Structural Eng. / Earthquake Eng. Vol. 5. No. 2, 293s- 302s. October 1988
Japan Society of Civil Engineers (Proc. of JSCE No. 398/I-10)

SOLUTIONS TO STRETCHING AND BENDING OF TRANSVERSELY ISOTROPIC, CIRCULAR THICK PLATES AND THEIR APPLICATION

*By Isamu A. OKUMURA**

A solution to a state of plane stress and a solution to a state of generalized plane stress of transversely isotropic, moderately thick plates in cylindrical coordinates are proposed. The solutions are derived from a generalized Elliott's solution which includes five potential functions. Expressions for components of displacement and stress in the solutions are presented in referring to non-axially symmetric problems of transversely isotropic, moderately thick circular or annular plates. As an application of the solutions, an axi-symmetric bending of a transversely isotropic, moderately thick circular plate subjected to a partial load is analyzed.

Keywords : elasticity, transverse isotropy, circular thick plate, annular thick plate

1. INTRODUCTION

The latest studies on two-dimensional or three-dimensional elasticity problems have been turning to those on anisotropic solids. Orthotropic and transversely isotropic solids among various classes of anisotropic solids are mainly treated from the practical necessity. A research on three-dimensional elasticity solutions to orthotropic solids is very complicated, because orthotropic solids have nine independent elastic constants. Therefore, three-dimensional elasticity solutions to have been found so far seem to be few except for those by Hata[1] and by Sonoda and Horikawa[2]. On the other hand, a research on three-dimensional elasticity solutions to transversely isotropic solids is comparatively simple and has been done by Elliott[3] and Lodge[4], because the number of independent elastic constants is only five. However, Elliott's solution yields the contraction of the solution that two independent potential functions are reduced to one potential function under equal roots of a quadratic equation. The author has proposed a generalized Elliott's solution in a previous paper[5] as a solution to make up this deficiency.

Though the three-dimensional elasticity solutions as stated above are due to be theoretically applicable to analyses of stretching and bending of thick plates, they are not practically applicable to analyses of moderately thick plates which are usually called thick plates because of the difficulty in numerical calculations. Therefore, a simplified and widely practicable elasticity solution is highly required as a solution to anisotropic, moderately thick plates. Studies on thin plates have been done by Lekhnitskii[6], Reissner and Stavsky[7] and by Wu and Vinson[8]. Also, studies on the theory of bending of orthotropic, rectangular thick plates have been done by Girkmann[9] and Panc[10]. The theory of moderately thick plates to have been stated in Love's book[11] seems to be the most exact in the theory of stretching and bending of

* Member of JSCE, Dr. Eng., Professor, Department of Civil Engineering, Kitami Institute of Technology (Koen-cho, Kitami).

isotropic, thick plates. The theory consists of a plane stress solution and a generalized plane stress solution[11]. If the theory is extended to that of anisotropic, thick plates, it becomes a strong means to analyze orthotropic or transversely isotropic, thick plates. However, studies on cylindrically anisotropic or transversely isotropic, moderately thick plates in cylindrical coordinates seem to have been hardly found even at the present time. The author only knows a study on a three-dimensional, thermal stress analysis of a transversely isotropic, very thick circular disc by Noda and Takeuti[12].

From the above point of view, this paper proposes a plane stress solution and a generalized plane stress solution to be available for analyses of stretching and bending of transversely isotropic, moderately thick plates in cylindrical coordinates. The derivation of the solutions is very complicated in contrast to that of thin plates, because displacements and stresses vary with the thickness coordinates in case of thick plates. The author has previously derived a plane stress solution and a generalized plane stress solution to isotropic, thick plates in cylindrical coordinates by making use of stress functions[13]. However, in this paper, the generalized Elliott's solution in place of stress functions is used for the derivation of the solutions, because it is difficult for transversely isotropic solids to derive the solutions from the direct use of the equations of equilibrium and the compatibility conditions in terms of components of stress. Therefore, in this paper, five potential functions included in the generalized Elliott's solution are ingeniously determined and relations between arbitrary constants included in the potential functions are determined from certain conditions of stresses to be satisfied. The determination of the potential functions is highly important to the construction of the solutions and is a point to need due consideration. The solutions are concretely stated to non-axially symmetric problems. An axi-symmetric bending of a transversely isotropic, moderately thick circular plate is analyzed as an application of the solutions.

2. A GENERALIZED ELLIOTT'S SOLUTION

We shall use a generalized Elliott's solution in the previous paper[5]. If we use cylindrical coordinates (r, θ, z) such that the axis of z is taken parallel to the axis of elastic symmetry, the solution is expressed in terms of components of displacement, that is, u_r, u_θ and u_z and elastic constants of transversely isotropic solids, that is, $c_{\alpha\beta}$ as

$$u_r=\frac{\partial}{\partial r}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]+\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad \text{(1·a)}$$

$$u_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\phi_{01}+\phi_{03}+\gamma_1\left(r\frac{\partial\phi_1}{\partial r}+z\frac{\partial\phi_3}{\partial z}\right)-\gamma_2\phi_1-\gamma_3\phi_3\right]-\frac{\partial\psi}{\partial r}, \quad \text{(1·b)}$$

$$u_z=\frac{\partial}{\partial z}\left[k_1(\phi_{01}-\gamma_3\phi_1)+k_2(\phi_{03}-\gamma_2\phi_3)+\gamma_1\left(k_1 r\frac{\partial\phi_1}{\partial r}+k_2 z\frac{\partial\phi_3}{\partial z}\right)\right], \quad \text{(1·c)}$$

in which

$$\nabla_1^2\phi_{01}+\nu_1\frac{\partial^2\phi_{01}}{\partial z^2}=0,\quad \nabla_1^2\phi_{03}+\nu_2\frac{\partial^2\phi_{03}}{\partial z^2}=0,\quad \nabla_1^2\phi_1+\nu_2\frac{\partial^2\phi_1}{\partial z^2}=0,\quad \nabla_1^2\phi_3+\nu_1\frac{\partial^2\phi_3}{\partial z^2}=0;$$

