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AN AUTOMATIC ARC LENGTH CONTROL ALGORITHM FOR TRACING EQUILIBRIUM PATHS OF NONLINEAR STRUCTURES

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An automatic arc length control algorithm for tracing smooth equilibrium paths of nonlinear structures is developed utilizing curvature of the paths. An example on a reticulated space elastic truss structure is presented to demonstrate the efficiency of the proposed algorithm.

Keywords : *arc length control, equilibrium path, space curve*

1. INTRODUCTION

The governing equilibrium equations of a discrete nonlinear structure are generally expressed in the system of n simultaneous nonlinear equations with $(n+1)$ unknowns in terms of n components of displacement or position vector and a loading intensity[1]. One of the best numerical procedure for obtaining the equilibrium paths of structures in $(n+1)$ dimensional space seems to be the so-called arc length control[2)~6)] with the Newton-Raphson method. However, the selection of an appropriate value of arc length which controls a convergence rate of the iteration procedure needs consideration[1]. A number of proposals for selecting this arc length have been reported[2)~4)]. All of them use the number of iterations needed to get a solution on the equilibrium path for a given convergence tolerance to decide the arc length for the next step. The basic concept behind this procedure is to obtain solutions by a nearly constant number of iterative cycles along the equilibrium path[2]. The concept seems to be reasonable, but it is reported that the procedure has been made rather intuitively without theoretical basis[3]. This paper presents an automatic arc length control criterion and hence algorithm based on the rational consideration utilizing Taylor expansion of a space curve, which indicates that an arc length should be controlled by the curvature. In order to show an efficiency of this algorithm, numerical examples on a reticulated space elastic truss are illustrated. The comparison among the results of this truss obtained by the proposed procedure and those by Itoh et. al.[3] and Ramm[4] is also discussed.

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2. GOVERNING EQUATIONS

The discretized governing equilibrium equations of nonlinear structures subjected to general static and conservative loading can be written by employing Cartesian coordinates, for simplicity, as[1)]

$$\int_0^f f_i(t)\,dt = K_i(x) \qquad (i=1,\ 2,\ \ldots,\ n) \quad \cdots\cdots (1)$$

where f=a parameter to specify the external loading path; $f_i=i$−th component of external loading pattern vector treated as a given value; $x=<x_1,\ x_2,\ \ldots,\ x_n>^T$ which is displacement or position vector; $K_i=i$−th component of internal force vector; and n=number of degrees of freedom of the system.

To simplify the formulation of the problem without losing generality, the external force term on the left-hand side of Eq. (1) can be idealized as a piecewise linear function of f. With this idealization, $f_i(f)$ becomes constant at each piecewise linear part and of which magnitudes and dimensions are so selected that f could represent the loading intensity. Since Eq. (1) has $(n+1)$ unknowns when f is treated as an unknown parameter, an additional equation is necessary to obtain solutions. Arc length control utilizes, as this additional equation, the arc length from a known point p, which can be expressed as

$$\sum_{i=1}^{n}(\alpha_i)^2(x_i-x_i^p)^2+(\alpha_f)^2(f-f^p)^2-(\Delta s^p)^2=0 \quad \cdots\cdots (2)$$

where α_i and α_f=arbitrary constants, equating the dimensions of x_i and f to that of Δs ; x_i^p and f^p= known solutions at p; and Δs^p=arc length along the equilibrium path from point p. By specifying the value of Δs^p, the Newton-Raphson iteration is performed for the $(n+1)$ nonlinear equations of Eqs. (1) and (2)[1)] until the converged solutions of x_i and f are found within a specified tolerance.

3. AUTOMATIC ARC LENGTH CONTROL ALGORITHM

In general, the solutions of Eq. (1) can be represented by the space curves in the $(n+1)$ dimensional Euclidean space of x_i and f. By denoting s as the length of a space curve measured from the point corresponding to the initial loading free state, the position vector of point $\boldsymbol{p}(s+\Delta s)$ on the $(n+1)$ dimensional space curve can be expanded into the Taylor series about the point $\boldsymbol{p}(s)$ as[7)]

