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TWO DIMENSIONAL FINITE ELEMENT FLOW ANALYSIS USING THE VELOCITY CORRECTION METHOD

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In this paper, a finite element method based on the velocity correction method is presented. However there arise difficulties about the boundary conditions which have been used for the solution of the pressure Poisson equation[1),3)]. For example, in the calculations of the conventional analysis, uniform pressure ($P=0$) or normal gradient of pressure equal to zero ($\partial P/\partial n=0$) has been adopted especially on the open boundary which is artificially introduced as the limitation of the calculation domain for the sake of analysis convenience. However, these boundary conditions are false because the pressure on the boundary can not be prescribed in general for the time-dependent problems. In order to improve these boundary conditions, a new approximation method is proposed in which the boundary pressure Poisson equation is solved for each time step.

The method presented in this paper employes the linear interpolation functions based on the quardrilateral isoparametric elements.

Keywords : finite element method, velocity correction method, flow analysis, boundary condition

1. INTRODUCTION

For the numerical analysis of the time-dependent incompressible viscous flow, several finite element methods based on the fractional step method have been successfully presented. In this paper, the velocity correction method will be presented which is classified into a branch of the fractional step method[7),2)].

It is well known that the velocity correction method was introduced by Chorin[4)] (1968) for a finite difference scheme. After the several improvements conducted by a plenty of investigators, Donea et al.[6)] (1982) have developed the practically useful procedure for the finite element method. They calculated several models for example : driven cavity flow using 4-node linear element, axisymmetric flow using 9-node quardrilateral element and the natural convection flow. Furthermore. Schneider et al.[5)] have presented the method using equal order pressure and velocity interpolations in time-dependent problem based on the velocity correction procedure. They used isoparametric quadrilateral finite element for the several test models. Kawahara et al.[8)] applied the scheme to the density flow in a tank with sloping bottom, in which linear triangular elements were used for all velocity, pressure and potential variables. Moreover, Kawahara and Ramaswamy[9)] have presented Lagrangian-Eulerian finite element method in which they used the velocity correction method effectively and calculated free surface problems which involved sloshing in a tank, solitary wave propagation et al.

In this paper, the finite element method based on the velocity correction method is presented. The advantages to utilize the velocity correction method can be described as follows.

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a) Incompressible continuity is sufficient for numerical results.
b) Both mixed interpolation method and smoothing technique for pressure are not necessary for the present method.
c) Computing performance is more efficient than the implicit scheme.

In case of incompressible flow, the equation of momentum can be written by velocity and pressure as unknown variables. Contrary to this, the equation of continuity does not include the pressure as an unknown variable. In the sense of numerical analysis the fact that the pressure is not included in the equation of continuity means computational inconvenience because computation can not continue using the pressure as a field variable. Therefore, it is neccessary to formulate the equation for pressure as a field variable. The differentiations of momentum equation using continuity equation leads to the pressure Poisson equation. The velocity correction method proceeds in the following manner. The intermediate velocity components can be obtained by the explicit discretized form of momentum equation. where the pressure gradient term is not included. Then, the pressure can be calculated by the pressure Poisson equation. Based on this pressure, the corrected velocity can be obtained for the next time step. In this computation, the boundary conditions must be introduced to coincide the solutions to be the same ones as those of the original equations.

In the fractional step methods, it is important to focus the research on the boundary condition for the open boundary, especially on out-flow or flow-through problem. In the present paper, the boundary pressure Poisson equation is introduced and how to solve the equation system is described.

2. THE VELOCITY CORRECTION METHOD

The algorithm which is used for the velocity correction method is described in this section. Non-dimensional form of the time-dependent incompressible Navier-Stokes equations are written using time t, as follows,

$$\frac{\partial U}{\partial t}+\nabla P+U\cdot\nabla U-\frac{1}{Re}\nabla^2 U=0 \quad (1)$$

$$\nabla\cdot U=0 \quad (2)$$

where, U and P are velocity and pressure respectively, and Re means Reynolds number.

