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BIFURCATION BEHAVIOR OF AN AXISYMMETRIC ELASTIC SPACE TRUSS

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Bifurcation behavioral characteristics of a cone-shaped axisymmetric elastic space truss made of n elastic members with n-axes of symmetry are studied. Equilibrium equations of the truss are investigated using cylindrical coordinates to verify the existence of $2n$ bifurcation paths. The number of paths increases proportionally to the member number n. The equilibrium equations show this increase of bifurcation paths by the vanishing of lower-order terms, resulting in non-vanishing terms with higher-order nonlinearity. The geometric symmetry of the truss results in rotational symmetry of the equilibrium equations and of the bifurcation paths.

Keywords : bifurcation behavior, symmetry, equilibrium equations, space trusses

1. INTRODUCTION

A method for obtaining bifurcation paths of discretized structural systems with the use of Taylor expansion of equilibrium equations has been studied by Nishino et al. [1),2)]. In these papers, they referred to bifurcation behavior of three-bar and four-bar space trusses subjected to a vertical loading at the top node. A comment is also given without any proof for the existence of infinitely many number of Euler bifurcation paths for a column with circular cross-section, though the existence is obvious. The three-bar truss with three axes of line symmetry has six bifurcation paths; the four-bar truss with four axes has eight paths; the circular column with infinite number of axes of symmetry has infinite number of paths. Thus higher geometric symmetry tends to yield greater number of bifurcation paths.

This note is aimed to investigate bifurcation behavior of n-bar axisymmetric elastic space truss shown in Fig. 1, which includes the three and four-bar trusses as special cases. Interrelationships between the number of bifurcation paths and the level of geometric symmetry are studied. The bifurcation behavior of a circular cone is studied as the limiting case where the member number n increases to infinity. Since the main object of this note is bifurcation behavior of the axisymmetric truss, Euler buckling of each member is disregarded, i.e., assumed as not to occur.

2. BIFURCATION BEHAVIOR OF AN AXISYMMETRIC ELASTIC TRUSS

The existence of $2n$ bifurcation paths will be verified first for the n-bar axisymmetric elastic truss

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shown in Fig. 1, subjected to a vertical loading f applied to the crown, i.e., to the node 0. The total potential energy function U of the truss in cylindrical coordinates can be written as,

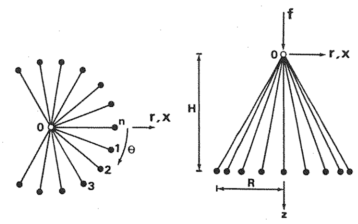

Fig. 1 n-Bar Axisymmetric Space Truss.

$$U=\frac{EA}{2L}\sum_{i=1}^{n}(L_i-L)^2-f\cdot w \quad (1)$$

where

$$\left.\begin{array}{l} L_i=\{(u\cos v-R\cos\alpha_i)^2+(u\sin v-R\sin\alpha_i)^2+w^2\}^{0.5} \\ \alpha_i=2\pi i/n \quad (i=1,2,\cdots,n,\ n\geqq 3) \end{array}\right\} \quad (2\cdot\text{a, b})$$

in which E=modulus of elasticity; A=cross-sectional area of the bars; L=initial length of the bars; L_i =length of the i-th bar after displacement; R=radius of the truss; r, θ, and z=right-handed cylindrical coordinates with origin at crown; and u, v, and w=displacements of the node 0 in this coordinates.

