Structural Eng. / Earthquake Eng. Vol. 7. No. 1, 47s-53s. April 1990
Japan Society of Civil Engineers (Proc. of JSCE No. 416 / I-13)

GEOMETRIC STIFFNESS MATRIX TO ANALYZE THE LATERAL-TORSIONAL BUCKLING OF CURVED MEMBERS

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The curved member is assumed to be an assemblage of straight members connected to each other at nodal points whose coordinates are introduced in the initial configuration. Although it has been recognized by many researchers that the straight beam element cannot always model the curved member properly, the present work proves that this conclusion is not true provided that the usual geometric stiffness of the thin-walled straight beam element is adjusted by taking into consideration the out-of-balance of the internal forces at the joint of two adjacent elements meeting at an angle

Keywords : *thin-walled straight beam element, lateral-torsional buckling, quasi- and semitangential moment*

1. INTRODUCTION

Closed form solution for the out-of-plane instability of circular arch caused by uniform bending moment applied in the plane of the curvature has been already established by Vlasov[1] in 1961. The corresponding results have been shown in Ref 2). when the warping of the cross section is neglected.

Recently, this problem has been reanalyzed by Yoo[3] who obtained closed form solution which is not consistent with the previously cited ones. Moreover, a finite element solution of this problem modeled by 16 straight beam elements developed by Hasegawa *et al.*[7] has confirmed the solution by Yoo. An earlier finite element solution based on straight beam element model for curved beams was presented by Bazant and El-Nimeiri[6]. They concluded that the straight beam element cannot model the curved beam, especially when the curvature is large or the arch is slender. This was also the opinion of Hayashi and Iwasaki[8] particularly when the curved beam is under initial bending.

Hayashi and Iwasaki arrived at these conclusions after solving this problem numerically using both curved and straight beam elements. They found that the use of the curved beam element model which is naturally more accurate than the other one leads to results qualitatively and quantitatively consistent with those of Vlasov. However, results not consistent neither in quality nor in quantity with those of Vlasov and similar to those of Yoo have been obtained when the straight beam element model is used. Other numerical solutions for this same problem were obtained by Rajasekaran and Ramm[5]. In order to show the incorrectness of Yoo's solution, they modeled the curved beam by the degenerated plate/shell elements.

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The computed critical moment was in good agreement with that of Vlasov.

Papangelis and Trahair[9] have also contributed to this problem by an independent closed form solution which turns out to be more or less in agreement with that of Vlasov. Another closed form solution has been reported recently by Yang and Kuo[4]. They compared their solution with those of Yoo, Vlazov and Timoshenko. Although their results seem to be consistent with those of Vlasov and Timoshenko on a semi-logarithmic plot, the errors are as high as 90% for the moderate curvature cases. They attributed the discrepancy in Yoo's formulation and that of Hasegawa *et al.* to the neglect of the potential energy due to shear stress. However it must be noted that the shear is not involved in this particular problem.

Now, from this short review, one can conclude that any solution not consistent with that of Vlasov is wrong. But the fact that the straight beam element model gives completly deviated results, remains still an unclear point that needs further research before recourse to the development of a curved element which is certainly more complicated.

To this purpose, the present work develops a finite element formulation using straight beam element slightly different from the usual one in the context that it takes into consideration the balance of each nodal point. Its accuracy is demonstrated by application to the lateral buckling of circular arch under uniform bending.

2. BASIC EQUATIONS

(1) Kinematics

It is well known that the theory of torsion and flexure of thin-walled members can be greatly simplified if the following kinematic assumptions are taken into consideration : (1) Apart from warping, the cross section is assumed to be rigid on its own plane ; (2) The shear strain of the middle surface is zero ; and (3) stresses normal to the plane of the wall are neglected. According to the first assumption, the lateral displacements of an arbitrary point P of the cross section are given in terms of those along the reference axis (indicated by the subscript 0) by, (Fig. 1)

$$v = v_0 - y(1-\cos\phi) - z\sin\phi \quad \cdots\cdots (1\cdot a)$$

$$w = w_0 - z(1-\cos\phi) + y\sin\phi \quad \cdots\cdots (1\cdot b)$$

in relatively small flexural displacements. The displacement in the longitudinal direction at that point may also be given in terms of the deformation of the reference axis, as follows

