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DIRECT LAGRANGIAN NONLINEAR ANALYSIS OF ELASTIC SPACE RODS USING TRANSFER MATRIX TECHNIQUE

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A numerical method using transfer matrix technique is developed to obtain solutions for space rods considering finite displacements. The present method is characterized by the point that the numerical solutions are directly derived from the highly nonlinear governing equations with Lagrangian expressions. The governing equations are based on the theory of finite displacement with small strains and no restrictions are made on the magnitude of displacements.

Keywords : *space frame, finite rotation, large displacement, transfer matrix method*

1. INTRODUCTION

In the Lagrangian formulation of finite displacement theory for space frames, it is a common practice to derive the governing differential equations, using the ordinary displacement components defined in terms of the coordinates fixed in space[1)~4)]. However, the formulation of this kind makes the governing equations highly nonlinear and complicated chiefly due to the finite rotations in space. Therefore, different from plane frames, it is very difficult to derive accurate governing equations for space frames without introducing any restrictions on the magnitude of displacements. Further, even when the accurate governing equations are derived, it is far more difficult to obtain numerical solutions directly from these governing equations, to say nothing of analytical solutions. For this reason, the approximate method with the separation of rigid body displacements, i. e. the method using moving coordinates[5)~9)], are primarily used to obtain numerical solutions for space frames which undergo large displacements.

Thus, so far as space frames are concerned, versatile numerical solutions have not yet been derived directly from the highly nonlinear governing equations with Lagrangian expressions. However, the solutions of this kind are important specifically to the point that the validity and the accuracy of the approximate method can be properly evaluated by these solutions.

We have already presented a new formulation of finite displacement theory for space rods based on Lagrangian approach[10),11)]. This formulation is characterized by the point that new deformation components are adopted as basic unknowns in lieu of the ordinary displacement components. With the new formulation, the exact governing equations were successfully obtained not only for the theory of finite displacements

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with small strains but also for that with finite strains through variational calculus. These theories introduce no restrictions on the magnitude of displacements except the customary beam assumptions, i. e. no change of cross sectional shapes and the Bernoulli-Euler hypothesis. Although the deformation components except extensional rate were firstly introduced by Love[12], his formulation ignores the elongation of the centroidal axis of rods. Further, his theory was not derived based on the variational calculus, thus lacking in the precise expressions of boundary conditions.

In this paper, we further present a numerical method which directly solves the governing differential equations with Lagrangian expression. Since the theory of finite displacements with finite strains is very much complicated and impractical, the numerical method is developed primarily for the theory of finite displacements with small strains[10]. The present method utilizes transfer matrix technique, and the field transfer equations are derived by making use of the Taylor expansion with respect to the element length[7),10),13),14)]. It should be noted here that the numerical analysis based on the accurate Lagrangian governing equations becomes possible mainly because the governing equations are drastically simplified by the adoption of the new deformation components.

For the ease of mathematical manipulations in this paper, space frames are assumed to consist of straight members with doubly symmetric cross section.

2. GOVERNING EQUATIONS FOR SPACE RODS UNDER SMALL STRAINS

Herein, the derivations of the governing equations for space rods based on the new formulation are explained briefly, since the detailed explanations have already been made in Refs. 10) and 11).

Consider a straight member subjected to distributed external forces as shown in Fig. 1. Rectangular Cartesian coordinate system (x, y, z) with base vectors $(\boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{g}_z)$ is introduced at the initial configuration of the member. The coordinate z is taken along the centroidal axis of the member, and the coordinates (x, y) with their origin at the centroid are chosen such that the coordinates coincide with the doubly symmetrical axes of the cross section.

In lieu of the customary displacement components, our new formulation introduces four deformation components of $\hat{\kappa}_x$, $\hat{\kappa}_y$, $\hat{\tau}$ and $\sqrt{\hat{g}_0}-1$, which represent the deformation of the centroidal axis. These deformation components are mathematically defined by

