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APPROXIMATE EXPLICIT FORMULAS FOR COMPLEX MODES OF TWO-DEGREE-OF-FREEDOM (2DOF) SYSTEM

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As an application of the general technique presented in a companion paper, the complex modes of nonproportionally damped close-coupled two-degree-of-freedom system are approximated through second-order perturbation. Assuming the nonproportionality to be either moderate or weak, the pseudo natural frequencies, modal damping ratios, and complex "modes" are expressed in terms of natural frequencies and mode shapes of the counterpart undamped 2DOF system, and of respective damping ratios of the two single-degree-of-freedom (SDOF) subsystems. The effects of damping nonproportionality are contained in eight nondimensional perturbation coefficients. These perturbations depend only on the ratio of masses, ratio of natural frequencies, and damping ratios of the two SDOF subsystems. The approximate explicit formulas avoid the complications of Ferrari's classical solution for the quartic characteristic equation. Parametric studies illustrate the convenient use of the present formulas as well as check their accuracy.
Keywords: 2DOF system, nonproportional damping, complex modes, perturbation

1. INTRODUCTION

Modal methods of dynamic analysis of multi-degree-of-freedom (MDOF) systems are generally very powerful and convenient; but some computational and conceptual difficulties arise when the damping distribution in the system is not proportional. In the latter case, the "modes" are complex instead of being real mode shapes. There are many examples of nonproportionally damped structures, including: structure with tuned mass damper (TMD); structure with localized high-damping elements or members; structure-equipment system; soil-structure system and base-isolated structure. Even a two-degree-of-freedom (2DOF) model, which is the simplest possible MDOF model of any structure, when nonproportionally damped becomes prohibitively complicated to analyze for modes in exact closed form. The associated quadratic eigenvalue problem requires the solution of a quartic characteristic equation, say by the classical Ferrari method[1]. However, when the general perturbation technique recently proposed by the authors[2] is applied, only the quadratic characteristic equation of the counterpart undamped system has to be solved directly. This can be done in closed form far simpler than a quartic. In this paper are presented the approximate explicit formulas for perturbed complex "modes", pseudo natural frequencies and modal damping ratios. Some example parametric studies follow, to illustrate the usefulness of the formulas as well as check their accuracy.

2. DESCRIPTION OF THE TWO-DEGREE-OF-FREEDOM (2DOF) SYSTEM

Fig. 1 shows the mechanical analogue of close-coupled 2DOF system, consisting of two masses m_a and m_b, and corresponding springs and dashpots. The subsystem natural frequencies ω_a and ω_b, and damping

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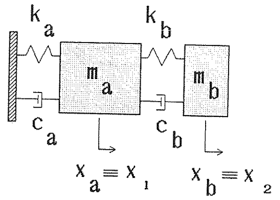

Fig. 1 Mechanical analogue of close-coupled 2DOF system.

ratios ξ_a and ξ_b are defined as :

$$\omega_s=\sqrt{k_s/m_s} \qquad \xi_s=c_s/\sqrt{4\,k_s m_s} \qquad s=a,\,b \cdots\cdots\cdots (1),(2)$$

To enhance the generality of the resulting formulas, three non-dimensional ratios are further defined, namely mass ratio μ, ratio of damping ratios (or proportionality parameter) ν, and frequency ratio (or tuning parameter) τ, as in Eqs. (3)~(5) below.

$$\mu=m_b/m_a \qquad \nu=\xi_b/\xi_a \qquad \tau=\omega_b/\omega_a \cdots\cdots\cdots (3),(4),(5)$$

These ratios describe the properties of subsystem "b" relative to corresponding properties of subsytem "a". The matrix equation of free motion may then be written as in Eqs. (6) in terms of these ratios and the properties of "b", namely m_b, ε_b and ω_b.

$$\boldsymbol{M}\ddot{x}+\boldsymbol{C}\ddot{x}+\boldsymbol{K}x=\boldsymbol{0} \cdots\cdots (6\cdot\mathrm{a})$$

$$\boldsymbol{M}=\frac{m_b}{\mu}\begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix} \qquad \boldsymbol{C}=\frac{2\,\omega_b\xi_b m_b}{\mu}\begin{bmatrix}\frac{1}{\tau\nu}+\mu & -\mu\\ -\mu & \mu\end{bmatrix} \cdots\cdots (6\cdot\mathrm{b,\,c})$$

$$\boldsymbol{K}=\frac{\omega_b^2 m_b}{\mu}\begin{bmatrix}\frac{1}{\tau^2}+\mu & -\mu\\ -\mu & \mu\end{bmatrix} \qquad x=<x_1 \quad x_2>^T=<x_a \quad x_b>^T \cdots\cdots (6\cdot\mathrm{d,\,e})$$