$$u = u_0 - y\psi_0 + z\chi_0 - \omega\phi' \quad \cdots\cdots (1\cdot c)$$

in which u_0 is the longitudinal displacement of the point on the reference axis x ; $(\ )'$ denotes the derivative with respect to x ; ψ_0 and χ_0 are the slopes of the reference axis in the x-y and z-x planes, respectively ; ϕ is the rotation of the cross section about x-axis (see Fig. 1) ; y and z are the principal centroidal axes making a right-hand Cartesian coordinate system with x ; and ω is the normalized warping function. Although the following approach applies for members with the non-symmetric cross section, only doubly symmetric I section is used here for simplicity, and emphasis is put on the modification of the geometric stiffness matrix. A relation between the rotations ψ_0 and χ_0 and the lateral displacements can be obtained through the second assumption as

$$\psi_0 = v_0'\cos\phi + w_0'\sin\phi, \quad \chi_0 = v_0'\sin\phi - w_0'\cos\phi \cdots (2)$$

Then the non-zero components of Green's strain are only the normal strain as

$$\varepsilon = u_0' - y v_0'' - z w_0'' - \omega\phi'' - y\phi w_0'' + z\phi v_0'' + [v_0'^2 + w_0'^2 + (y^2+z^2)\phi'^2]/2 \quad \cdots\cdots (3\cdot a)$$

and the shear strain of the St. Venant type as

$$\gamma_s = 2n\phi' \quad \cdots\cdots (3\cdot b)$$

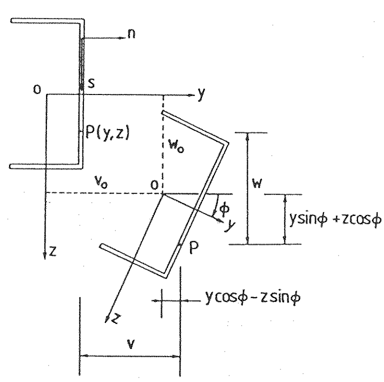


Fig. 1 Displacements is a cross section.

in which n is another coordinate taken along the thickness of the wall and originated on the middle surface. It should be noted that in the derivation of Eq. (3), the trigonometric functions of ϕ are expanded by the Taylor series and the third and higher order quantities are neglected.

(2) Virtual work principle

In the absence of the body forces, the virtual work equation of a body with the volume V and its surface S is given by

$$\int_V(\sigma^0_{ij}+\sigma_{ij})\delta e_{ij}dV-\int_S(T^0_i+T_i)\delta u_i dS=0 \quad (4)$$

in which σ_{ij} and e_{ij} are the incremental components of the second Piola-Kirchhoff stress tensor and the Green strain tensor, respectively; T_i and u_i denote the external load components applied at S and their corresponding displacements. All these quantities are measured from the reference state where σ^0_{ij} and T^0_i are acting. For convenience sake, e_{ij} and u_i are separated into their linear and quadratic parts indicated by the superscripts (l) and (q), as

$$e_{ij}=e^l_{ij}+e^q_{ij}\,;\; u_i=u^l_i+u^q_i \quad (5)$$

Substituting Eq. (5) into Eq. (4) and taking into account that the system is in equilibrium with respect to the small displacements near the reference state, we can simplify Eq. (4) as

$$\int_V(\sigma_{ij}\delta e^l_{ij}+\sigma^0_{ij}\delta e^q_{ij})dV-\int_S(T_i\delta u^l_i+T^0_i\delta u^q_i)dS=0 \quad (6)$$

It should be noted that in the derivation of Eq. (6), only the quadratic terms remain including $T^0_i\delta u^q_i$ which has been usually neglected, and will play an important role.