$$\nabla_1^2\psi+\nu_3\frac{\partial^2\psi}{\partial z^2}=0,\quad \nabla_1^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \quad \text{(2·a~e)}$$

$$\gamma_1=\begin{cases}1 & [\nu_1=\nu_2]\\ 0 & [\nu_1\neq\nu_2]\end{cases},\quad \gamma_2=\begin{cases}\dfrac{2c_{11}\nu_2}{c_{11}\nu_2-c_{44}} & [\nu_1=\nu_2]\\ \dfrac{2\nu_2}{\nu_1-\nu_2}\cdot\dfrac{c_{11}\nu_1-c_{44}}{c_{11}\nu_2-c_{44}} & [\nu_1\neq\nu_2]\end{cases},\quad \gamma_3=\begin{cases}0 & [\nu_1=\nu_2]\\ \dfrac{2\nu_2}{\nu_1-\nu_2} & [\nu_1\neq\nu_2]\end{cases}, \quad \text{(3·a~c)}$$

and ν_1 and ν_2 are the roots of

$$c_{11}c_{44}\nu^2+[c_{13}(c_{13}+2c_{44})-c_{11}c_{33}]\nu+c_{33}c_{44}=0, \quad \text{(4)}$$

and k_1, k_2 and ν_3 are the following parameters and ratio of the elastic constants, respectively:

$$k_1=\frac{c_{11}\nu_1-c_{44}}{c_{13}+c_{44}},\quad k_2=\frac{c_{11}\nu_2-c_{44}}{c_{13}+c_{44}},\quad \nu_3=\frac{c_{44}}{c_{66}}. \quad \text{(5·a~c)}$$

We will call the above solution by the name of the generalized Elliott's solution.

The generalized Hooke's law of transversely isotropic solids is expressed as

$$\sigma_{rr}=c_{11}\varepsilon_{rr}+c_{12}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz},\quad \sigma_{\theta\theta}=c_{12}\varepsilon_{rr}+c_{11}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz},\quad \sigma_{zz}=c_{13}\varepsilon_{rr}+c_{13}\varepsilon_{\theta\theta}+c_{33}\varepsilon_{zz};$$
$$\sigma_{\theta z}=2\,c_{44}\varepsilon_{\theta z},\quad \sigma_{zr}=2\,c_{44}\varepsilon_{zr},\quad \sigma_{r\theta}=2\,c_{66}\varepsilon_{r\theta},\qquad (6\cdot a\sim f)$$

in which

$$c_{66}=\frac{1}{2}(c_{11}-c_{12}),\qquad (7)$$

and σ_{ij} and ε_{ij} are stress tensor and strain tensor, respectively.

Components of strain are expressed in the form

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta},\quad \varepsilon_{zz}=\frac{\partial u_z}{\partial z},\quad \varepsilon_{\theta z}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right);$$
$$\varepsilon_{zr}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right).\qquad (8\cdot a\sim f)$$

Thus, the generalized Elliott's solution, the generalized Hooke's law and the components of strain which are needed for the derivation of a plane stress solution and a generalized plane stress solution have been stated.

3. A PLANE STRESS SOLUTION

We denote the thickness of a plate under consideration by h and put the origin of the cylindrical coordinates on the middle plane of the plate. A plane stress solution which will be sought here is a solution satisfying the following conditions :

$$\sigma_{zz}=0,\quad \sigma_{\theta z}=0,\quad \sigma_{zr}=0\,.\qquad (9\cdot a\sim c)$$

We put the potential functions satisfying Eqs. (2·a, c, e) in the form

$$\phi_{01}=\sum_{m=0}^{\infty}D_m^{(1)}\cos m\theta r^m+\sum_{m=0}^{\infty}D_m^{(3)}\cos m\theta\left[\frac{r^{m+2}}{2(m+1)}-\frac{r^m z^2}{\nu_1}\right],\qquad (10\cdot a)$$

$$\phi_1=\sum_{m=0}^{\infty}F_m^{(1)}\cos m\theta\left[\frac{r^{m+2}}{2(m+1)}-\frac{r^m z^2}{\nu_2}\right],\quad \psi=\sum_{m=0}^{\infty}E_m^{(1)}\sin m\theta\left[\frac{r^{m+2}}{2(m+1)}-\frac{r^m z^2}{\nu_3}\right],\qquad (10\cdot b,\ c)$$

in which $D_m^{(1)}$ to $E_m^{(1)}$ are arbitrary constants with superscripts (1) or (3) for the distinction.

Expressions for the components of displacement are obtained from the substitution of Eqs. (10·a~c) in Eqs. (1·a~c). Also, expressions for components of stress are obtained from Eqs. (6·a~f) and (8·a~f) with the aid of the expressions for the components of displacement. For economy of space, only expressions for the components of stress which are needed for the derivation of the solution will be stated.

Equation (10·a) yields

$$\sigma_{zz}^{(0,1)}=2\left(c_{13}-c_{33}\frac{k_1}{\nu_1}\right)\sum_{m=0}^{\infty}D_m^{(3)}\cos m\theta\ r^m,\qquad (11\cdot a)$$

$$\sigma_{\theta z}^{(0,1)}=2\,c_{44}\frac{1+k_1}{\nu_1}z\sum_{m=0}^{\infty}D_m^{(3)}m\sin m\theta\ r^{m-1},\qquad (11\cdot b)$$

$$\sigma_{zr}^{(0,1)}=-2\,c_{44}\frac{1+k_1}{\nu_1}z\sum_{m=0}^{\infty}D_m^{(3)}m\cos m\theta\ r^{m-1}.\qquad (11\cdot c)$$

Equation (10·b) yields

$$\sigma_{zz}^{(0,2)}=2\sum_{m=0}^{\infty}F_m^{(1)}\cos m\theta\left\{c_{13}[\gamma_1(m+2)-\gamma_2]-c_{33}k_1\frac{\gamma_1 m-\gamma_3}{\nu_2}\right\}r^m,\qquad (12\cdot a)$$

$$\sigma_{\theta z}^{(0,2)}=\frac{2\,c_{44}}{\nu_2}z\sum_{m=0}^{\infty}F_m^{(1)}m\sin m\theta[\gamma_1 m(1+k_1)-(\gamma_2+k_1\gamma_3)]\,r^{m-1},\qquad (12\cdot b)$$

$$\sigma_{zr}^{(0,2)}=-\frac{2\,c_{44}}{\nu_2}z\sum_{m=0}^{\infty}F_m^{(1)}m\cos m\theta[\gamma_1 m(1+k_1)-(\gamma_2+k_1\gamma_3)]\,r^{m-1}.\qquad (12\cdot c)$$