Applying the forward Euler scheme on the time derivative term if velocity; i.e. $\partial U/\partial t \doteqdot (U^{m+1}-U^m)/\Delta t$, and approximating pressure as $P\doteqdot P^{m+1}$, eq. (1) is rewritten in the following form.

$$U^{m+1}+\Delta t\nabla P^{m+1}=U^m+\Delta t\left(\frac{1}{Re}\nabla^2 U^m-U^m\cdot\nabla U^m\right) \quad (3)$$

Taking the both sides of eq. (3) and substituting the continuity condition; $\nabla\cdot U^{m+1}=0$, the pressure Poisson equation can be obtained.

$$\nabla^2 P^{m+1}=\frac{1}{\Delta t}\nabla\cdot\tilde{U} \quad (4)$$

where,

$$\tilde{U}=U^m+\Delta t\left(\frac{1}{Re}\nabla^2 U^m-U^m\cdot\nabla U^m\right) \quad (5)$$

Here, $\tilde{U}$ means intermediate velocity and plays a role of predictor. And P^{m+1} is the corrector for U^{m+1}.

3. FINITE ELEMENT METHOD

For the approximation of spatial functions, the finite element method is used. The weighted residual equations of eq. (5), (4) and (3) are formulated as eqs. (6), (7) and (8), respectively,

$$\int_\Omega \tilde{U}^*\tilde{U}\,d\Omega=\int_\Omega \tilde{U}^*U^m d\Omega$$
$$+\Delta t\left(\frac{1}{Re}\int_\Gamma \tilde{U}^*\nabla U^m d\Gamma-\frac{1}{Re}\int_\Omega \nabla\tilde{U}^*\nabla U^m d\Omega-\int_\Omega \tilde{U}^*U^m\cdot\nabla U^m d\Omega\right) \quad (6)$$

$$\int_\Omega \nabla P^* \nabla P^{m+1} d\Omega = \int_\Gamma P^* \nabla P^{m+1} d\Gamma - \frac{1}{\Delta t} \int_\Omega P^* \nabla \cdot \tilde{U} d\Omega \quad (7)$$

$$\int_\Omega U^* U^{m+1} d\Omega = \int_\Omega U^* \tilde{U} d\Omega - \Delta t \int_\Omega U^* \nabla P^{m+1} d\Omega \quad (8)$$

where, $\tilde{U}^*$, U^*, and P^* are weighting functions for intermediate velocity, primitive velocity and pressure, respectively. Both Laplacian terms of velocity and pressure are decomposed by the formula of integration by parts. The boundary integral term for pressure is treated on the right hand side and the reason is described in the next section. The standard Galerkin method is employed for the discretization of the basic equations. The interpolation and weighting functions are defined as follows.

$\tilde{U} = [N]\{\tilde{U}\}$ ············ (9) $\quad U = [N]\{U\}$ ············ (11) $\quad P = [N]\{P\}$ ············ (13)

$\tilde{U}^* = [N]\{\tilde{U}^*\}$ ············ (10) $\quad U^* = [N]\{U^*\}$ ············ (12) $\quad P^* = [N]\{P^*\}$ ············ (14)

where, N is the inaterpolation function. In this paper, for the convenience of formulation of the boundary condition on pressure, the bilinear function based on the four node quardrilateral element is employed and the Gauss integration points are 3×3 for all variables.

The finite element equations can be formulated as follows,

$$M\tilde{U} = MU^m + \Delta t \left[\frac{1}{Re}(C-S)U^m - A(U^m)U^m \right] \quad (15)$$

$$SP^{m+1} = CP^{m+1} - \frac{1}{\Delta t} H\tilde{U} \quad (16)$$

$$MU^{m+1} = M\tilde{U} - \Delta t H P^{m+1} \quad (17)$$

where,

$M = \int_\Omega N^T N d\Omega$ ······· (18) $\quad C = \int_\Gamma N^T \nabla N d\Gamma$ ······ (20) $\quad H = \int_\Omega N^T \nabla N d\Omega$ ······ (22)

$S = \int_\Omega \nabla N^T \nabla N d\Omega$ ······· (19) $\quad A = \int_\Omega N^T N \cdot \nabla N d\Omega$ ······· (21)

For the sake of calculation efficiency, lumped mass matrix is used instead of consistent mass matrix in the eqs. (15) and (17).