(3) Stiffness equation

The stiffness equation of a thin-walled straight beam element concerning the out-of-plane instability considers only v and ϕ as non-zero displacements. These are regarded as the lateral-torsional buckling displacements measured from an already stressed reference state with the normal stress σ^0_x due to in-plane loading, which is defined by

$$\sigma^0_x=\frac{N^0}{A}+\frac{M^0_z}{I_{zz}}z \quad (7)$$

in which A=area of the cross section; $I_{zz}=\int_A z^2dA$ is the moment of inertia about the centroidal axis y; and N^0 (positive in tension) and M^0_z are the initial axial force and bending moment about y axis, respectively. Substituting Eqs. (3) and (7) into Eq. (6) and taking into account the linearity of the material lead to

$$\int_0^l(EI_{yy}v''_0\delta v''_0+EI_\omega\phi''\delta\phi''+GJ\phi'\delta\phi')dx+$$
$$\int_0^l(N^0v'_0\delta v'_0+N^0I_p\phi'\delta\phi'+M^0_z\phi\delta v''_0+M^0_z v''_0\delta\phi)dx-\int_S(T_i\delta u^l_i+T^0_i\delta u^q_i)dS=0 \quad (8)$$

in which $I_{yy}=\int_A y^2dA$; $I_\omega=\int_A\omega^2dA$; $J=\int_A(2n)^2dA$; $I_p=(I_{yy}+I_{zz})/A$; E is Young's modulus and G is the shear modulus. Two models of displacement functions are considered. The first model uses the interpolation function $[\boldsymbol{n}_3]=\left[1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3}\,;\,-x+\frac{2x^2}{l}-\frac{x^3}{l^2}\,;\,\frac{3x^2}{l^2}-\frac{2x^3}{l^3}\,;\,\frac{x^2}{l}-\frac{x^3}{l^2}\right]$ for both v_0 and ϕ in order to take the warping effects into account. This model must correspond to the results by Vlasov. On the other hand, the second model neglects the warping and uses the function $[\boldsymbol{n}_1]=\left[1-\frac{x}{l}\,;\,\frac{x}{l}\right]$ for ϕ, which will lead to the results by Timoshenko. In both models, the final form of the stiffness equation can be written as

$$[\boldsymbol{K}_e+\boldsymbol{K}_g]\boldsymbol{r}=\boldsymbol{R} \quad (9)$$

where, in model 1, the nodal displacement vector $\boldsymbol{r}$ is

$$\boldsymbol{r}^T=[v_i\;\; -v'_i\;\; v_j\;\; -v'_j\,;\,\phi_i\;\; -\phi'_i\;\; \phi_j\;\; -\phi'_j] \quad (10\cdot a)$$

and in model 2

$$\boldsymbol{r}^T=[v_i\;\; -v'_i\;\; v_j\;\; -v'_j\,;\,\phi_i\;\; \phi_j] \quad (10\cdot b)$$

Explicit expressions of the matrices $\boldsymbol{K}_e$ and $\boldsymbol{K}_g$ are given in the APPENDIX. For the linear buckling analysis, the current loading step is considered as the buckled state. Hence the increments of the applied forces are kept zero, and the first term of the last integral of Eq. (8) vanishes. However the second term will have non-zero components. By using Eq. (7) and the quadratic terms in Eq. (1·c), it can be easily shown that $\boldsymbol{R}$ reduces to

$$\boldsymbol{R}^T=[0\ \ -M^0_{zi}\phi_i\ \ 0\ \ -M^0_{zj}\phi_j\ ;M^0_{zi}v'_i\ \ 0\ \ M^0_{zj}v'_j\ \ 0]\ ;\ \text{i. e.}\ \boldsymbol{R}=\boldsymbol{K}_q\boldsymbol{r} \quad\cdots\cdots (11)$$

where $\boldsymbol{K}_q$ is also given in the APPENDIX. Therefore taking into consideration Eq. (11), the stiffness equation Eq. (9) becomes

$$[\boldsymbol{K}_e+\boldsymbol{K}_{gq}]\,\boldsymbol{r}=0 \quad\cdots\cdots (12)$$

where

$$\boldsymbol{K}_{gq}=\boldsymbol{K}_g-\boldsymbol{K}_q \quad\cdots\cdots (13)$$

The last matrix $\boldsymbol{K}_q$ which stems from the quadratic components of the kinematics has never been introduced by any researcher. Note that the bending moment M^0_z of Eq. (7) is assumed to vary linearly along the length of the element and has the form

$$M^0_z=-M^0_{zi}\left(1-\frac{x}{l}\right)+M^0_{zj}\left(\frac{x}{l}\right) \quad\cdots\cdots (14)$$

in which M^0_{zi} and M^0_{zj} are its corresponding values at the left and right ends of the element, respectively.