Equation (10·c) yields

$$\sigma_{zz}^{(0,3)}=0, \qquad (13\cdot a)$$

$$\sigma_{\theta z}^{(0,3)}=\frac{2\,c_{44}}{\nu_3}\,z\sum_{m=0}^{\infty}E_m^{(1)}\,m\sin m\theta r^{m-1}, \quad \sigma_{zr}^{(0,3)}=-\frac{2\,c_{44}}{\nu_3}\,z\sum_{m=0}^{\infty}E_m^{(1)}\,m\cos m\theta r^{m-1}. \qquad (13\cdot b,\ c)$$

In the first place, if we impose the following condition on the component of stress from condition (9·a) :

$$\sigma_{zz}^{(0,1)}+\sigma_{zz}^{(0,2)}=0, \qquad (14)$$

we obtain the following relation between the arbitrary constants :

$$D_m^{(3)}=-F_m^{(1)}\left(\frac{\nu_1}{\nu_2}\right)\left\{\frac{c_{13}}{c_{13}\nu_1-c_{33}k_1}[\nu_1\gamma_3+\nu_2(2\,\gamma_1-\gamma_2)]+\gamma_1 m-\gamma_3\right\}. \qquad (15)$$

In the second place, if we substitute the above arbitrary constant into Eqs. (11·b, c) and impose the following conditions on the components of stress from conditions (9·b, c) :

$$\sigma_{\theta z}^{(0,1)}+\sigma_{\theta z}^{(0,2)}+\sigma_{\theta z}^{(0,3)}=0, \quad \sigma_{zr}^{(0,1)}+\sigma_{zr}^{(0,2)}+\sigma_{zr}^{(0,3)}=0, \qquad (16\cdot a,\ b)$$

we obtain the following relation :

$$E_m^{(1)}=F_m^{(1)}\left(\frac{\nu_3}{\nu_2}\right)\left\{\frac{c_{13}(1+k_1)}{c_{13}\nu_1-c_{33}k_1}[\nu_1\gamma_3+\nu_2(2\,\gamma_1-\gamma_2)]+\gamma_2-\gamma_3\right\}. \qquad (17)$$

Thus, relations (15) and (17) between the arbitrary constants which have to satisfy the conditions of the plane stress solution have been obtained. If we use these relations and the following relations :

$$\nu_1\gamma_3+\nu_2(2\,\gamma_1-\gamma_2)=-\frac{2\,c_{44}\nu_2}{c_{11}\nu_2-c_{44}}, \quad \gamma_2-\gamma_3=\frac{2\,c_{11}\nu_2}{c_{11}\nu_2-c_{44}}, \qquad (18\cdot a,\ b)$$

we obtain the plane stress solution as the results :

$$u_r=u_r^{(0,1)}+u_r^{(0,2)}+u_r^{(0,3)}$$
$$=\sum_{m=0}^{\infty}\cos m\theta\left\{D_m^{(1)}m r^{m-1}+F_m^{(1)}\gamma_5\left[\frac{r^{m+1}}{m+1}\{\gamma_6[\nu_1(m+2)-\nu_3 m(1+k_1)]-(m+2)+m\nu_3\nu_4\}\right.\right.$$
$$\left.\left.+2\,k_1\gamma_6 m r^{m-1}z^2\right]\right\}, \qquad (19\cdot a)$$

$$u_\theta=u_\theta^{(0,1)}+u_\theta^{(0,2)}+u_\theta^{(0,3)}$$
$$=\sum_{m=0}^{\infty}(-)\sin m\theta\left\{D_m^{(1)}m r^{m-1}+F_m^{(1)}\gamma_5\left[\frac{r^{m+1}}{m+1}\{\gamma_6[\nu_1 m-\nu_3(1+k_1)(m+2)]-m+\nu_3\nu_4(m+2)\}\right.\right.$$
$$\left.\left.+2\,k_1\gamma_6 m r^{m-1}z^2\right]\right\}, \qquad (19\cdot b)$$

$$u_z=u_z^{(0,1)}+u_z^{(0,2)}=4\,k_1\gamma_5\gamma_6 z\sum_{m=0}^{\infty}(-)\,F_m^{(1)}\cos m\theta r^m, \qquad (19\cdot c)$$

$$\sigma_{rr}=\sigma_{rr}^{(0,1)}+\sigma_{rr}^{(0,2)}+\sigma_{rr}^{(0,3)}$$
$$=\sum_{m=0}^{\infty}\cos m\theta\{D_m^{(1)}m(m-1)(c_{11}-c_{12})\,r^{m-2}+F_m^{(1)}\gamma_5[r^m\{\gamma_6[m(c_{11}-c_{12})\{\nu_1-\nu_3(1+k_1)\}$$
$$+2\{\nu_1(c_{11}+c_{12})-2\,c_{13}k_1\}]-c_{11}(m+2)+c_{12}(m-2)+m\nu_3\nu_4(c_{11}-c_{12})\}$$
$$+2\,m(m-1)(c_{11}-c_{12})\,k_1\gamma_6 r^{m-2}z^2]\}, \qquad (19\cdot d)$$

$$\sigma_{\theta\theta}=\sigma_{\theta\theta}^{(0,1)}+\sigma_{\theta\theta}^{(0,2)}+\sigma_{\theta\theta}^{(0,3)}$$
$$=\sum_{m=0}^{\infty}\cos m\theta\{D_m^{(1)}m(m-1)(c_{12}-c_{11})\,r^{m-2}+F_m^{(1)}\gamma_5[r^m\{\gamma_6[m(c_{12}-c_{11})\{\nu_1-\nu_3(1+k_1)\}$$
$$+2\{\nu_1(c_{12}+c_{11})-2\,c_{13}k_1\}]-c_{12}(m+2)+c_{11}(m-2)+m\nu_3\nu_4(c_{12}-c_{11})\}$$
$$+2\,m(m-1)(c_{12}-c_{11})\,k_1\gamma_6 r^{m-2}z^2]\}, \qquad (19\cdot e)$$