$$\boldsymbol{p}(s+\Delta s)=\boldsymbol{p}(s)+\boldsymbol{e}_1(s)\Delta s+\kappa(s)\,\boldsymbol{e}_2(s)\frac{\Delta s^2}{2}$$
$$+\left[-\{\kappa(s)\}^2\boldsymbol{e}_1(s)+\frac{d\kappa(s)}{ds}\,\boldsymbol{e}_2(s)+\kappa(s)\,\tau(s)\,\boldsymbol{e}_3(s)\right]\frac{\Delta s^3}{3!}+$$
$$\text{(higher order terms)} \quad \cdots\cdots (3)$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$=unit tangent, principal normal and binormal vectors, respectively, which are the three particular unit vectors defined by Frenet-Serret formula in $(n+1)$ dimensional Euclidean space; $\kappa(s)$ and $\tau(s)$=curvature and torsion of space curves. Noting that the second term on the right-hand side of Eq. (3) is the linear extrapolation from the known point $\boldsymbol{p}$, the deviation of the initial values from the exact solution for a small quantity of Δs is most influenced by the third term. It would be most reasonable to control the arc length based on this term in relation to the second term. Noting both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors and considering only positive Δs as usual in arc length control, the quotient of the magnitude of the third term by that of the second term in Eq. (3) results in

$$\frac{\kappa(s)\Delta s^2/2}{\Delta s}=\frac{\kappa(s)\Delta s}{2} \quad \cdots\cdots (4)$$

The arc length could be controlled by keeping the right-hand term of Eq. (4) at a constant level. Expressing this arbitrary constant value by $C/2$, Eq. (4) leads to

$$\Delta s=\frac{C}{\kappa(s)} \quad \cdots\cdots (5)$$

With this control, the arc length is shortened, when the bent of the curve increases, and hence more closely located discrete points on the curve can be obtained. On the other hand, when the curve becomes straight, i. e., when the curvature approaches zero, the excessive value of Δs obtained by Eq. (5) may fail to trace the neighboring point on the equilibrium path. To avoid this failure, the maximum magnitude of the arc length is better to be limited. In computing an equilibrium path by arc length control, the initial arc length has to be given to start the computation. Since structures are expected to show the least nonlinear response at the loading free initial state, it would be natural to limit the maximum arc length by this initial length. Designating the arc length employed for starting the automatic arc length control algorithm by Δs^0, the magnitude determined by Eq. (5) is limited by this Δs^0 when it exceeds this value, as

$$\Delta s \leqq \Delta s^0 \quad \cdots\cdots (6)$$

For the space curve in $(n+1)$ dimensional space, $\kappa(s)$ can be expressed by[7]

$$\kappa(s)=[\sum_{i=1}^{n}(\alpha_i)^2\{d^2x_i(s)/ds^2\}^2+(\alpha_f)^2\{d^2f(s)/ds^2\}^2]^{1/2} \quad \cdots\cdots (7)$$

Since the discrete points on equilibrium path are numerically obtained, $\{d^2x_i(s)/ds^2\}$ and $\{d^2f(s)/ds^2\}$ have to be calculated numerically through the adoption of an interpolation function of $x_i(s)$ and $f(s)$.

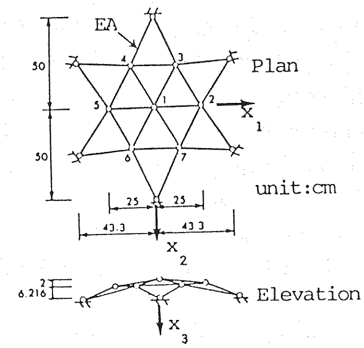


Fig. 1 Reticulated Space Truss

4. NUMERICAL EXAMPLES

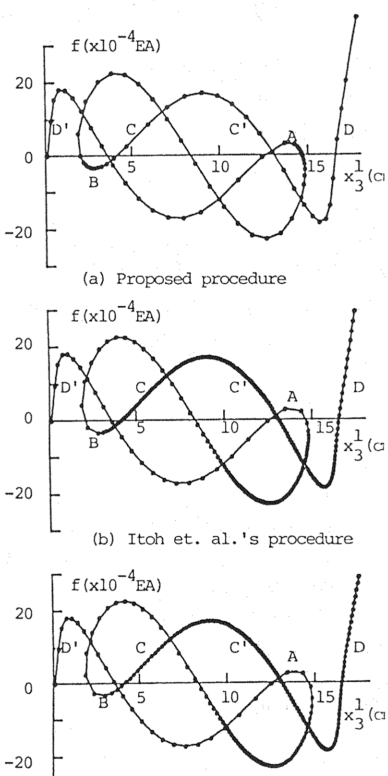
To construct an interpolation function of $x_i(s)$ and $f(s)$, any standard numerical procedures can be utilized. In this study, the Newton's interpolation method[8] is chosen, and a variety of orders of polynomial functions have been tried. Numerical results showed no significant improvement for any polynomials higher than the third order compared to the increase of numerical computation. Because of this, the third order polynomial has been used to evaluate curvatures. The selection of two parameters, i. e., C and Δs^0 which control the are lengths of an equilibrium path was made on trial basis.