3. EXPLICIT FORMULAS FOR PERTURBED EIGENVECTORS AND EIGENVALUES

Eqs. (6·c) and (6·d) show that, if $\nu=\tau$ then $\boldsymbol{C}$ becomes proportional to $\boldsymbol{K}$, and the system becomes proportionally damped. In that case, the natural modes $\boldsymbol{y}_{0j}$ are unchanged by the existence of damping. Otherwise, the eigenvectors of Eq. (6·a) are complex and different from the natural modes. The eigenvalues, if the system were proportionally damped, would be $\lambda_{0j}=-\omega_{0j}\xi_{0j}+i\omega_{0j}\sqrt{1-\xi_{0j}^2}$, where $i=\sqrt{-1}$, ω_{0j}=natural frequency, and ξ_{0j}=modal damping ratio. When damping is nonproportional, ω_{0j} and ξ_{0j} also are changed. In this paper, nonproportionality is assumed to be either weak or moderate, and the nonproportionality effects are expressed as modifications or perturbations on $\boldsymbol{y}_{0j}$, ω_{0j} and ξ_{0j}.

The pertubation technique[2)] expresses the complex eigenvectors and eigenvalues of Eq. (6·a) in terms of the natural modes and frequencies of the counterpart undamped system, and the actual damping matrix. The natural frequencies ω_{01} and ω_{02} of the associated undamped system are obtainable as roots of the characteristic equation $\det(-\omega_0^2\boldsymbol{M}+\boldsymbol{K})=0$, which is quadratic in the case of 2DOF system. The corresponding modes $\boldsymbol{y}_{01}$ and $\boldsymbol{y}_{02}$ are next obtained by substituting ω_{01} and ω_{02}, in turn, into the matrix eigenvalue equation, i.e., $(-\omega_{0j}^2\boldsymbol{M}+\boldsymbol{K})\,\boldsymbol{y}_{0j}=\boldsymbol{0}$. $\boldsymbol{y}_{0j}$ is normalized such that $\boldsymbol{y}_{0j}^T\boldsymbol{M}\boldsymbol{y}_{0j}=1$. The (unperturbed) damping ratios ξ_{01} and ξ_{02} are subsequently obtained by transforming the damping matrix $\boldsymbol{C}$ through the modal matrix $\boldsymbol{Y}_0$, whose j-th column is $\boldsymbol{y}_{0j}$; the j-th diagonal element in the transformed matrix is $2\omega_{0j}\xi_{0j}$. On substituting $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ from Eqs. (6·b) to (6·d), the resulting explicit formulas come out as :

$$s_1^2=\frac{1}{2}[(1+\tau^2+\mu\tau^2)-\sqrt{(1+\tau^2+\mu\tau^2)^2-4\,\tau^2}] \cdots\cdots (7\cdot\mathrm{a})$$

$$s_2^2=\frac{1}{2}[(1+\tau^2+\mu\tau^2)+\sqrt{(1+\tau^2+\mu\tau^2)^2-4\,\tau^2}] \cdots\cdots (7\cdot\mathrm{b})$$

$$\omega_{0j}=s_j\omega_b/\tau, \quad j=1,2 \cdots\cdots (8)$$

$$\boldsymbol{y}_{0j}=\frac{\sqrt{\mu}}{\sqrt{[1+\mu\{\tau^2/(\tau^2-s_j^2)\}^2]\,m_b}}\begin{Bmatrix}1\\ \dfrac{\tau^2}{\tau^2-s_j^2}\end{Bmatrix} \cdots\cdots (9)$$

$$\xi_{0j}=\xi_b\frac{(\tau^2-s_j^2)^2+\mu\nu\tau s_j^4}{\nu s_j\{(\tau^2-s_j^2)^2+\mu\tau^4\}} \cdots\cdots (10)$$

The nonproportionality-induced perturbations are redefined in the following formats[2)] :

$$\lambda_j = -\omega_j \xi_j + i\omega_j \sqrt{1-\xi_j^2}, \quad j=1,2 \qquad (11)$$

$$\omega_j = \omega_{0j}\sqrt{1+\alpha_j}, \qquad (12)$$

$$\xi_j = \xi_{0j}\sqrt{1+\beta_j} \qquad (13)$$

$$\boldsymbol{y}_j = (\boldsymbol{y}_{0j} + \zeta_{jk}\boldsymbol{y}_{0k}) + i(\eta_{jk}\boldsymbol{y}_{0k}), \quad (j,k)=(1,2),(2,1) \qquad (14)$$

where the nondimensional perturbations α_j, β_j, ζ_{jk} and η_{jk} are real-valued functions of the nondimensionalized parameters μ, ν, τ and ξ_b. Note that the properties m_b and ω_b do not affect the perturbations, although they influence ω_{0j} and $\boldsymbol{y}_{0j}$ (Eqs. (8), (9)). The explicit formulas for the perturbations are given by Eqs. (15) ~ (20). In the terminology of Ref. 2), α_j (Eq. (15)) and β_j (Eq. (16)) are the nonproportionality-induced perturbations on the j-th natural frequency and modal damping ratio, respectively. ζ_{jk} (Eq. (17)) and η_{jk} (Eq. (18)) are the modal perturbations that "couple" the natural modes to form the complex eigenvectors.