4. BOUNDARY CONDITION FOR PRESSURE

In case that the pressure Poisson equation is applied on the whole domain of Ω, it is necessary to introduce the boundary conditions for pressure derived from the following equations on Γ, where, $\Gamma = \partial\Omega$, $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$

$$\frac{\partial P^{m+1}}{\partial n} = \frac{1}{Re} \nabla^2 U_n^m - U^m \cdot \nabla U_n^m - (U_n^{m+1} - U_n^m)/\Delta t \quad (23)$$

$$\frac{\partial P^{m+1}}{\partial \tau} = \frac{1}{Re} \nabla^2 U_\tau^m - U^m \cdot \nabla U_\tau^m - (U_\tau^{m+1} - U_\tau^m)/\Delta t \quad (24)$$

$$\frac{\partial U_n^{m+1}}{\partial n} + \frac{\partial U_\tau^{m+1}}{\partial \tau} = 0 \quad (25)$$

in which Γ_1, Γ_2 and Γ_3 are boundaries on which velocity is given (inlet), velocity is zero (wall) and velocity is unknown (outlet) respectively.

These equations are obtained by the normal and tangential projections of eqs. (1) and (2) onto Γ, where coordinate system is shown in Fig. 1. If the boundary condition of pressure is considered, normal gradient term of pressure can be evaluated by eq. (23) because the velocity U^{m+1} and U^m are given on Γ_1 and Γ_2. While in general time-dependent problems, neither pressure gradient nor pressure itself on the open boundary can be prescribed because U^{m+1} on Γ_3 is unknown. Therefore, a new approximation method to calculate the pressure on Γ_3 by using the similar procedure as the inner domain of Ω is necessary and proposed in the following manner.

From eqs. (23) and (24), the boundary pressure Poisson equation is derived using eq. (25) as follows,

$$\left(\frac{\partial^2}{\partial n^2}+\frac{\partial^2}{\partial \tau^2}\right)P^{m+1}=\frac{1}{\Delta t}\,\nabla\cdot\tilde{U} \qquad (26\cdot a)$$

where, eq. (26·b) can be introduced as follows when $\nabla\cdot U^m=0$ is applied on the right-hand side.

$$\left(\frac{\partial^2}{\partial n^2}+\frac{\partial^2}{\partial \tau^2}\right)P^{m+1}=2\frac{\partial U_n^m}{\partial n}\frac{\partial U_\tau^m}{\partial \tau}-2\frac{\partial U_n^m}{\partial \tau}\frac{\partial U_\tau^m}{\partial n} \qquad (26\cdot b)$$

Eq. (26·b) is equivalent to the pressure Poisson eq. which can be introduced from steady state basic eq. Eqs. (26·a, b) are similarly discretized by the finite element method in the region adjacent to Γ_3 and it is necessary to solve the finite element equation for each time step. However, if the weighted residual equation would have been formulated using the strict boundary integration, the additional Dirichlet condition for pressure would be needed on the boundary in order to solve the equation. Then, the pressure values of the nodes on which Dirichlet condition were given would not vary according as time proceeds. To avoid this inconvenience, the adjacent nodes to the open boundary are utilized approximately. The integration of eq. (26·a, or b) can be considered as those over the adjacent elements to the boundary. The discretized form of eq. (26) in the region adjacent to Γ_3 can be written as eq. (27). Every nodal pressure on Γ_3 is calculated for each time step using eq. (27).