$$\frac{d}{dz}\begin{Bmatrix}\hat{\boldsymbol{i}}_{x0}\\ \hat{\boldsymbol{i}}_{y0}\\ \hat{\boldsymbol{i}}_{z0}\end{Bmatrix}=[\hat{D}]\begin{Bmatrix}\hat{\boldsymbol{i}}_{x0}\\ \hat{\boldsymbol{i}}_{y0}\\ \hat{\boldsymbol{i}}_{z0}\end{Bmatrix},\quad \hat{\boldsymbol{g}}_{z0}=\sqrt{\hat{g}_0}\,\hat{\boldsymbol{i}}_{z0},\quad [\hat{D}]=\begin{bmatrix}0, & \hat{\tau}, & -\hat{\kappa}_y\\ -\hat{\tau}, & 0, & \hat{\kappa}_x\\ \hat{\kappa}_y, & -\hat{\kappa}_x, & 0\end{bmatrix} \quad\cdots\cdots (1\cdot a\sim c)$$

where $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ are the unit vectors obtained by normalizing the deformed base vectors $(\hat{\boldsymbol{g}}_{x0}, \hat{\boldsymbol{g}}_{y0}, \hat{\boldsymbol{g}}_{z0})$ on the centroidal axis which are orthogonal due to the beam assumptions. Physically, $(\hat{\kappa}_x/\sqrt{\hat{g}_0}, \hat{\kappa}_y/\sqrt{\hat{g}_0})$, $\hat{\tau}/\sqrt{\hat{g}_0}$, and $\sqrt{\hat{g}_0}-1$ correspond to the components of curvature in the directions of the deformed symmetrical axes of cross section, the torsional rate and the extensional rate, respectively, of the deformed centroidal axis. External distributed force $\boldsymbol{p}$ and distributed moment $\boldsymbol{m}$ applied on the centroidal axis as shown in Fig. 1 are expressed by the components in the directions of the vectors $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ respectively as $(\hat{p}_x, \hat{p}_y, \hat{p}_z)$ and $(\hat{m}_x, \hat{m}_y, \hat{m}_z)$.

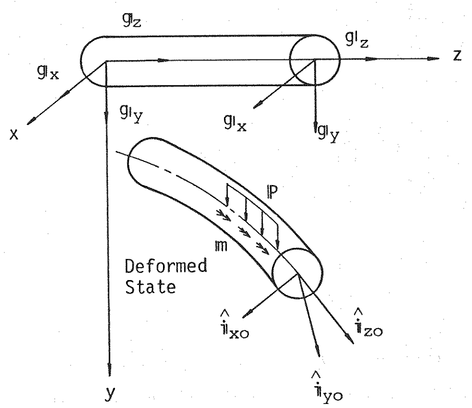
With the four deformation components along with the external force components defined above, the governing equations for the finite displacement theory of space rods are derived through the principle of virtual work by introducing the condition of small strains. The condition of small strains is given as follows in terms of the deformation components

$$|\sqrt{\hat{g}_0}-1|\ll 1,\quad |\hat{\kappa}_x y|\ll 1,\quad |\hat{\kappa}_y x|\ll 1,\quad |\hat{\tau} y|\ll 1,\quad |\hat{\tau} x|\ll 1 \quad\cdots\cdots (2\cdot a\sim c)$$

The constitutive relations are assumed to follow the linear elastic relations as

$$\sigma_{zz}=E e_{zz},\quad \sigma_{zx}=2G e_{zx},\quad \sigma_{zy}=2G e_{zy} \quad\cdots\cdots (3\cdot a\sim c)$$

in which $(\sigma_{zz}, \sigma_{zx}, \sigma_{zy})$ and (e_{zz}, e_{zx}, e_{zy}) are 2nd Piola-Kirchhoff stress tensor and Green strain tensor,



$\mathbb{P} = \hat{P}_x \hat{\mathbb{i}}_{xo} + \hat{P}_y \hat{\mathbb{i}}_{yo} + \hat{P}_z \hat{\mathbb{i}}_{zo}$

$\mathbb{m} = \hat{m}_x \hat{\mathbb{i}}_{xo} + \hat{m}_y \hat{\mathbb{i}}_{yo} + \hat{m}_z \hat{\mathbb{i}}_{zo}$

Fig. 1 Coordinate Systems for Direct Lagrangian Expressions.

Table 1 Governing Equations with Lagrangian Expression.