3. CORRECTION OF THE GEOMETRIC STIFFNESS MATRIX

As mentioned before, the curved beam will be considered as an assemblage of straight elements which are connected to each other at the nodal points. As far as the out-of-plane problems are concerned, it has been shown by Argyris *et al.* [10] that the joint of two members meeting at an angle subjected to out-of-plane rotations will be in imbalance because the bending moment resulted from the stress distribution given by Eq. (7) behaves quasitangentially. Consider the two-member planar frame and let it be subjected to a constant bending moment applied at the free end as shown in Fig. 2. A_1 and A_2 are, respectively, the cross sections of element 1 and 2 meeting at an angle $(\pi-\alpha)$. The quasitangential moments[11] on these sections are the moments which are decomposed into a couple force of its magnitude M acting on a unit length rigid lever as shown in Figs. 2 and 3. When the joint undergoes a rotation ϕ applied with respect to the longitudinal axis of element 2, it can be seen from the table in Fig. 3 that as long as the meeting angle is

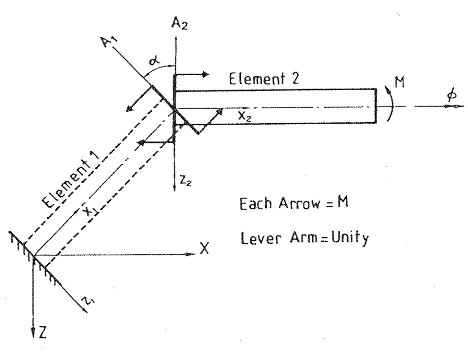
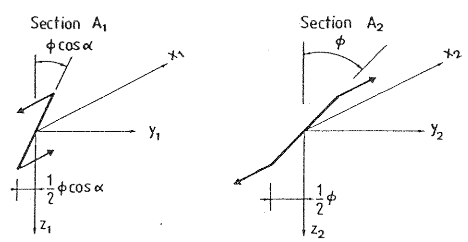


Fig. 2 Planar frame under a quasitangential bending moment.



$$F_x = F_y = F_z = 0$$
$$M_x = M\phi \cos\alpha \sin\alpha$$
$$M_y = M - M = 0$$
$$M_z = -M\phi + M\phi \cos^2\alpha$$

Fig. 3 Imbalance at the joint when moment is decomposed quasitangentially.

different from π (case of straight beam), there will be always an imbalance of moments at the joint. However, if we consider that the bending moment is not quasitangential but semitangential[11], the moment is decomposed into two couple forces of $M/2$ as in Fig. 4, it can be proven that the joint retains its equilibrium even in the presence of the out-of-plane rotations. This can be demonstrated in Fig. 4, where two members under the uniform bending are connected by the right angle. Therefore, it seems necessary to make modification on the nodal force vector $\boldsymbol{R}$ in Eq. (11) by replacing the quasitangential bending moment by a semitangential one as Argyris *et al.*[10] have shown. The geometrical consideration leads to the fact that the presence of an in-plane semitangential bending moment M_z^0 results in two out-of-plane moments M_x and M_y due to the out-of-plane rotations, given by[12]

$$M_x=\frac{1}{2}M_z^0\theta_z\,;\,M_y=\frac{1}{2}M_z^0\phi \cdots\cdots\cdots\cdots\cdots\cdots (15)$$

where θ_z is equivalent to v_0' in our formulation. Although the similar expressions naturally result from the quadratic terms of the displacement in Eq. (11), the moment is treated as the quasitangential quantity and "1/2" in Eq. (15) does not appear. Since $\boldsymbol{R}$ in Eq. (11) is simply proportional to M_z^0, the modification can be done by replacing M_z^0 by $M_z^0/2$. Hence follows the corrected stiffness equation as

$$[\boldsymbol{K}_e+\boldsymbol{K}_{gs}]\,\boldsymbol{r}=0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (16)$$