$$\alpha_j = \frac{\rho^2}{A_{jk}}[s_k^2 - s_j^2)] + \frac{\rho^4}{A_{jk}^2}\left[\frac{1}{4}(s_k^2-s_j^2)^2 + \frac{1}{4}\frac{\xi_{0j}^2}{1-\xi_{0j}^2}(s_k^2+s_j^2)^2 - \frac{\xi_{0k}\xi_{0j}}{1-\xi_{0j}^2}(s_k^2+s_j^2)s_k s_j + \frac{\xi_{0k}^2}{1-\xi_{0j}^2}s_k^2 s_j^2\right] \qquad (15)$$

$$\beta_j = \frac{1}{1+\alpha_j}\left\{\frac{2\rho^2}{A_{jk}}\left[s_k^2 - \frac{\xi_{0k}}{\xi_{0j}}s_k s_j\right] + \frac{\rho^4}{A_{jk}^2}\left[s_k^2 - \frac{\xi_{0k}}{\xi_{0j}}s_k s_j\right]^2 - \alpha_j\right\} \qquad (16)$$

$$\zeta_{jk} = \frac{\rho}{A_{jk}}[\xi_{0j}s_j(s_k^2+s_j^2) - 2\xi_{0k}s_k s_j^2] \qquad (17)$$

$$\eta_{jk} = \frac{\rho}{A_{jk}}[-\sqrt{1-\xi_{0j}^2}\,s_j(s_k^2-s_j^2)] \qquad (18)$$

$$A_{jk} = 2(s_k^2+s_j^2)(\xi_{0k}s_k - \xi_{0j}s_j)^2 + (s_k^2-s_j^2)(|s_k^2-s_j^2| - 2\{\xi_{0k}^2 s_k^2 - \xi_{0j}^2 s_j^2\}) \qquad (19)$$

$$\rho = 2\xi_b\left(\frac{\tau-\nu}{\tau\nu}\right)\sqrt{\frac{\mu\tau^4}{(1+\tau^2+\mu\tau^2)^2 - 4\tau^2}} \qquad (20)$$

The perturbations (Eq. (15) ~ (18)) become zero when the nondimensional quantity ρ is zero. As may be seen in Eq. (20), the nonproportionality so disappears when ν, the ratio of subsystem damping ratios (Eq. (4)), is equal to τ. Hence ν can also be described as damping proportionality parameter of close-coupled 2DOF system. However, Eq. (20) also indicates that once ν and τ are unequal, i. e. damping is nonproportional, the degree of nonproportionality (as reflected by the perturbations α_j, β_j, ζ_{jk}, and η_{jk}) is influenced not only by ν and τ but also by μ and ξ_b. The following section is on parametric studies of several example systems.

4. PARAMETRIC STUDIES OF SEVERAL 2DOF SYSTEMS

(1) System parameters

Four systems are studied, namely : (A) structure and TMD ; (B) structure and equipment ; (C) ground and superstructure ; and (D) base isolator and superstructure. The parameters μ, ν, τ, and ξ_b are listed in Table 1, where it is indicated also that one of the four parameters is varied in each system, to demonstrate how the perturbations would vary correspondingly. In systems, A, C and D, the structural damping ratio considered is 2.0 %. For the structure with equipment or secondary subsystem, i. e. system B, 5.0 % structural damping is considered.

The tuned mass damper (TMD) in system A has a mass ratio μ that can be reasonably expected for massive civil engineering structures, i. e. 1.0 %. The damping ratio ξ_b of 6.4 % is optimum when $\tau=$ 0.9868, by Warburton and Ayorinde's optimization criterion[3]. The value of τ is parametrically varied around this optimally tuned case.

In the structure-equipment system, i. e. system B, only the tuned condition ($\tau=1.0$) is analyzed, this being the potentially most damaging situation for the equipment or secondary subsystem. The equipment is considered to be much less damped than the structure ($\nu=0.2$), and ranging from extremely light to very light compared to the structure $(0.002<\mu<0.01)$.