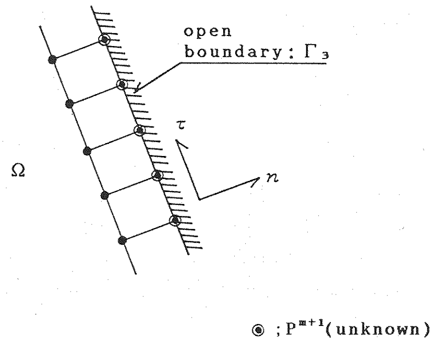

Fig. 1 the elements adjacent to the open boundary and the local co-ordinate system.

$$[L]\begin{Bmatrix}P^{m+1}\\ P^m\end{Bmatrix}=\begin{Bmatrix}f^m\\ 0\end{Bmatrix} \qquad (27)$$

where, P^{m+1} on Γ_3 are unknown and P^m on the adjacent nodes are known, and L means Laplacian matrix, f^m is velocity gradient vector on the right hand side of eq. (26). The pressure P^m of the adjacent nodes plays a role as predictor because these nodal points locate on the nearest to the boundary. Pressure in the whole domain can be calculated by eq. (16) imposing the pressure on Γ_3 calculated by eq. (27). The boundary pressure varies automatically for every time step and at the same time the nodal pressure on the adjacent nodes is corrected to $m+1$ time step.

It is worthwhile that the pressure Poisson equation can be solved without any artificial restrictions, because it is sufficient to add only velocity Dirichlet condition to basic equation systems in the time-dependent problems. In the present method, only initial value of pressure are necessary to solve the boundary pressure Poisson equation (27).

5. CALCULATION PROCEDURE

The calculation procedure of the present method is summarized as follows.

step 1 : give U^0 and P^0 as initial and boundary conditions
step 2 : solve intermediate velocity $\tilde{U}$ by eq. (15)
step 3 : solve P^{m+1} on Γ_3 by eq. (27)
step 4 : solve P^{m+1} in the whole domain by eq. (16) with P^{m+1} on Γ_3
step 5 : calculate U^{m+1} by eq. (17) with Dirichlet condition for velocity
step 6 : go to step 2 and continue

Time increment is related to numerical stability and must be originally obeyed under CFL condition (see 11)). However, the finite element mesh which is used in the practical calculations is irregular in general, therefore it is difficult to determine the time increment following the criterion derrived by the regular mesh pattern. By the present method, the following fact has been obtained by the authors numerical experiments. It is necessary to use the time increment of which value is nearly inverse number of Re number.

Since the coefficient matrices of eqs. (16) and (27) are symmetric, these equations can be easily solved by the sky line method or any other direct solver. Only once the forward decomposition of coefficient matrix is necessitated out of time-loop. The back-ward substitution is necessary in each time-loop. Following this procedure it is possible to solve the non-linear equation systems only by linear algebraic equations.

6. NUMERICAL RESULTS

Fig. 2 shows the schematical sketch which is the Karman vortex shedding problem around a square column. Γ_1 means the boundary of velocity Dirichlet condition, and Γ_2 is also the same type but velocity is always fixed to be zero. Γ_3 means the outlet where velocity and pressure are unknown. Thus, it is necessary to solve the boundary pressure Poisson equation on Γ_3.

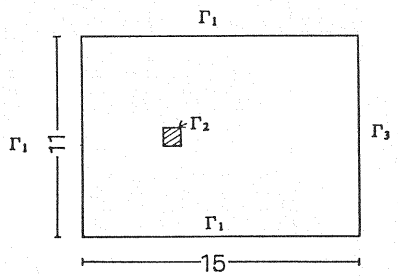

Fig. 2 Karman vortex shedding problem around a square column.

In this calculation, Reynolds number is set as 1 000. in which inlet velocity, width of the column and kinematic viscosity are 1.0, 1.0 and 0.001, respectively. Initial values of velocity and pressure were set to be zero for all nodes.

Fig. 3 (a) shows the velocity distribution of the time sequence. In early time steps, symmetric twin vortices are showm behind the column. According as the time steps proceed, the vortices grow up, then gradually move into un-symmetric form, and finally flow down-ward apart from the column. Quasi-steady state vortex shedding is shown in the later time steps.

In order to check the mass conservation, divergence of velocity is calculated. The results are less than 10^{-3} for all nodes as the numerical results.