where $\boldsymbol{K}_{gs}=\boldsymbol{K}_g-\frac{1}{2}\boldsymbol{K}_q.$

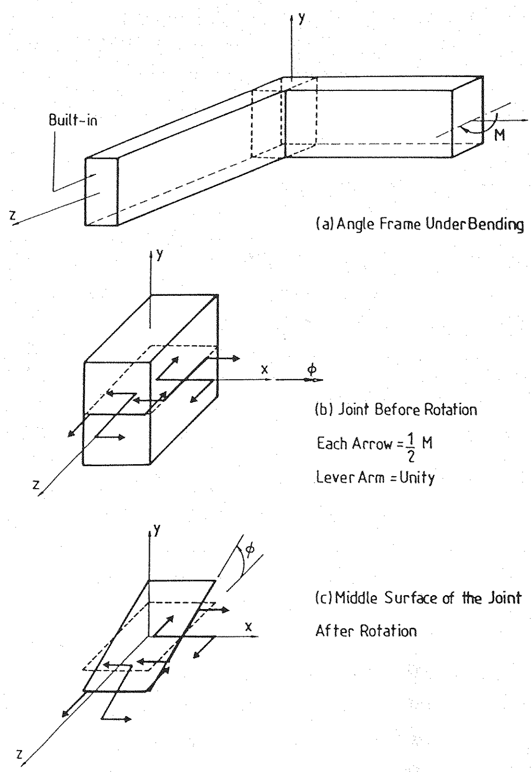


Fig. 4 Equilibrium of the joint in case of semitangential moment.

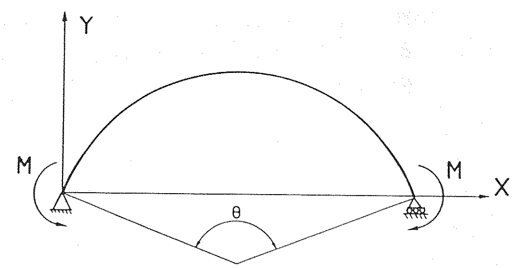


Fig. 5 Circular arch in uniform bending.

4. APPLICATION TO THE LATERAL-TORSIONAL BUCKLING OF CIRCULAR ARCHES

Using Eq. (16), the critical moment for the arch shown in Fig. 5 is computed for certain subtended angles θ. The applied moment in this figure is reckoned positive. The ordinary procedure of FEM using an appropriate number of straight elements leads to the eigenvalue problem in this special case, because M_z^0 is constant along the arch. If the boundary conditions are changed, the bending moment may no longer uniform in the arch. In such a case, the in-plane problem is solved for the unit applied forces to obtain the distribution of the internal forces. Then the eigenvalue problem using the obtained distribution of the internal forces may be solved.

Table 1 Critical positive moment (in kN·m).

Researcher	Subtended angle in degree				
	0.05	10.0	30.0	50.0	90.0
Vlasov[1]	346.8	590.2	1257.1	1996.3	3519.2
Timoshenko[2]	312.8	561.0	1241.0	1986.0	3513.0
Yoo[3]	345.8	345.9	339.3	323.8	266.1
Yang[4]	347.8	905.7	2343.0	3756.2	6121.5
Rajasekaran and Ramm[5)]					
PSE*					3838.3
SBE**					5238.2
Bazant[6]	347.8	909.5	2438.5	4101.9	6820.1
Hasegawa[7]	345.8	345.9	339.4	324.8	261.1
Present :					
Model 1	346.8	589.8	1257.3	1998.1	3529.8
Model 2	312.9	561.2	1242.2	1988.1	3520.5

* Plate/Shell Element ; ** Straight Beam Element

Table 2 Critical negative moment (in kN·m).

Researcher	Subtended angle in degree				
	0.05	10.0	30.0	50.0	90.0
Vlasov[1]	344.9	202.0	92.5	55.3	25.5
Timoshenko[2]	310.9	173.1	76.2	45.2	20.8
Yoo[3]	345.8	343.8	333.6	315.0	253.0
Yang[4]	343.9	131.2	48.3	27.1	10.9
Rajasekaran and Ramm[5)]					
PSE*					24.1
SBE**					12.8
Bazant[6]	343.9	132.3	51.8	33.1	22.2
Hasegawa[7]	345.8	343.6	333.0	315.5	244.1
Present :					
Model 1	344.9	203.0	95.0	58.3	35.0
Model 2	311.0	174.0	78.5	49.1	30.2