In system C, tuning or resonance between ground natural frequency and structural frequency is likewise postulated, representing a significant case of soil-structure inertial interaction. There being a wide variability in soil damping, the ratio of damping ratios, ν, is parametrically varied $(0.10<\nu<2.00)$. Of the four systems, only the parametric study of system C includes a case where $\nu=\tau$, i.e. proportionally damped.

Table 1 Non-dimensional parameters of examples.

	System A	System B	System C	System D
'a'	structure	structure	ground	isolator
'b'	TMD	equipment	structure	structure
μ	0.01	0.002~0.01	0.30	5.00
ν	3.20	0.20	0.10~2.00	0.20
τ	0.70~1.30	1.00	1.00	1.00~3.00
ξ_b	0.064	0.01	0.02	0.02

Systems A, B and C all require a coupled model of the two subsystems, because of the tuning and associated strong interaction between the subsystems. In contrast, system D is designed such that the isolator frequency is much lower than the natural frequency of the superstructure, i.e., $\tau>1.0$. Nevertheless, a coupled model may be used in investigating the effects of isolators that are softer than the superstructure by different degrees $(1.0<\tau<3.0)$, and are relatively so highly damped as to make $\nu\ll\tau$.

These examples are chosen for the present study because they are commonly encountered in practice, and since they cover a wide region for the four parameters μ, ν, τ, and ξ_b.

(2) Analysis of perturbations

Fig. 2 (a)~(d) are plots of the eight perturbations : α_1, α_2, β_1, β_2, ζ_{12}, ζ_{21}, η_{12}, and η_{21}. Fig. 2 (a), for example, consists of two sets of graphs ; the upper graphs are for α_1, α_2, β_1, and β_2 versus the tuning parameter τ, which are eigenvalue-related, while the lower graphs show the eigenvector pertubations ζ_{12}, ζ_{21}, η_{12}, and η_{21}. Systems A, B, C and D correspond to figures (a), (b), (c), and (d), in that order. It should be noted that the vertical scale varies from graph to graph in Fig. 2.

A look at the overall trends of the plots in Fig. 2 reveals that the eigenvector perturbations, particularly

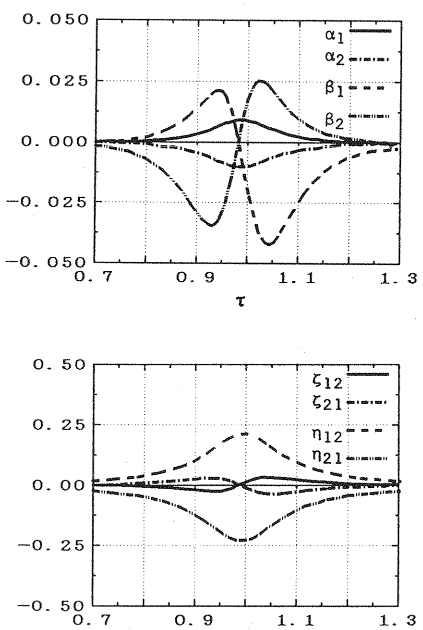

Fig. 2(a) Perturbations on structure with TMD (system A).

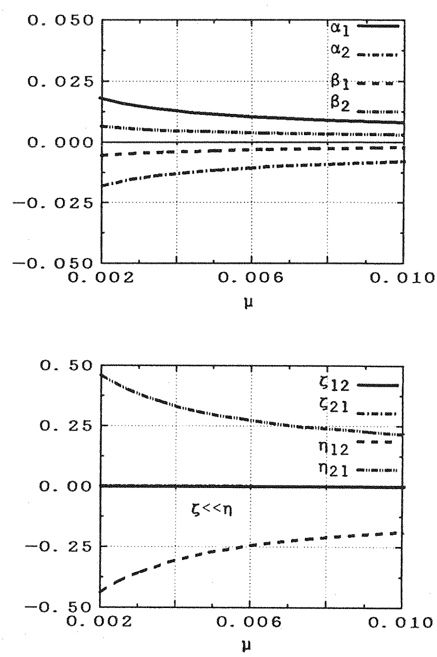

Fig. 2(b) Perturbations on structure with equipment (system B).

η_{12} and η_{21}, are much larger than the eigenvalue perturbations. It is to be recalled (Eq. (14)) that η_{12} and η_{21} are associated with the imaginary part of the complex eigenvectors.

Specifically, Fig. 2 (a) demonstrates that the respective magnitudes of the perturbations do not necessarily all increase or decrease simultaneously when one parameter— in this case, the tuning parameter τ—is changed. This observation is in contrast with Figs. 2(b) ~ (d). It is noteworthy that in the range of τ in Fig. 2 (a), there is a value of τ where some of the perturbations (namely, α_1, α_2, ζ_{12}, and ζ_{21}) are almost zero while the others (namely, β_1, β_2, η_{12}, and η_{21}) practically attain their respective peak magnitudes; and this τ is nearly equal to 0.9868 which is the "optimum" value that was numerically obtained by Warburton and Ayorinde [3].