The potential function U is expanded into infinitive power series of u and trigonometric functions of v. For this purpose, the length of the i-th member L_i is expanded into the following infinitive power series, which are convergent for u less than $(R^2+w^2)^{0.5}$,

$$L_i=\hat{L}\left\{1+\sum_{j=1}^{\infty}F(j)(u/\hat{L})^{2j}+\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\sum_{m=0}^{[(j-k)/2]}G(j,k,m)(u/\hat{L})^{j+k}\cos(j-k-2m)(\alpha_i-v)\right\} \quad (3)$$

where

$$\left.\begin{array}{l} \hat{L}=(R^2+w^2)^{0.5};\quad F(j)=(-1)^{j-1}\dfrac{(2j-3)!!}{j!2^j}; \\ G(j,k,m)=2\cdot(-1)^{j-k}F(j)(R/\hat{L})^{j-k}\dbinom{j}{k}\dbinom{j-k}{m} \end{array}\right\} \quad (4\cdot\text{a}\sim\text{c})$$

and $[n]$ is the Gauss symbol indicating the integer number which does not exceed the variable n. The following formulas[3] are employed in the derivation of Eq. (3),

$$\left.\begin{array}{l} (1+a)^{0.5}=1+\sum_{j=1}^{\infty}(-1)^{j-1}\dfrac{(2j-3)!!}{j!2^j}a^j \quad (-1\leqq a\leqq 1); \\ (2j-1)!!=1\cdot 3\cdots(2j-3)(2j-1);\quad (-1)!!=1; \\ (a+b)^j=\sum_{k=0}^{j}\dbinom{j}{k}a^k b^{j-k};\quad \cos^k\phi=\dfrac{1}{2^{k-1}}\sum_{m=0}^{[k/2]}\dbinom{k}{m}\cos(k-2m)\phi \end{array}\right\} \quad (5\cdot\text{a}\sim\text{e})$$

Considering Eq. (3) and the following trigonometric equality,

$$\sum_{i=1}^{n}\cos j(\alpha_i-v)=\begin{cases} 0 & \text{for } j\neq kn \\ n\cos jv & \text{for } j=kn \quad k=1,2,\cdots,\infty \end{cases} \quad (6)$$

the sum of L_i for all members are expressed as,

$$\sum_{i=1}^{n}L_i=n\hat{L}\left\{1+\sum_{j=1}^{\infty}F(j)(u/\hat{L})^{2j}+\sum_{j=1}^{\infty}\sum_{\substack{k=0\\ j-k=\text{even}}}^{j-1}G(j,k,(j-k)/2)(u/\hat{L})^{j+k}\right.$$
$$\left.+\sum_{m=1}^{\infty}\sum_{j=mn}^{\infty}\sum_{\substack{k=0\\ j-k-mn=\text{even}}}^{j-mn}G(j,k,(j-k-mn)/2)(u/\hat{L})^{j+k}\cos(mnv)\right\} \quad (7)$$

By changing the dummy variables j and k into $k+mn$ and $2j-k$, this equation can be rewritten as,

$$\sum_{i=1}^{n}L_i=n\hat{L}\left\{1+\sum_{j=1}^{\infty}\left[F(j)+\sum_{k=j+1}^{2j}G(k,2j-k,k-j)\right](u/\hat{L})^{2j}\right.$$
$$\left.+\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}G(k+mn,2j-k,k-j)(u/\hat{L})^{2j+mn}\cos(mnv)\right\} \quad (8)$$

Substituting Eq. (8) into Eq. (1) leads to an infinitive power series expression of U as,

$$U=\sum_{j=0}^{\infty}a_j u^{2j}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}b_{j,k,m}u^{2j+mn}\cos(mnv)-f\cdot w \quad (9)$$

where

$$\left.\begin{aligned}&a_0=\frac{nEA}{2L}(\hat{L}^2+L^2)-nEA\hat{L}\;;\quad a_1=\frac{nEA}{2L}-\frac{nEA}{\hat{L}}(F(1)+G(2,0,1))\;;\\&a_j=-nEA(\hat{L})^{1-2j}\left\{F(j)+\sum_{k=j+1}^{2j}G(k,2j-k,k-j)\right\}\quad(j\geqq 2)\;;\\&b_{j,k,m}=-nEA(\hat{L})^{1-2j-mn}G(k+mn,2j-k,k-j)\end{aligned}\right\}\qquad(10\cdot\text{a}\sim\text{d})$$