Fig. 4 shows the bird's-eye view of presuure distribution in time sequence. The higher values of the pressure are shown in the front side of the column, and lower values are observed at the left, right and rear sides of the column. As shown in these figures, Karman vortex can pass through the outlet boundary, and it has been possible to continue the stable calculation by the present method.

The details of pressure time sequence around the column are shown in Fig. 5. Vertical axis means pressure coefficient C_p, and horizontal axis is time t. Average C_p for each surface of the column is calculated by following equation.

$$C_p=(P_i-P_0)/\{\rho V_0^2/2\} \qquad (28)$$

where, P_i, P_0, V_0 and ρ mean calculated pressure, standard pressure (inlet), standard velocity (inlet), and density of fluid respectively. The Strouhal number is about 0.13, and pressure coefficients of each

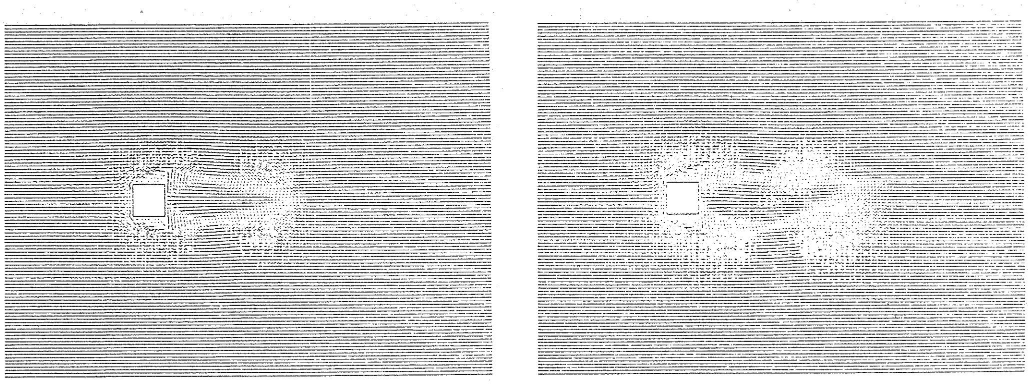

Fig. 3 (a) velocity vectors (early time step).