Fig. 2 (a) also demonstrates the point made just after Eq. (20) above. That is, while the condition $\tau-\nu=0$ is enough to ensure proportionality and render all perturbations zero, the quantity $\tau-\nu\neq0$ alone is not a direct measure of the perturbations. In Fig. 2 (a) where $\nu=3.2$, the difference $(\tau-\nu)$ varies monotonically from -2.5 (at $\tau=0.7$) to -1.9 (at $\tau=1.3$), yet the associated perturbations are largest at intermediate values of τ.

On the other hand, Fig. 2 (b) represents a system where the difference $(\tau-\nu)$ is fixed at 0.8, and only the mass ratio is varied. Fig. 2(b) shows clearly that, in this system, nonproportionality increases—and all perturbations monotonically grow—as the equipment becomes even lighter relative to the structure. In fact, for the lightest equipment considered here for system B ($\mu=0.002$), the perturbations are large enough for one to doubt in that case the applicability of the original assumption of this perturbation technique, namely, weak to moderate nonproportionality. This question of accuracy is further discussed in the next section.

In Fig. 2(c), meanwhile, all perturbations are zero, as expected, when $\nu=1.0=\tau$. Perhaps less readily expected is the existence of two different trends for the perturbations in the range $\nu<1.0$ and the range $\nu>1.0$. The results in Fig. 2 (c) indicate that the nonproportionality-induced perturbations are greater when the superstructure is less damped than the ground ($\nu<1.0$), rather than when it is the other way around ($\nu>1.0$). It may be noted that, indeed, reported studies of soil-structure interaction cover

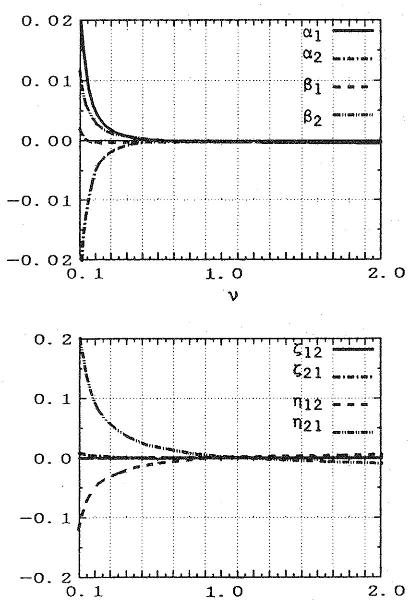

Fig. 2(c) Perturbations on ground and structure system (C).

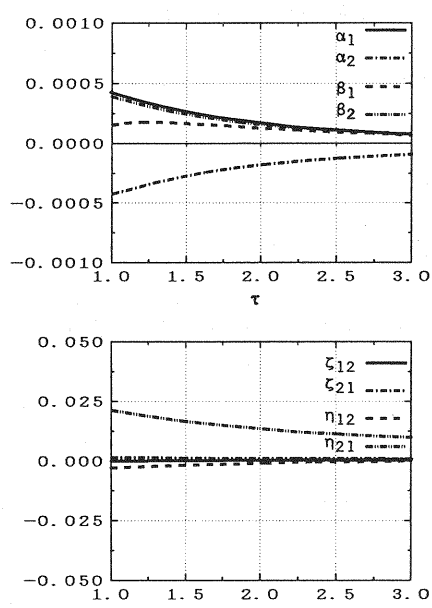

Fig. 2(d) Perturbations on isolator and structure system (D).

mostly cases of $\nu<1.0$. Damping nonproportionality has been almost invariably judged to be significant by those investigations. Fig. 2(c) serves to verify this trend, and also to display it in a wider context by showing also the dissimilar trend for $\nu>1.0$.

Looking at Figs. 2(a) and (c), it may verified that neither the difference $(\tau-\nu)$ nor its absolute value alone is sufficient indirect measure of the degree of damping nonproportionality.

Lastly, Fig. 2(d) demonstrates a system where τ and ν are unequal and very different, yet the perturbations are very small. Nevertheless, as τ is decreased from right to left in the graphs, the perturbations increase moderately. It must be pointed out that these increases in the perturbations are associated in Fig. 2(d) with decreasing difference $(\tau-\nu)$.

Since $\tau=1.0$ corresponds practically to maximum perturbations for system D in the range of τ in Fig. 2 (d), it may be of interest to consider a modified isolator-structure system with $\tau=1.0$ but $\nu=0.1$ (half of the value for system D). Separate calculations which are not shown here, indicate that at $\tau=1.0$ the effect of halving the ratio ν to 0.1 (i.e., doubling the damping of the isolator) is practically to double the perturbations. Still, the nonproportionality effects are small in comparison with systems A, B and C.

