.0005 \leqq h/a \leqq 0.01$) and is sufficiently accurate for practical purposes. Thus, this equation may also serve as a check on other numerical methods such as the FEM and the BEM.

In the study, emphasis was placed on the free vibration problem of cylindrical shells. Therefore, it would be also necessary to investigate the behavior of these structures under dynamic forces, which will be the subject of a forthcoming paper by the authors.

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