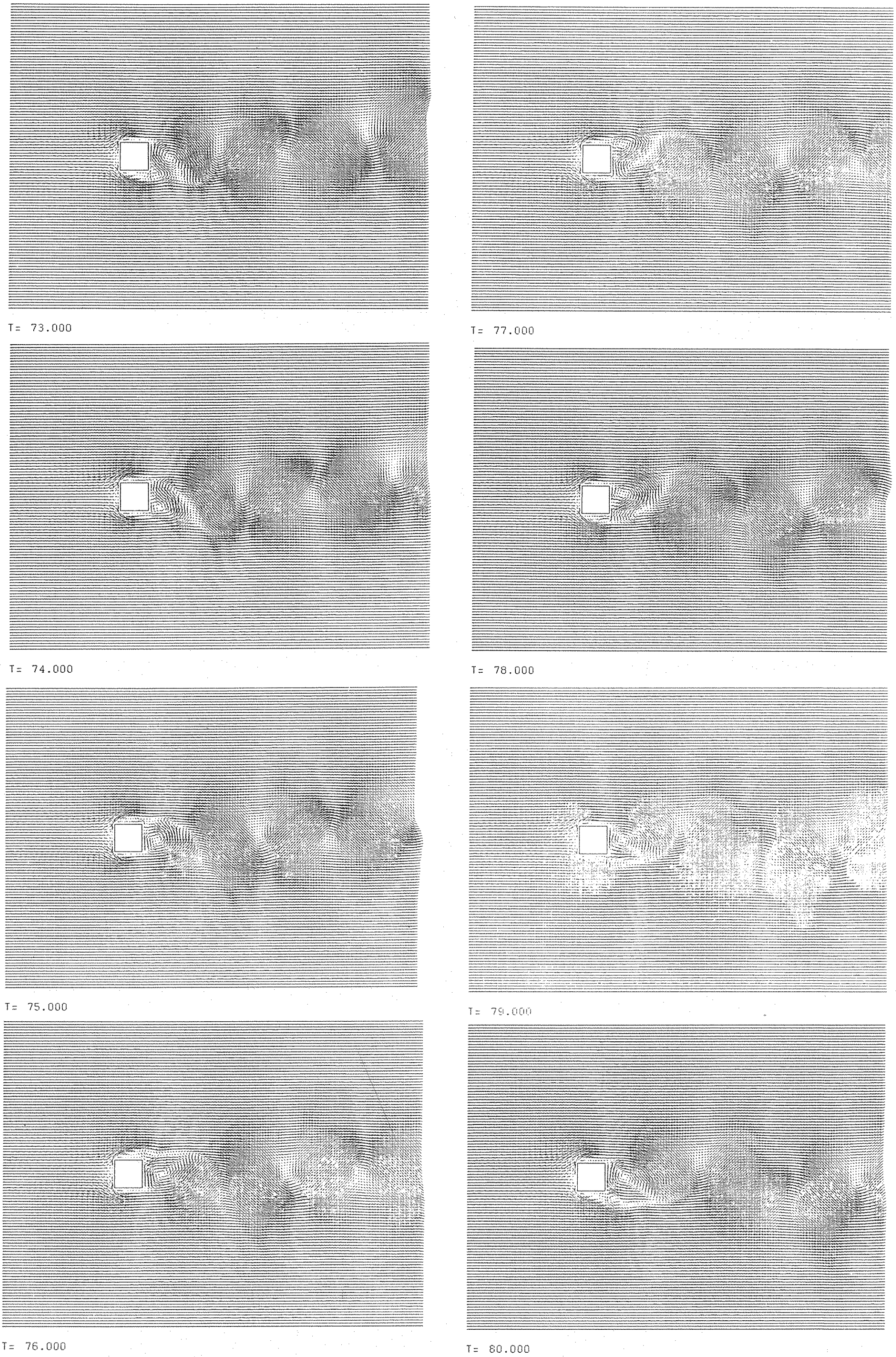

Fig. 3 (b) velocity vectors (later time step).