The above observations concerning magnitude of perturbations due to damping nonproportionality, are not all directly obtainable by mere examination of Eqs. (7), (10), (15)~(20). Nevertheles, these approximate explicit formulas are convenient for parametric study because they eliminate the need for numerically solving the eigenproblem of size-four matrices and also avoid complicated formulas from Ferrari's exact solution of the quartic characteristic equation. The present formulas have beeen written for ease of calculation, but it may be possible also to rewrite them in forms that would highlight the effect of each parameter. This latter task may not be very simple, however, since four parameters (μ, ν, τ and ε_b) are involved.

(3) Check on accuracy of perturbations

As noted above in connection with the large perturbations in system B (Fig. 2(b)), the applicability of the perturbation technique becomes questionable when it results to perturbations that are large enough to be comparable to unity in magnitude. As was done in the accuracy check for the numerical examples in Ref. 2), "exact" eigenvalues and eigenvectors may be obtained through the formulation that was pointed out by Foss[4], which involves a size-four matrix in the case of 2DOF system. These "exact"" values may be used as reference in calculating the errors in ω_1, ξ_1, $\boldsymbol{y}_1$, ω_2, ξ_2, and $\boldsymbol{y}_2$ as estimated by perturbation formulas in this paper.

Only for purposes of calculating errors, each complex eigenvector is renormalized in the manner shown in Fig. 3. That is, the element of vector $\boldsymbol{y}_j$ ($j=1$ or 2) that has the bigger modulus (i.e. square root of the sum of the squares of real and imaginary parts) is multiplied by a complex constant in order to make it pure real and equal to unity. This element of vector $\boldsymbol{y}_j$ then lies on the real axis of the Argand diagram of Fig. 3. The other eigenvector element, when multiplied by the same complex constant, takes a modulus L_j and argument θ_j, as shown in the same figure.

Percentage error is defined for each of the eight real scalars ω_1, ξ_1, L_1, θ_1, ω_2, ξ_2, L_2 and θ_2, in the following manner:

$$\varepsilon=\text{absolute value}\left(\frac{\text{exact}-\text{perturbed}}{\text{exact}}\right)\times 100\ \%$$

Errors for systems A and B are plotted in Figs. 4(a) and (b). Certain general observations may be made concerning the trend in the errors. For instance, it may be noted that the percentage error is roughly proportional to the absolute value of perturbation. In the upper half of Fig. 4(a), for example, the plot of $\varepsilon(\omega_1)$ resembles the plot of absolute value of α_1 from the upper half of Fig. 2(a). Recall that α_1 is the perturbation on ω_1.

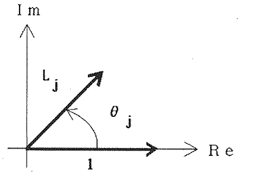

Fig. 3 Argand diagram of elements of renormalized eigenvector.

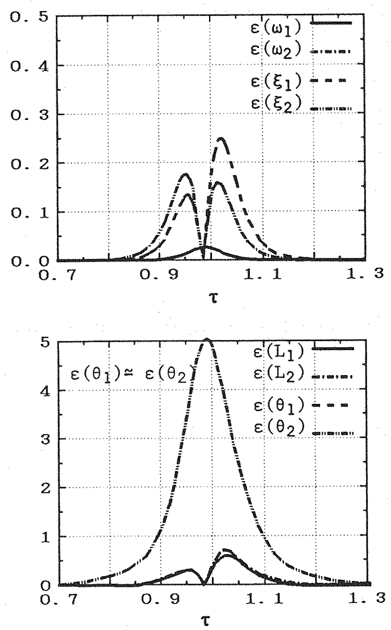

Fig. 4 (a) Percentage errors due to perturbation of structure-TMD system.

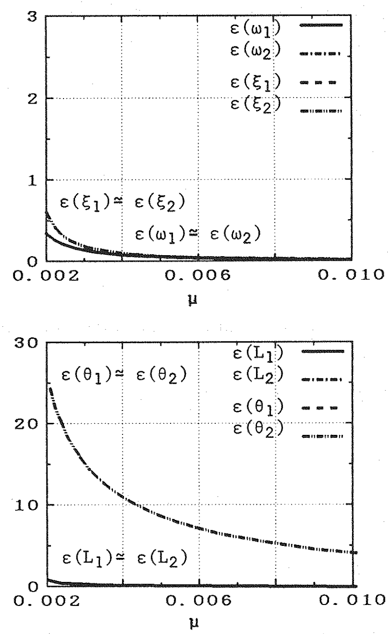

Fig. 4 (b) Percentage errors due to perturbation of structure-equipment system.