* Plate/Shell Element ; ** Straight Beam Element

For the sake of comparison with the results in other references, the following material and sectional properties are adopted : $E=200$ GN/m^2 ; $G=77.2$ GN/m^2 ; $A=92.9$ cm^2 ; $I_{yy}=11\,360$ cm^4 ; $I_{zz}=3\,870$ cm^4 ; $I_\omega=555\,900$ cm^6 ; $J=58.9$ cm^4 and the length of the arch is 10.24 m. The computed positive and negative critical moments for several subtended angles are shown in Tables 1 and 2, respectively. Here the case with the small subtended angle, i. e. $\theta=0.05$, is examined in order to show that the present new geometric stiffness matrix leads to the well-known, already established and confirmed results for the straight beams, and thus the new terms in this matrix play a significant role when there exists a very small but non-zero angle between elements. The rate of convergence of solutions has been found very rapid for both these two models, and all the results in the tables have been obtained using 16 elements.

5. SUMMARY AND CONCLUSIONS

A geometric stiffness matrix for analyzing the out-of-plane instability of thin-walled members with doubly symmetric cross section has been derived through the use of the virtual work equation of an initially stressed and equilibrated continua. The geometric stiffness is modified by considering the semitangential moment transfer between elements.

The accuracy and validity of this approach is examined by computing the critical moment of a circular arch and making comparison with the solutions of Vlasov and Timoshenko whose correctness has become very clear after the works of Rajasekaran and Ramm who used degenerated plate/shell element model and Hayashi and Iwasaki who used curved beam element model.

Finally, even without correcting the geometric stiffness matrix but taking into account the work of the external forces with the quadratic terms of displacements, results similar to those of Bazant and El-Nimeiri are obtained. Since these results are much better than those of Yoo and comparable to some extent with the exact solution of Vlasov and Timoshenko, it seems necessary to include the quadratic terms of the displacements in the derivation of the virtual work equation.

Acknowledgement : This work has been supported in part by the Grand-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture to Tohoku University.

APPENDIX

$$K_e=\begin{bmatrix} EI_{yy}k_{33}^{220} & 0 \\ \text{Symm.} & EI_{\omega}k_{33}^{220}+GJk_{33}^{110} \end{bmatrix};\ K_g=\begin{bmatrix} \frac{1}{2}(N_j^0-N_i^0)k_{33}^{110} & -M_{zi}^0 k_{33}^{200}+(M_{zi}^0+M_{zj}^0)k_{33}^{201} \\ \text{Symm.} & \frac{I_p}{2}(N_j^0-N_i^0)k_{33}^{110} \end{bmatrix}$$

where the submatrices are given by

$$k_{kl}^{mnp}=\int_0^l \{n_k\}^m [n_l]^n \left(\frac{x}{l}\right)^p dx$$

in which m or n denotes the degree of differentiation and p is the exponent power. Explicit expressions are given below :

$$k_{33}^{220}=\frac{1}{l^3}\begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix},\quad k_{33}^{110}=\frac{1}{30l}\begin{bmatrix} 36 & -3l & -36 & -3l \\ -3l & 4l^2 & 3l & -l^2 \\ -36 & 3l & 36 & 3l \\ -3l & -l^2 & 3l & 4l^2 \end{bmatrix}$$

$$k_{33}^{200}=\frac{1}{30l}\begin{bmatrix} -36 & 33l & 36 & 3l \\ 3l & -4l^2 & -3l & l^2 \\ 36 & -3l & -36 & -33l \\ 3l & l^2 & -3l & -4l^2 \end{bmatrix},\quad k_{33}^{201}=\frac{1}{30l}\begin{bmatrix} -3 & 6l & 3 & -3l \\ 0 & -l^2 & 0 & l^2 \\ 33 & -6l & -33 & -27l \\ 3l & 0 & -3l & -3l^2 \end{bmatrix}$$

All components of the matrix K_q are zero except the following : $K_q(2,5)=K_q(5,2)=-M_{zi}^0$; and $K_q(4,7)=K_q(7,4)=-M_{zj}^0$.

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(Received April 3 1989)