Equilibrium equations can be obtained by differentiating U with respect to u, v, and w, respectively,

$$\left.\begin{aligned}&U_{,u}=u\left[\sum_{j=1}^{\infty}2ja_ju^{2j-2}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}(2j+mn)b_{j,k,m}u^{2j+mn-2}\cos(mnv)\right]=0\\&U_{,v}=\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}mnb_{j,k,m}u^{2j+mn}\sin(mnv)=0\\&U_{,w}=\sum_{j=0}^{\infty}\frac{da_j}{dw}u^{2j}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}\frac{db_{j,k,m}}{dw}u^{2j+mn}\cos(mnv)-f=0\end{aligned}\right\}\qquad(11\cdot\text{a}\sim\text{c})$$

Eq. (11·a) can be satisfied by either $u=0$ or $v=\theta_i\equiv i\pi/n$ ($i=1, 2, \cdots, 2n$). The solution expressed by $u=0$, $f=da_0/dw$ and indefinite v of Eq. (11) represents the main load-displacement path.

For $v=\theta_i$ and $u\neq 0$, the values of u and w can be determined for the given f from the equations $U_{,u}/u=0$ and $U_{,w}=0$ from the implicit function theorem. In this manner, $v=\theta_i$ denotes $2n$ bifurcation paths branching radially from the main equilibrium path represented by $u=0$. The location of the bifurcation point of these $2n$ bifurcation paths on the main path can be obtained by substituting $u=0$ in $U_{,u}/u=0$, which shows that the $2n$ paths bifurcate simultaneously from the same bifurcation point represented by $u=0$, $a_1(w)=0$ and $f=da_0/dw$ for any value of n. Fig. 2 shows a plane view of the directions of these bifurcation paths, which branch either toward a vertex of the n-bar truss, corresponding to θ_{2j} ($j=1, 2, \cdots, n$), or toward a middle direction of two neighboring vertices, corresponding to θ_{2j-1}. Rotational symmetry among the $2n$ paths can be shown by substituting this $v=\theta_i$ into Eqs. (11·a, c), which leads to,

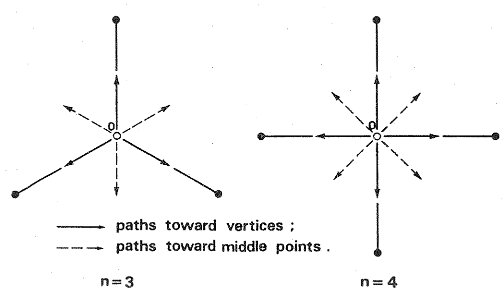

Fig. 2 Directions of Bifurcation Paths of the n-bar Trusses.

$$\left.\begin{aligned}&U_{,u}/u\,|_{v=\theta_i}=\sum_{j=1}^{\infty}2ja_ju^{2j-2}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}(2j+mn)b_{j,k,m}u^{2j+mn-2}c_i=0\\&U_{,w}\,|_{v=\theta_i}=\sum_{j=0}^{\infty}\frac{da_j}{dw}u^{2j}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}\frac{db_{j,k,m}}{dw}u^{2j+mn}c_i-f=0\end{aligned}\right\}\qquad(12\cdot\text{a, b})$$

where

$$c_i=\begin{cases}(-1)^m & \text{for odd } i\\ 1 & \text{for even } i\end{cases}\qquad(13)$$

Thus Eq. (11) yields two distinct sets of u, w, and f values as the solutions. This shows the presence of two different types of bifurcation paths : those directing toward a vertex and those toward a middle point. The paths belonging to the same type, having the same u, w, and f values with the difference of rotational direction by $2\pi j/n$ ($j=1, 2, \cdots, n$) radians, are rotationally symmetric.