Similarly, the plot of $\varepsilon(\theta_1)$ in the lower half of Fig. 4 (a) resembles the plot of absolute value of η_{12} from the lower part of Fig. 2 (a).

The errors in θ_j, meaning the errors in η_{jk}, are the greatest. This trend is to be expected, in light of the preceding paragraph, and the observation from the parametric studies above, that the perturbations η_{12} and η_{21} dominate the others. As noted also in the numerical examples of Ref. 2), the magnitude of perturbation itself is an indication of the error involved in the formula. In this sense, the present perturbation technique generates its own accuracy check. The parametric studies in this paper and numerical examples in Ref. 2), suggest that the percentage error in θ does not exceed 10 % if the magnitude of perturbation η does not exceed 0.3. The errors in the other perturbations are then much smaller.

Table 2 is a summary of the range of error introduced by perturbation to each of the eight quantites ω_1, ξ_1, L_1, θ_1, ω_2, ξ_2, L_2 and θ_2. For system A, for example, the respective peaks of the plots in Fig. 4 (a) are listed as "maximum" percentage errors. For system B (or Fig. 4 (b)), the errors within the range of μ considered, do not reach zero; hence the respective lowest points of the plots in Fig. 4 (b) are listed in brackets as "minimum" percentage error.

Both Fig. 4 and Table 2 indicate that the present perturbation formulas may not be accurate enough for structure-equipment system when the mass ratio is extremely small. In principle, one may improve the

Table 2 Summary of maximum [and minimum] percentage errors in parameter range considered for examples.

	System A	System B	System C	System D
max $\varepsilon(\omega_1)$ [min]	0.028 % [0.000]	0.345 % [0.018]	0.047 % [0.000]	0.000 % [0.000]
max $\varepsilon(\xi_1)$ [min]	0.248 [0.000]	0.636 [0.019]	0.019 [0.000]	0.000 [0.000]
max $\varepsilon(L_1)$ [min]	0.616 [0.000]	0.760 [0.018]	0.023 [0.000]	0.000 [0.000]
max $\varepsilon(\theta_1)$ [min]	5.011 [0.000]	24.160 [4.093]	2.758 [0.008]	0.022 [0.000]
max $\varepsilon(\omega_2)$ [min]	0.026 [0.000]	0.338 [0.017]	0.029 [0.000]	0.000 [0.000]
max $\varepsilon(\xi_2)$ [min]	0.175 [0.000]	0.610 [0.018]	0.014 [0.000]	0.000 [0.000]
max $\varepsilon(L_2)$ [min]	0.675 [0.000]	0.768 [0.021]	0.020 [0.000]	0.000 [0.000]
max $\varepsilon(\theta_2)$ [min]	4.975 [0.000]	23.900 [4.072]	2.711 [0.008]	0.022 [0.000]

formulas by extending the order of perturbation beyond second order[2]. This does not ensure, however, that the resulting formulas would not be too long and involved. The formulas developed by Igusa and Der Kiureghian[5] may be used instead. Said formulas have been specifically derived under the assumption that the mass ratio μ is small (e. g., $\mu<0.01$ in the examples of Ref. 5)). However, it must be remembered that these formulas also assume that the overall damping is low (e. g., $\xi_a+\xi_b<0.06$ in the examples of Ref. 5). Also, the formulas are not nearly as explicit as Eqs. (7), (10), (15)～(20) of the present paper.

5. SUMMARY AND CONCLUSION

As a direct application of the general technique presented in a companion paper[2], the pseudo natural frequencies, modal damping ratios, and complex "modes" of close-coupled two-degree-of-freedom (2DOF) system with moderately nonproportional damping have been approximated in terms of natural frequencies and mode shapes of the counterpart undamped system, and of damping ratios of the two single-degree-of-freedom (SDOF) subsystems. The effects of damping nonproportionality have been expressed as nondimensional perturbations α_1, β_1, ζ_{12}, η_{12}, α_2, β_2, ζ_{21} and η_{21}.

The explicit formulas for complex modes, frequencies, and damping ratios, which include the perturbations, are convenient for parametric studies. It may be possible also to rewrite these explicit formulas in order to highlight either separate or joint influence of the following nondimensional parameters : mass ratio μ, damping proportionality parameter ν, tuning parameter τ, and damping ratio of b-subsystem ξ_b.

Parametric studies were performed on four commonly encountered systems : A) structure with TMD ; B) structure and equipment ; C) soil-structure system ; and D) base-isolated structure. These systems have varying degrees of damping nonproportionality. The parameters considered in the study covered a wide region of μ, ν, τ and ξ_b.