$$\sigma_{r\theta}=\sigma_{r\theta}^{(0,1)}+\sigma_{r\theta}^{(0,2)}+\sigma_{r\theta}^{(0,3)}$$
$$=2\,c_{66}\sum_{m=0}^{\infty}(-)\,m\sin m\theta\{D_m^{(1)}(m-1)\,r^{m-2}-F_m^{(1)}\gamma_5[r^m\{1+\gamma_6[\nu_3(1+k_1)-\nu_1]-\nu_3\nu_4\}$$
$$-2(m-1)\,k_1\gamma_6 r^{m-2}z^2]\}, \qquad (19\cdot f)$$

in which

$$\gamma_5=\frac{c_{44}}{c_{11}\nu_2-c_{44}},\quad \gamma_6=\frac{c_{13}}{c_{13}\nu_1-c_{33}k_1},\quad \nu_4=\frac{c_{11}}{c_{44}}. \qquad (20\cdot a\sim c)$$

Though the solutions with minus exponents of power may be obtained from putting $m=-m$ in Eqs. (10·a) to (19·f) for $m\geq 2$, they are omitted on account of limited space.

Thus, a plane stress solution to a transversely isotropic, moderately thick plate has been stated.

4. A GENERALIZED PLANE STRESS SOLUTION

We denote the thickness of a plate by h and put the origin of the cylindrical coordinates on the middle plane of the plate in the same manner as the plane stress solution. A generalized plane stress solution which will be sought here is a solution satisfying the following conditions :

$$\sigma_{zz}=0,\quad (\sigma_{\theta z})_{z=\pm h/2}=0,\quad (\sigma_{zr})_{z=\pm h/2}=0. \qquad (21\cdot a\sim c)$$

We put the potential functions satisfying Eqs. (2·b, d, e) in the form

$$\phi_{03}=\sum_{m=0}^{\infty} A_m^{(1)}\cos m\theta r^m z+\sum_{m=0}^{\infty} A_m^{(3)}\cos m\theta\left[\frac{r^{m+2}}{2(m+1)}z-\frac{r^m z^3}{3\nu_2}\right], \qquad (22\cdot a)$$

$$\phi_3=\sum_{m=0}^{\infty} C_m^{(3)}\cos m\theta r^m z+\sum_{m=0}^{\infty} C_m^{(1)}\cos m\theta\left[\frac{r^{m+2}}{2(m+1)}z-\frac{r^m z^3}{3\nu_1}\right],\quad \psi=\sum_{m=0}^{\infty} B_m^{(1)}\sin m\theta r^m z, \qquad (22\cdot b,\ c)$$

in which $A_m^{(1)}$ to $B_m^{(1)}$ are arbitrary constants.

Equation (22·a) yields

$$\sigma_{zz}^{(0,1)}=2\left(c_{13}-c_{33}\frac{k_2}{\nu_2}\right)z\sum_{m=0}^{\infty} A_m^{(3)}\cos m\theta r^m, \qquad (23\cdot a)$$

$$\sigma_{\theta z}^{(0,1)}=c_{44}(1+k_2)\left\{\sum_{m=0}^{\infty}(-)A_m^{(1)}m\sin m\theta r^{m-1}+\sum_{m=0}^{\infty}(-)A_m^{(3)}m\sin m\theta\left[\frac{r^{m+1}}{2(m+1)}-\frac{r^{m-1}z^2}{\nu_2}\right]\right\}, \qquad (23\cdot b)$$

$$\sigma_{zr}^{(0,1)}=c_{44}(1+k_2)\left\{\sum_{m=0}^{\infty} A_m^{(1)}m\cos m\theta r^{m-1}+\sum_{m=0}^{\infty} A_m^{(3)}\cos m\theta\left[\frac{m+2}{2(m+1)}r^{m+1}-\frac{m}{\nu_2}r^{m-1}z^2\right]\right\}. \qquad (23\cdot c)$$

Equation (22·b) yields

$$u_r^{(0,2)}=(\gamma_1-\gamma_3)z\sum_{m=0}^{\infty} C_m^{(3)}m\cos m\theta r^{m-1}+\sum_{m=0}^{\infty} C_m^{(1)}\cos m\theta\left[\frac{m+2}{2(m+1)}(\gamma_1-\gamma_3)r^{m+1}z-\frac{m}{3\nu_1}(3\gamma_1-\gamma_3)r^{m-1}z^3\right], \qquad (24\cdot a)$$

$$u_\theta^{(0,2)}=(\gamma_1-\gamma_3)z\sum_{m=0}^{\infty}(-)C_m^{(3)}m\sin m\theta r^{m-1}+\sum_{m=0}^{\infty}(-)C_m^{(1)}m\sin m\theta\left[\frac{\gamma_1-\gamma_3}{2(m+1)}r^{m+1}z-\frac{3\gamma_1-\gamma_3}{3\nu_1}r^{m-1}z^3\right], \qquad (24\cdot b)$$

$$\sigma_{zz}^{(0,2)}=2\left[c_{13}(\gamma_1-\gamma_3)-c_{33}k_2\frac{3\gamma_1-\gamma_2}{\nu_1}\right]z\sum_{m=0}^{\infty} C_m^{(1)}\cos m\theta r^m, \qquad (24\cdot c)$$

$$\sigma_{\theta z}^{(0,2)}=c_{44}[\gamma_1(1+k_2)-(\gamma_3+k_2\gamma_2)]\sum_{m=0}^{\infty}(-)C_m^{(3)}m\sin m\theta r^{m-1}+c_{44}\sum_{m=0}^{\infty}(-)C_m^{(1)}m\sin m\theta$$
$$\cdot\left\{\frac{r^{m+1}}{2(m+1)}[\gamma_1(1+k_2)-(\gamma_3+k_2\gamma_2)]-\frac{r^{m-1}z^2}{\nu_1}[3\gamma_1(1+k_2)-(\gamma_3+k_2\gamma_2)]\right\}, \qquad (24\cdot d)$$

$$\sigma_{zr}^{(0,2)}=c_{44}[\gamma_1(1+k_2)-(\gamma_3+k_2\gamma_2)]\sum_{m=0}^{\infty} C_m^{(3)}m\cos m\theta r^{m-1}+c_{44}\sum_{m=0}^{\infty} C_m^{(1)}\cos m\theta$$
$$\cdot\left\{\frac{m+2}{2(m+1)}[\gamma_1(1+k_2)-(\gamma_3+k_2\gamma_2)]r^{m+1}-\frac{m}{\nu_1}[3\gamma_1(1+k_2)-(\gamma_3+k_2\gamma_2)]r^{m-1}z^2\right\}. \qquad (24\cdot e)$$