By employing available stiffness equations for geometrically nonlinear truss element[5], a reticulated space truss exhibiting rather complicated main equilibrium path[5] is selected as the demonstrative example to show the effectiveness of the proposed automatic algorithm expressed by Eqs. (5) and (6). **Fig. 1** shows the initial configuration of the truss which is subjected to a vertical loading at each node of the same intensity, except at node 1 where reduced intensity of one half is applied[5].

Fig. 2 (a) shows the results obtained by the proposed procedure where C and Δs^0 are chosen to be 0.133 and 1.879, respectively. **Figs. 2** (b) and 2 (c) show similar results obtained by the procedures of Itoh et. al.[3] and Ramm[4], in which their criteria for selecting arc length Δs are expressed by

$$\Delta s=\frac{\alpha+N_sN_c}{\alpha+N_c^2}\Delta s^* ; \quad \Delta s=\{N_s/N_c\}^{1/2}\Delta s^* \quad \cdots\cdots (8\cdot a, b)$$

where N_c=the number of iterations required to get the previous solution with arc length Δs^*; N_s=the number of iterations aimed



(a) Proposed procedure

(b) Itoh et. al.'s procedure

(c) Ramm's procedure

Fig. 2 Comparison of Results of Main Equilibrium Path.

at; and α=a parameter related to changing rate of the arc length[3]. In order to trace automatically the main equilibrium path of this particular truss, Eqs. (8) are utilized for the computation of Figs. 2(b) and 2(c) with the values of $N_s=3$, $\alpha=3$ and Δs^* at the first step equal to Δs^0 used for the computation of Fig. 2(a) It is noted that the solutions on equilibrium paths in Fig. 2 are represented by the dots, and are obtained by employing the same bounds for arc length ($\Delta s^{max}=\Delta s^0$, $\Delta s^{min}=0.085$) and the same tolerance of 10^{-5} for convergence of all $(n+1)$ components. A number of combinations were tried for Δs^0, C, N_s and α. Among them, Fig. 2 are the results obtained by using the initial arc length with which the difference of the three results are most pronounced. For other selection of the initial arc lengths, the results are not as pronounced as those in Fig. 2, but the same tendencies discussed below have always existed.

A glance of Figs. 2(a), 2(b) and 2(c) would easily convince of the superiority of the proposed procedure to the other two. At the part where the equilibrium path deviates from the straight line the most, i. e., at such parts as A and B in Fig. 2, the proposed controlling procedure is computing with closer arc lengths than the other two, which is preferable in order to get better shape of the equilibrium path. On the other hand, at the part where the equilibrium path is not deviating much from the straight line such as the parts C and C′, the proposed procedure selects arc lengths much longer than the other two. At the symmetrical parts D and the starting part D′, the proposed procedure selects similar arc length with the initial length because of Eq. (6), otherwise it could select even longer lengths, while the others select much shorter arc lengths at the part D than at the part D′.

At the parts A and B, the average number of iterations are more or less of the same order among the three procedures, while the standard deviation of the number of iterations of the proposed procedure is much less compared to the others.

5. CONCLUDING REMARKS

Criterion for an automatic arc length control algorithm is presented for tracing smooth equilibrium paths of nonlinear structures based on the terms appearing in the Taylor expansion of a space curve. Curvature of an equilibrium curve is proposed as the criteria for the selection of arc langths. An illustrative example on tracing rather complicated equilibrium path of a space reticulated elastic truss indicates better performance of the proposed algorithm than the others reported in literature to trace automatically the entire smooth equilibrium path in three aspects as; 1) solutions obtained at closer intervals at the parts where an equilibrium path deviates more from a straight line, 2) smaller deviation of iterative cycles especially at sharply bent parts, and 3) smaller number of total iterations to trace a complex equilibrium curve.

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