Among the significant findings of the parametric studies are :

1) The eigenvector perturbations are dominant. While natural frequencies and modal damping ratios may be hardly perturbed by damping nonproportionality, the natural modes are often converted into complex eigenvectors with significant imaginary parts.

2) While the condition $\nu=\tau$ is enough to remove nonproportionality, neither the difference $(\nu-\tau)$ nor its absolute value alone is sufficient as direct measure of perturbations. All the parameters μ, ν, τ and ξ_b have some influence.

3) A change in one parameter (μ, ν, τ, or ε_b) does not necessarily mean simultaneous increases or reductions in the magnitudes of all perturbations.

4) Nonproportionality and the associated perturbations tend to be significant when the two subsystems are tuned, i. e., $\tau=1.0$.

5) Comparing any two systems with identical τ, μ, and ν (where $\nu\neq\tau$), the system with larger ε_b—hence with higher overall damping—has stronger nonproportionality. The present formulas may be used even in such system where the overall damping is subcritical but high[2]. This is a major advantage over other previously reported perturbation formulas.

6) When the subsystem frequencies are tuned but the subsystem dampings are unequal, nonproportionality tends to increase with a decrease in mass ratio μ. For tuned structure-equipment system with extremely small mass ratio and low overall damping, other specialized perturbation formulas[5] may be more accurate.

APPENDIX-LIST OF MAJOR SYMBOLS

Matrices (size 2×2)

$\boldsymbol{C}$: damping matrix (Eq. (6·c))

$\boldsymbol{K}$: stiffness matrix (Eq. (6·d))
$\boldsymbol{M}$: mass or inertia matrix (Eq. (6·b))
$\boldsymbol{Y}_0$: modal matrix where column j is natural mode $\boldsymbol{y}_{0j}$

Vectors (size 2×1)

$\boldsymbol{x}$: displacement (Eq. (6·e))
$\boldsymbol{y}_j$: complex j-th eigenvector (Eq. (14))
$\boldsymbol{y}_{0j}$: j-th natural mode, mode shape, or real eigenvector (Eq. (9))

Scalars pertaining to subsystem s (s=a or b)

c_s : viscous damping coefficient
k_s : stiffness coefficient
m_s : mass
ω_s : natural frequency (Eq. (1))
ξ_s : damping ratio (Eq. (2))

Scalars relating subsystems a and b

μ : mass ratio (Eq. (3))
ν : ratio of damping ratios, or proportionality parameter (Eq. (4))
τ : frequency ratio, or tuning parameter (Eq. (5))
ρ : (Eq. (20))

Scalars pertaining to mode j (j=1 or 2)

s_j : (Eq. (7))
L_j : modulus of smaller element of renormalized j-th complex eigenvector (Fig. 3)
θ_j : phase angle of smaller element of renormalized j-th complex eigrnvector (Fig. 3)
α_j : perturbation on natural frequency (Eqs. (12) and (15))
β_j : perturbation on damping ratio (Eqs. (13) and (16))
λ_j : complex eigenvalue (Eq. (11))
λ_{0j} : complex unperturbed eigenvalue
ξ_{0j} : damping ratio when proportionally damped (Eq. (10))
ξ_j : pseudo damping ratio (Eq. (13))
ω_{0j} : natural frequency (Eqs. (7) and (8))
ω_j : pseudo natural frequency (Eq. (12))

Scalars relating modes j and j ((j, k) =(1, 2) or (2, 1))

A_{jk}= (Eq. (19))
ζ_{jk}=perturbation coefficient on real part of j-th mode (Eqs. (14) and (17))
η_{jk}=perturbation coefficient on imaginary part of j-th mode (Eqs. (14) and (18))

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REFERENCES

1) Aleksandrov, A. D. et al. (eds.) : Mathematics - Its Contents, Methods, and Meanings, MIT Press, Vol. I, pp 266-270, 1963.
2) Pacheco, B. M. and Fujino, Y. : Perturbation Technique to Approximate the Effect of Damping Nonproportionality in Modal Dynamic Analysis, JSCE J. Struct. Eng. / Earthq. Eng., Vol. 6, No. 1, pp. 169 s-178 s, 1989.
3) Warburton, G. B. and Ayorinde, E. O. : Optimum Absorber Parameters for Simple Systems, Earthq. Eng. Struct. Dyn., Vol. 8, pp. 197-217, 1980.
4) Foss, K. A. : Coordinates which Uncouple the Equations of Motion of Damped Linear Dynamic Systems, ASME J. App. Mech.,

Vol. 25, pp. 361-364, 1958.

5) Igusa, T. and Der Kiureghian, A. : Dynamic Characterization of Two-Degree-of-Freedom Equipment-Structure Systems, ASCE J. Eng. Mech., Vol. 111, No. 1, pp. 1-19, 1985.

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