Equation (22·c) yields

$$u_r^{(0,3)}=z\sum_{m=0}^{\infty} B_m^{(1)}m\cos m\theta r^{m-1},\quad u_\theta^{(0,3)}=z\sum_{m=0}^{\infty}(-)B_m^{(1)}m\sin m\theta r^{m-1},\quad \sigma_{zz}^{(0,3)}=0;$$

$$\sigma_{\theta z}^{(0,3)}=c_{44}\sum_{m=0}^{\infty}(-)B_m^{(1)}m\sin m\theta r^{m-1},\quad \sigma_{zr}^{(0,3)}=c_{44}\sum_{m=0}^{\infty}B_m^{(1)}m\cos m\theta r^{m-1}. \tag{25·a∼e}$$

In the first place, if we impose the following condition on the component of stress from condition (21·a):

$$\sigma_{zz}^{(0,1)}+\sigma_{zz}^{(0,2)}=0, \tag{26}$$

we obtain the following relation between the arbitrary constants:

$$A_m^{(3)}=C_m^{(1)}\left(\frac{\nu_2}{\nu_1}\right)\left\{\frac{c_{13}}{c_{13}\nu_2-c_{33}k_2}[\nu_1\gamma_3+\nu_2(2\gamma_1-\gamma_2)]-(3\gamma_1-\gamma_2)\right\}. \tag{27}$$

In the second place, the sums of Eqs. (24·a, b) and (25·a, b) yield

$$u_r^{(0,2)}+u_r^{(0,3)}=z\sum_{m=0}^{\infty}[B_m^{(1)}+C_m^{(3)}(\gamma_1-\gamma_3)]\,m\cos m\theta r^{m-1}+\cdots, \tag{28·a}$$

$$u_\theta^{(0,2)}+u_\theta^{(0,3)}=z\sum_{m=0}^{\infty}(-)[B_m^{(1)}+C_m^{(3)}(\gamma_1-\gamma_3)]\,m\sin m\theta r^{m-1}+\cdots. \tag{28·b}$$

The first terms of the summation notations in the above equations are needless solutions. Therefore, we exclude their terms and so obtain the following relation:

$$B_m^{(1)}=-C_m^{(3)}(\gamma_1-\gamma_3). \tag{29}$$

Lastly, if we substitute relations (27) and (29) into Eqs. (23·b, c) and (25·d, e), respectively and impose the following conditions on the components of stress from conditions (21·b, c):

$$(\sigma_{\theta z}^{(0,1)}+\sigma_{\theta z}^{(0,2)}+\sigma_{\theta z}^{(0,3)})_{z=\pm h/2}=0,\quad (\sigma_{zr}^{(0,1)}+\sigma_{zr}^{(0,2)}+\sigma_{zr}^{(0,3)})_{z=\pm h/2}=0, \tag{30·a, b}$$

we have

$$A_m^{(1)}(1+k_2)+C_m^{(3)}k_2(\gamma_1-\gamma_2)=\frac{h^2}{4}\cdot\frac{C_m^{(1)}}{\nu_1}\left\{\gamma_2-\gamma_3+\frac{c_{13}(1+k_2)}{c_{13}\nu_2-c_{33}k_2}[\nu_1\gamma_3+\nu_2(2\gamma_1-\gamma_2)]\right\}. \tag{31}$$

From the above equation, we obtain the following relation:

$$C_m^{(3)}=-\frac{A_m^{(1)}(1+k_2)}{k_2(\gamma_1-\gamma_2)}+\frac{h^2C_m^{(1)}}{4\nu_1k_2(\gamma_1-\gamma_2)}\left\{\gamma_2-\gamma_3+\frac{c_{13}(1+k_2)}{c_{13}\nu_2-c_{33}k_2}[\nu_1\gamma_3+\nu_2(2\gamma_1-\gamma_2)]\right\}. \tag{32}$$

Thus, relations (27), (29) and (32) between the arbitrary constants which have to satisfy the conditions of the generalized plane stress solution have been obtained. If we use these relations, Eqs. (18·a, b) and Eqs. (20·a, c), we obtain the generalized plane stress solution as the results:

$$u_r=u_r^{(0,1)}+u_r^{(0,2)}+u_r^{(0,3)}$$

$$=z\sum_{m=0}^{\infty}\cos m\theta\left\{A_m^{(1)}mr^{m-1}-C_m^{(1)}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5\left[\frac{m+2}{m+1}r^{m+1}(\nu_2\gamma_7-1)-\frac{2m}{3}r^{m-1}z^2(\gamma_7-\nu_4)\right]\right\}, \tag{33·a}$$

$$u_\theta=u_\theta^{(0,1)}+u_\theta^{(0,2)}+u_\theta^{(0,3)}$$

$$=z\sum_{m=0}^{\infty}(-)m\sin m\theta\left\{A_m^{(1)}r^{m-1}-C_m^{(1)}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5\left[\frac{r^{m+1}}{m+1}(\nu_2\gamma_7-1)-\frac{2}{3}r^{m-1}z^2(\gamma_7-\nu_4)\right]\right\}, \tag{33·b}$$