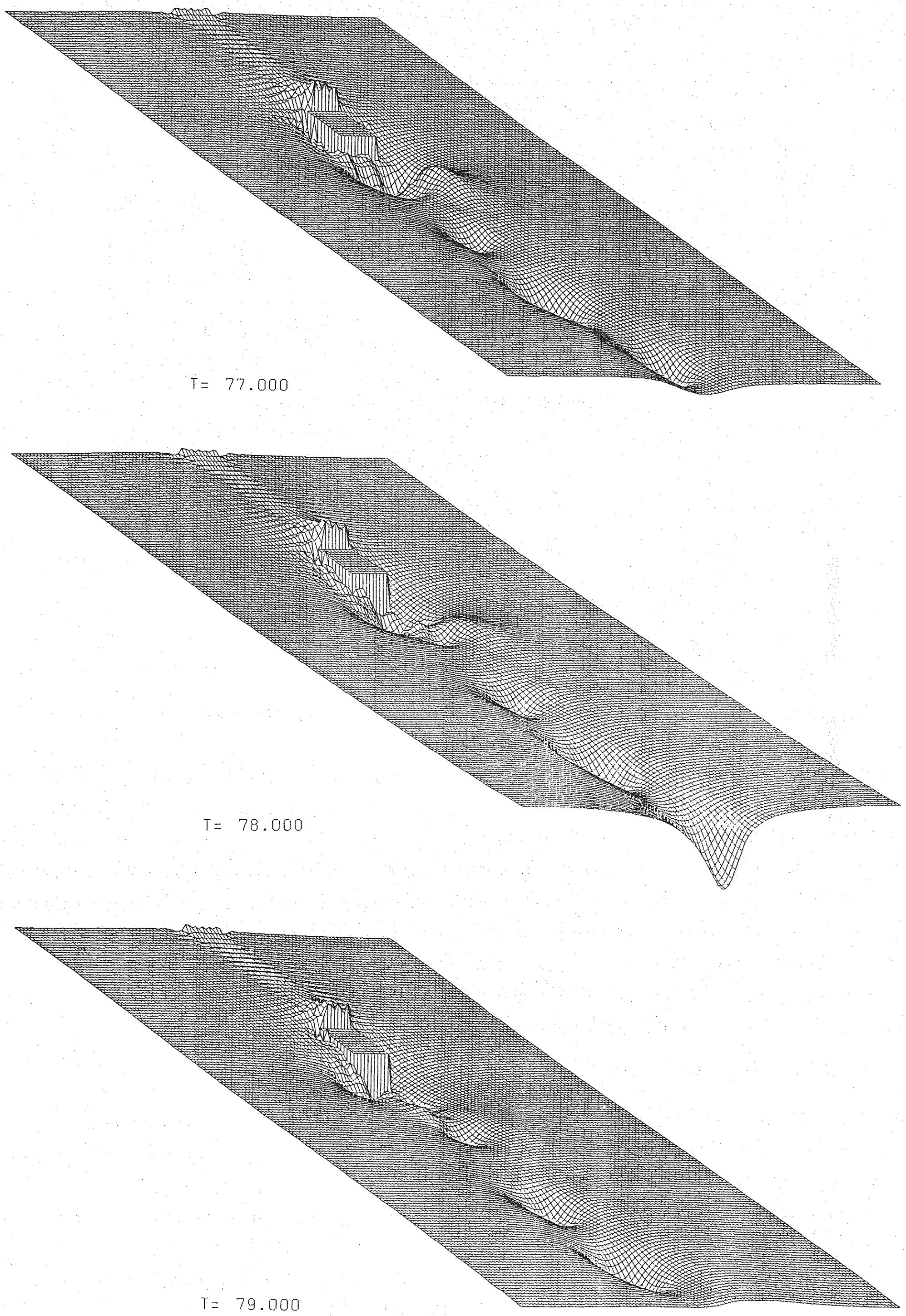

Fig. 4 presurre distribution by the bird's-eye view.

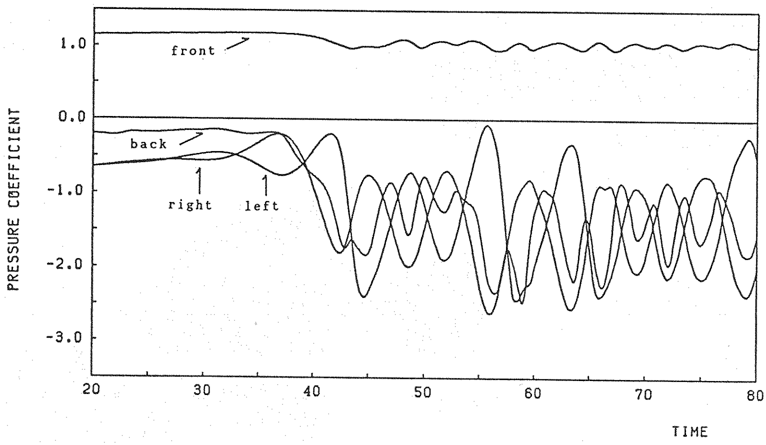

Fig. 5 pressure coefficient for each surface (space mean).

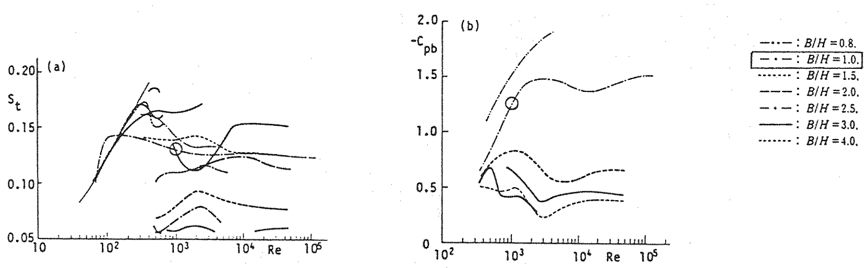

Fig. 6 the experimental results from reference 10).

surface are as follows in the present calculation from time 50 to time 80 in Fig. 5.