The load f on the bifurcation paths can be expressed as a function of u from Eq. (12·b),

$$\left.\begin{aligned}&f(u)|_{v=\theta_{2i}}=\sum_{j=0}^{\infty}\frac{da_j}{dw}u^{2j}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}\frac{db_{j,k,m}}{dw}u^{2j+mn}\\&f(u)|_{v=\theta_{2i-1}}=\sum_{j=0}^{\infty}\frac{da_j}{dw}u^{2j}-\sum_{j=0}^{\infty}\sum_{k=j}^{2j}\sum_{m=1}^{\infty}(-1)^m\frac{db_{j,k,m}}{dw}u^{2j+mn}\end{aligned}\right\}\qquad(14\cdot\text{a, b})$$

These equations denote the bifurcated paths. Thus f is an even function of u for even n, for which $2j+mn$ is even, but not for odd n. The bifurcation point for these paths, accordingly, are symmetric for even n and are asymmetric for odd n. These facts are due to rotational symmetry among those bifurcation paths.

3. DISCUSSIONS AND CONCLUDING REMARKS

Bifurcation behavior of the n-bar truss from the main equilibrium path is discussed based on the results obtained in the previous section. The number of bifurcation paths $2n$ is proportional to the number of members n and hence the level of geometric symmetry of the truss. The equilibrium equations Eq. (11) expressed an increase of paths by increasing the frequencies of its trigonometric terms, $\sin(mnv)$ and $\cos(mnv)$, proportionally to n. In order to realize such an increase of frequency, the equations enhanced its nonlinearity by vanishing sinusoidal terms lower than $u^{n-2}\sin(nv)$ and $u^{n-2}\cos(nv)$.

The use of cylindrical coordinates is vital to identifying periodic nature of equilibrium equations. Although the power series expression of the equations in this coordinates is highly complex even for the n-bar truss with the simplest geometric configuration, the use of the total potential energy function is of great assistance in deriving them.

Geometric symmetry played an important role in bifurcation behavior of the n-bar space truss. The geometry of the truss (see Fig. 1) is preserved through $2\pi i/n$ ($i=1, 2, \cdots, n$) radian rotations and through reflections in the straight lines $\theta=-\pi i/n$. Such geometric symmetry is reflected on the equilibrium equations Eq. (11) in terms of periodic nature of trigonometric functions. These equations, which can be preserved by the transformation of v to $v+2\pi i/n$ and to $2\pi i/n-v$, are invariant regarding those rotations and reflections. As a result of this, directions of the bifurcation paths, branching out toward either the vertices or middle points of the vertices, correlate with geometric symmetry of the truss.

In a limit when n goes infinity, the n-bar truss becomes an axisymmetric circular cone. Hence the equilibrium equation $U_{,v}=0$ (Eq. (11·b)) is satisfied for all values of v. The number of bifurcation paths, which equals $2n$, becomes infinite. The envelope of the bifurcation paths forms a plane. These results obtained herein for the circular cone supplement the discussion carried out in Ref. 1) without a rigorous proof that a column with circular cross-section has infinite number of paths.

The $2n$ bifurcation paths of the n-bar truss have been obtained in a heuristic manner so that one cannot assure that they are the only bifurcation paths. Nonetheless at the least for the three-bar and four-bar trusses, which are special cases of the n-bar truss, the $2n$ paths are the only paths branching at the bifurcation point as proven in Ref. 1). The conclusions drawn through the simple n-bar truss, which has bifurcation modes that are line symmetric with respect to an axis, possess only limited generality. Other more realistic truss domes have bifurcation modes with more than one-axis of line symmetries[4]. The number of paths and the types of points for the truss nonetheless fully agreed with those obtained for one-axis symmetric bifurcation modes branching from the main equilibrium paths of regular n-gonal-shaped reticulated truss domes with n from five up to ten[4], for which the number of bifurcation paths equals $2n$ and the relevant bifurcation points are symmetric when n is even and asymmetric when n is odd. In this fashion, bifurcation behavior of truss domes has strong correlation with their geometric symmetry. Future studies on bifurcation behavior should put an emphasis on geometric symmetry.

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