$$u_z=u_z^{(0,1)}+u_z^{(0,2)}$$

$$=\sum_{m=0}^{\infty}(-)\cos m\theta\left\{A_m^{(1)}r^m+C_m^{(1)}k_2\left(\frac{\nu_2}{\nu_1}\right)\gamma_5\left[\frac{r^{m+2}}{m+1}(\nu_2\gamma_7-1+\nu_1\nu_4)\right.\right.$$

$$\left.\left.+r^m\left\{\frac{h^2}{2k_2}[\gamma_7(1+k_2)-\nu_4]-2\gamma_7z^2\right\}\right]\right\}, \tag{33·c}$$

$$\sigma_{rr}=\sigma_{rr}^{(0,1)}+\sigma_{rr}^{(0,2)}+\sigma_{rr}^{(0,3)}$$

$$=z\sum_{m=0}^{\infty}\cos m\theta\left\{A_m^{(1)}m(m-1)(c_{11}-c_{12})r^{m-2}-C_m^{(1)}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5\left[r^m\{\gamma_7[\nu_2\{c_{11}(m+2)-c_{12}(m-2)\}\right.\right.$$

$$\left.\left.-4c_{13}k_2]-c_{11}(m+2)+c_{12}(m-2)\}-\frac{2m(m-1)}{3}(c_{11}-c_{12})r^{m-2}z^2(\gamma_7-\nu_4)\right]\right\}, \tag{33·d}$$

$$\sigma_{\theta\theta}=\sigma_{\theta\theta}^{(0,1)}+\sigma_{\theta\theta}^{(0,2)}+\sigma_{\theta\theta}^{(0,3)}$$

$$=z\sum_{m=0}^{\infty}\cos m\theta\left\{A_m^{(1)}m(m-1)(c_{12}-c_{11})r^{m-2}-C_m^{(1)}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5\left[r^m\{\gamma_7[\nu_2\{c_{12}(m+2)-c_{11}(m-2)\}\right.\right.$$

$$-4\,c_{13}k_2]-c_{12}(m+2)+c_{11}(m-2)\}-\frac{2\,m\,(m-1)}{3}(c_{12}-c_{11})\,r^{m-2}\,z^2(\gamma_7-\nu_4)\Big]\Big\}, \qquad (33\cdot e)$$

$$\sigma_{\theta z}=\sigma_{\theta z}^{(0,1)}+\sigma_{\theta z}^{(0,2)}+\sigma_{\theta z}^{(0,3)}$$

$$=\frac{c_{44}}{2}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5[\gamma_7(1+k_2)-\nu_4](h^2-4\,z^2)\sum_{m=0}^{\infty}C_m^{(1)}m\sin m\theta\;r^{m-1}, \qquad (33\cdot f)$$

$$\sigma_{zr}=\sigma_{zr}^{(0,1)}+\sigma_{zr}^{(0,2)}+\sigma_{zr}^{(0,3)}$$

$$=\frac{c_{44}}{2}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5[\gamma_7(1+k_2)-\nu_4](h^2-4\,z^2)\sum_{m=0}^{\infty}(-)\,C_m^{(1)}m\cos m\theta\;r^{m-1}, \qquad (33\cdot g)$$

$$\sigma_{r\theta}=\sigma_{r\theta}^{(0,1)}+\sigma_{r\theta}^{(0,2)}+\sigma_{r\theta}^{(0,3)}$$

$$=2\,c_{66}\,z\sum_{m=0}^{\infty}(-)\,m\sin m\theta\left\{A_m^{(1)}(m-1)\,r^{m-2}-C_m^{(1)}\left(\frac{\nu_2}{\nu_1}\right)\gamma_5\Big[r^m(\nu_2\gamma_7-1)\right.$$

$$\left.-\frac{2(m-1)}{3}\,r^{m-2}z^2(\gamma_7-\nu_4)\Big]\right\}, \qquad (33\cdot h)$$

in which

$$\gamma_7=\frac{c_{13}}{c_{13}\nu_2-c_{33}k_2}. \qquad (34)$$

Though the solutions with minus exponents of power may be obtained from putting $m=-m$ in Eqs. (22·a) to (33·h) for $m\geq 2$, they are omitted.

Thus, a generalized plane stress solution to a transversely isotropic, moderately thick plate has been stated.

5. AN APPLICATION OF THE PRESENT SOLUTIONS TO AN AXI-SYMMETRIC BENDING OF A CIRCULAR THICK PLATE

As an application of the solutions stated in Chaps. 3 and 4, we analyze an axi-symmetric bending of a transversely isotropic, moderately thick circular plate as shown in Fig. 1.

(1) A homogeneous solution

If we put $m=0$ in Eqs. (19·a~f) and (33·a~h), we obtain the plane and generalized plane stress solutions to axi-symmetric problems of transversely isotropic, moderately thick plates. We use their solutions as a homogeneous solution here. From adding the plane and generalized plane stress solutions, expressions for components of displacement and stress to have a direct bearing on boundary conditions are as follows:

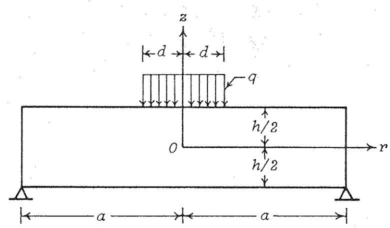

Fig. 1 Coordinate system of circular plate.

$$u_z^{(0)}=-4\,F_1k_1\gamma_5\gamma_6z+A_1-C_1\left(\frac{\nu_2}{\nu_1}\right)k_2\gamma_5\left\{r^2(\nu_2\gamma_7-1+\nu_1\nu_4)+\frac{h^2}{2\,k_2}[\gamma_7(1+k_2)-\nu_4]-2\,\gamma_7z^2\right\}, \qquad (35\cdot a)$$

$$\sigma_{rr}^{(0)}=-2\,F_1\gamma_5\{c_{11}+c_{12}-\gamma_6[\nu_1(c_{11}+c_{12})-2\,c_{13}k_1]\}$$

$$-2\,C_1\left(\frac{\nu_2}{\nu_1}\right)\gamma_5z\left\{\gamma_7[\nu_2(c_{11}+c_{12})-2\,c_{13}k_2]-(c_{11}+c_{12})\right\}, \qquad (35\cdot b)$$

in which F_1, A_1 and C_1 are arbitrary constants to be determined from given boundary conditions on the circular edge.