C_{pf} (front) $=1.0$ $\quad$ C_{pb} (back) $=-1.5$ $\quad$ C_{pl} (left) $=-1.4$ $\quad$ C_{pr} (right) $=-1.4$

These results are compared with the experiments[10]. In Fig. 6, Strouhal number and back side pressure coefficient are shown from the reference[10]. The experimental results on Reynolds number 10^3 with square prism are denoted by circle mark. The Strouhal number of present calculation is well in agreement with the experimental one, whereas the pressure coefficient is slightly different. In the present calculation, grid Reynolds number is more than 10^2, therefore, it would be necessary to use more refined mesh around the wall, on the other hand, the research on the wall boundary condition is important for numerical methods. This is one of the future study problems.

7. CONCLUDING REMARKS

In the present paper, the finite element method based on the velocity correction method is presented. The standard Galerkin formulation using bilinear interpolation function is employed, The studies on the present method can be summarized as follows.

(1) A new approximation method for the open boundary is presented for the time-dependent incompressible viscous flow problem using the velocity correction method.

(2) By the present method, it has been shown that the vortex shedding problems can be calculated stably without any artificial restriction for pressure Poisson equation.

(3) Comparing with experimental results, Strouhal number is well in agreement, however, there are some problems about pressure coefficient on the wall boundary. This is one of the future studies.

(4) In previous methods, mixed interpolation or smoothing technique has been adopted for the sake of numerical instability, however, it is demonstrated that on the spatial interpolation for the finite element

method, the linear interpolation and same order interpolation for both velocity and pressure can be effectively used throughout the present calculations. This is one of the main advantages of the velocity correction method.

REFERENCES

1) Gresho, P. and Sani, R. : On Pressure Boundary Condition for the Incompressible Navier-Stokes Equations, Int. J. Num. Meth. Fluid, Vol. 7, pp. 1111~1145, 1987.
2) Shimura, M. and Kawahara, M. : Three Dimensional Analysis of Flow in Clean Room Using the Velocity Correction method, Proc. the sixth Intrnational Symposium on Finited Element Method in Flow Problems, Eds. by Glowinski et al. Antibes, 1986.
3) Shimura, M. and Kawahara, M. : Finite Element Flow Analysis Using the Velocity Correction Method, Proc. NUMETA 87, Eds. by G. N. Pande et al. Swansea, 1987.
4) Chorin : Numerical Solution of the Navier-Stokes Equations, Math. Comp., pp. 745~762, 22, 1968.
5) Schneider, G. E. and Rathby, G. D. : Finite Element Analysis of Incompressible Fluid Flow Incorporation Equal Oder Pressure and Velocity Interpolation, Comp. Meth. in Fluids, pp. 49~83, 1980.
6) Donea, J. Giuliani, S. and Laval, H. : Finite Element Solution of the Unsteady Navier-Stokes Equations by a Fractional Step Method, Comp. Meth. Apl. Mech. Eng., pp. 53~73, 1982.
7) Mizukami, A. and Tsutiya : A Finite Element Method for the Three Dimensional Non-Steady Navier-Stokes Equations, Int. J. Numer. Meth. Fluids, 4, 1984.
8) Kawahara, M. and Ohmiya, K. : Finite Element Analysis of Density Flow Using the Velocity Correction Method, Int. J. Numer. Meth. Fluids, 5, 1985.
9) Kawahara, M. Ramaswamy, B. and Anju, A. : Lagrangian Finite Element Method, Proc. Japan Soc. Civil Engineers, 1986.
10) Okajima, A. and Sugitani, K. : Strouhal Number and Back Side Pressure Coefficient for Rectangular Prisms : Influence of Reynolds Number, Proc. Japan Soc. Mech. End., pp. 2004~2012, 1984 (in Japanese).
11) Roache, P. J. : Computational Fluid Dynamics, by Hermora Publishers, 1976.

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