(2) A particular solution

A particular solution is needed for satisfying loading conditions on the upper and lower faces of the plate, because the homogeneous solution can not satisfy their conditions. The potential functions to be needed for the axi-symmetric bending are as follows:

$$\phi_{03}=\sum_{s=1}^{\infty}J_0(\alpha_sr)\left(A_s^{(1)}\cosh\frac{\alpha_sz}{\sqrt{\nu_2}}+L_s^{(1)}\sinh\frac{\alpha_sz}{\sqrt{\nu_2}}\right), \qquad (36\cdot a)$$

$$\phi_3=\sum_{s=1}^{\infty} J_0(\alpha_s r)\left(C_s^{(1)}\cosh\frac{\alpha_s z}{\sqrt{\nu_1}}+M_s^{(1)}\sinh\frac{\alpha_s z}{\sqrt{\nu_1}}\right), \quad (36\cdot b)$$

in which $A_s^{(1)}$ to $M_s^{(1)}$ are arbitrary constants to be determined from given loading conditions on the upper and lower faces and $\alpha_s=\lambda_s/a$.

Expressions for components of displacement are obtained from the substitution of Eqs. (36·a, b) in Eqs. (1·a, c). Also, expressions for components of stress are obtained from Eqs. (6·a, b, c, e) and (8·a, b, c, e) with the aid of the expressions for the components of displacement. Their components are distinguished from the homogeneous solution by superscript (1).

(3) Loading and boundary conditions

We consider the circular thick plate whose upper face is subjected to a partially distributed uniform load

Table 1 Elastic constants $c_{\alpha\beta}$ (in units of 10GPa).

$c_{\alpha\beta}$	c_{44}	c_{11}	c_{33}	c_{12}	c_{13}
Magnesium	1.64	5.97	6.17	2.62	2.17
Cadmium	1.56	11.0	4.69	4.04	3.83
Isotropy	1.0	3.0	3.0	1.0	1.0

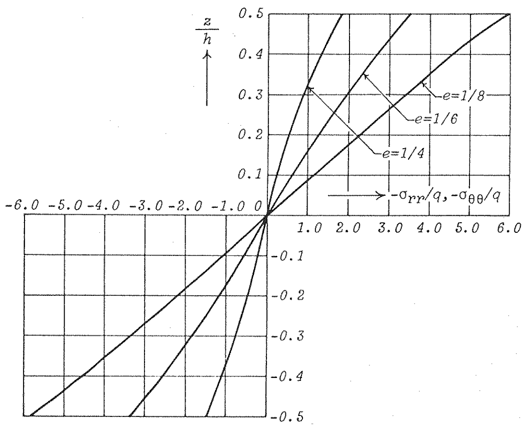

Fig. 2 Stress distribution of σ_{rr} (Magnesium, $r=0$, $e=h/(2a)$).

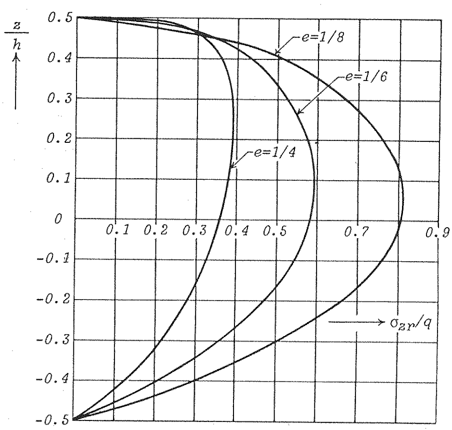

Fig. 3 Stress distribution of σ_{zr} (Magnesium, $r=0.3a$, $e=h/(2a)$).

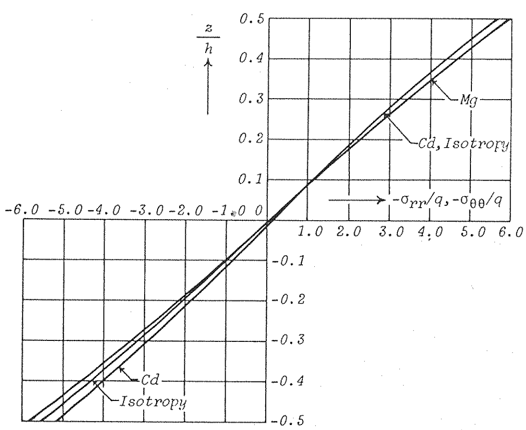

Fig. 4 Comparison of σ_{rr} between Magnesium, Cadmium and Isotropy ($r=0$, $e=h/(2a)=1/8$).

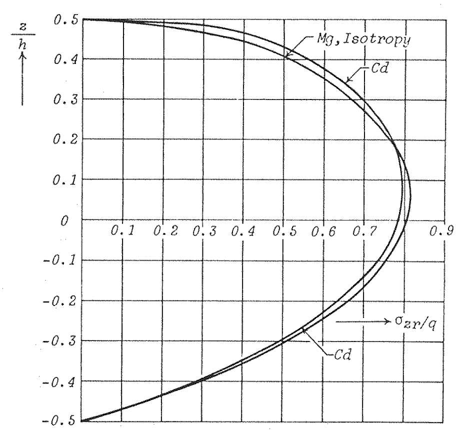

Fig. 5 Comparison of σ_{zr} between Magnesium, Cadmium and Isotropy ($r=0.3a$, $e=h/(2a)=1/8$).

over a circular area and whose lower face is free from surface tractions. Then, loading conditions become

$$\text{on } z=\frac{h}{2}, \quad \sigma_{zr}=0, \quad \sigma_{zz}=-p(r), \qquad (37\cdot\text{a, b})$$

$$\text{on } z=-\frac{h}{2}, \quad \sigma_{zr}=0, \quad \sigma_{zz}=0, \qquad (37\cdot\text{c, d})$$

in which

$$p(r)=\begin{cases} q & [0\leq r<d] \\ 0 & [d<r\leq a] \end{cases}=\sum_{s=1}^{\infty} e_s J_0(\alpha_s r). \qquad (38)$$

The coefficient e_s in the above equation denotes a Fourier coefficient in a Bessel expansion of $p(r)$ and is expressed in the form

$$e_s=2q\left(\frac{d}{a}\right)\frac{J_1(\lambda_s d/a)}{\lambda_s J_1^2(\lambda_s)}, \quad \lambda_s=\text{root's of } J_0(\lambda_s)=0. \qquad (39\cdot\text{a, b})$$

From imposing loading conditions (37·a∼d) on the components of stress, that is, $\sigma_{zr}^{(1)}$ and $\sigma_{zz}^{(1)}$, the arbitrary constants in the particular solution are exactly determined.

If the circular edge of the plate under consideration is simply supported, boundary conditions become

$$\text{on } r=a, \quad T_{rr}=0, \quad (u_z)_{z=0}=0, \quad M_r=0, \qquad (40\cdot\text{a}\sim\text{c})$$

in which

$$T_{rr}=\int_{-h/2}^{h/2}(\sigma_{rr}^{(0)}+\sigma_{rr}^{(1)})\,dz, \quad M_r=\int_{-h/2}^{h/2} z\,(\sigma_{rr}^{(0)}+\sigma_{rr}^{(1)})\,dz. \qquad (41\cdot\text{a, b})$$

From boundary conditions (40·a∼c), the arbitrary constants in the homogeneous solution are exactly determined.

(4) Numerical results

Numerical calculations were made for a circular thick plate with $d/a=0.3$ and various values of $e=h/(2a)$, referring to two materials of transverse isotropy and a material of isotropy. The values of the elastic constants of magnesium crystal, cadmium crystal and an isotropic material with Poisson's ratio $\nu=0.25$ are given in Table 1[14]. Numerical results were obtained from taking the first 56 terms in the Fourier series. They are shown in Figs. 2 to 5.

Thus, an axi-symmetric bending of a transversely isotropic, moderately thick circular plate has been analyzed as an application of the plane and generalized plane stress solutions.

6. CONCLUDING REMARKS

From paying attention to the latest increase of transversely isotropic materials, this paper proposed a plane stress solution and a generalized plane stress solution to transversely isotropic, moderately thick plates in cylindrical coordinates and analyzed an axi-symmetric bending of a circular thick plate as an application of their solutions. The solutions were derived from the generalized Elliott's solution in the previous paper. As stated in Chap. 1, three-dimensional elasticity solutions are not practically applicable to analyses of stretching and bending of moderately thick plates which are usually called thick plates except for a particular case. Therefore, the simplified and widely practicable elasticity solutions presented in this paper are highly important to practical analyses of moderately thick plates and the theory of elasticity. It has been confirmed that the solutions were exactly coincident with plane and generalized plane stress solutions to isotropic, thick plates previously reported by the author, when the solutions were specialized into those to isotropic solids. The solutions are applicable to an analysis of stretching and particular bending of a circular thick plate or an annular thick plate in themselves. Also, the solutions are applicable to analyses of more general bending of the thick plates, when a particular solution is used together as stated in Chap. 5. Furthermore, the solutions are applicable to a thermal stress analysis of a transversely isotropic, moderately thick plate which has lately attracted considerable attention.

In consideration of the premises, the author may conclude that the plane and generalized plane stress solutions presented in this paper are fully useful for elasticity and thermoelasticity problems of transversely isotropic, moderately thick plates in cylindrical coordinates.

REFERENCES

1) Hata, K. : On one method for solving three-dimensional elasticity problems in orthotropic solids, Proc. of the 6th Japan Natl. Cong. Appl. Mech., pp. 43～46, 1956.
2) Sonoda, K. and Horikawa, T. : Displacement functions for an orthotropic elastic body and their applications to thick plate problems, Theoretical and Applied Mechanics, Vol. 29, pp. 117～126, 1981.
3) Elliott, H. A. : Three-dimensional stress distributions in hexagonal aeolotropic crystals, Proc. of the Cambridge Philosophical Society, Vol. 44, pp. 522～533, 1948.
4) Lodge, A. S. : The transformation to isotropic form of the equilibrium equations for a class of anisotropic elastic solids, Quart. J. of Mech. and Appl. Math., Vol. 8, pp. 211～225, 1955.
5) Okumura, I. A. : Generalization of Elliott's solution to transversely isotropic solids and its application, Proc. of JSCE, No. 386/Ⅰ-8, Structural Eng. /Earthquake Eng., Vol. 4, No. 2, pp. 185～195, 1987.
6) Lekhnitskii, S. G. : Anisotropic Plates, Chap. 2, Gordon and Breach Science Publishers, 1968.
7) Reissner, E. and Stavsky, Y. : Bending and stretching of certain types of heterogeneous aeolotropic elastic plates, J. of Appl. Mech., Vol. 28, No. 3, pp. 402～408, 1961.
8) Wu, C.-Ⅰ. and Vinson, J. R. : Influences of large amplitudes, transverse shear deformation, and rotatory inertia on lateral vibrations of transversely isotropic plates, J. of Appl. Mech., Vol. 36, No. 2, pp. 254～260, 1969.
9) Girkmann, K. : Flächentragwerke, 6. Aufl., Anhang, Springer-Verlag, 1978.
10) Panc, V. : Theories of Elastic Plates, Part Ⅱ, Noordhoff International Publishing, 1975.
11) Love, A. E. H. : A Treatise on the Mathematical Theory of Elasticity, 4th ed., Chap. 22, Dover Publications, 1944.
12) Noda, N. and Takeuti, Y. : Transient thermal stresses in a transversely isotropic thick circular plate, Theoretical and Applied Mechanics, Vol. 31, pp. 123～133, 1982.
13) Okumura, I. A. : On some solutions to plane stress problems in cylindrical coordinates, Mem. of the Kitami Institute of Technology, Vol. 12, No. 2, pp. 225～238, 1981 (in Japanese).
14) Huntington, H. B. : The elastic constants of crystals, Solid State Physics, edited by Seitz, F. and Turnbull, D., Vol. 7, p. 213, Academic Press, 1957.

(Received April 14 1988)