Equilibrium Equations	Stress Resultant vs. Deformation Components
$\hat{F}'_x - \hat{F}_y \hat{\tau} + \hat{F}_z \hat{\kappa}_y + \hat{p}_x = 0$	$\tilde{N} = EA(\sqrt{\hat{g}_0} - 1) + EJ\hat{\tau}^2/2$
$\hat{F}'_y + \hat{F}_x \hat{\tau} - \hat{F}_z \hat{\kappa}_x + \hat{p}_y = 0$	$\tilde{M}_x = -EI_x \hat{\kappa}_y$, $\tilde{M}_y = EI_y \hat{\kappa}_x$
$\hat{F}'_z + \hat{F}_y \hat{\kappa}_x - \hat{F}_x \hat{\kappa}_y + \hat{p}_z = 0$	$\tilde{M}_z = T_s + K\hat{\tau}$
$\tilde{M}'_z - \tilde{M}_y \hat{\kappa}_y - \tilde{M}_x \hat{\kappa}_x + \hat{m}_z = 0$	
where	where
$\hat{F}_x = \tilde{M}'_x - \tilde{M}_y \hat{\tau} + \tilde{M}_z \hat{\kappa}_x - \hat{m}_y$	$T_s = GJ\hat{\tau}$
$\hat{F}_y = \tilde{M}'_y + \tilde{M}_x \hat{\tau} + \tilde{M}_z \hat{\kappa}_y + \hat{m}_x$	$K = EJ(\sqrt{\hat{g}_0} - 1) + EJ_{rr}\hat{\tau}^2/2$
$\hat{F}_z = \tilde{N}$, $(\cdot)' = d(\cdot)/dz$	

Remarks : The following notations are used throughout Tables and Equations

$\tilde{M}_x = \int_A \sigma_{zz} x\, dA$, $\tilde{M}_y = \int_A \sigma_{zz} y\, dA$, $T_s = \int_A (\sigma_{zy} x - \sigma_{zx} y) dA$, $K = \int_A \sigma_{zz}(x^2+y^2) dA$

$A = \int_A dA$, $I_x = \int_A x^2 dA$, $I_y = \int_A y^2 dA$, $J = \int_A (x^2+y^2) dA$, $J_{rr} = \int_A (x^2+y^2)^2 dA$

respectively, defined in terms of the (x, y, z) coordinates. Utilizing the beam assumptions together with the condition of small strains, strain vs. deformation-relations are given by

$$e_{zz} = \sqrt{\hat{g}_0} - 1 + y\hat{\kappa}_x - x\hat{\kappa}_y + (x^2+y^2)\hat{\tau}^2/2, \quad e_{zx} = -y\hat{\tau}/2, \quad e_{zy} = x\hat{\tau}/2 \quad \cdots\cdots (4\cdot a \sim c)$$

Under the condition of small strains, constitutive relations of eqs. (3) with eqs. (4) exactly coincide with those defined between physical components of stress and strain.

The equilibrium equations along with the stress resultant vs. deformation-relations derived from the above procedures are summarized in Table 1. The constitutive relation between M_z and the deformation components can further be simplified considering the conditions given by eqs. (2). This relation can be expressed from Table 1 as

$$\tilde{M}_z = T_s + K\hat{\tau} = GJ\hat{\tau}\{1 + E(\sqrt{\hat{g}_0} - 1)/G + EJ_{rr}\hat{\tau}^2/2\,GJ\} \quad \cdots\cdots (5)$$

Since both $E(\sqrt{\hat{g}_0} - 1)/G$ and $EJ_{rr}\hat{\tau}^2/GJ$ have the order of the magnitude of strain, the 2nd and the 3rd terms of the very right side of eq. (5) can be ignored compared with unity, thus resulting in the following simplified constitutive relation.

$$\tilde{M}_z = GJ\hat{\tau} \quad \cdots\cdots (6)$$

In the present analysis, eq. (6) will be used in lieu of eq. (5).

In order to introduce geometrical boundary conditions as well as to analyze the deformed geometry, the deformation components have to be related to translational and rotational components.

Here, the translational displacements on the centroidal axis of the rod are expressed by the components (u_0, v_0, w_0) in the directions of the vectors $(\boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{g}_z)$, while the rotational displacements are evaluated by the following directional cosines $[l_{\hat{a}b}]$ of vectors $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ referred to the coordinate system (x, y, z).

$$\begin{Bmatrix} \hat{\boldsymbol{i}}_{x0} \\ \hat{\boldsymbol{i}}_{y0} \\ \hat{\boldsymbol{i}}_{z0} \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{Bmatrix}, \quad [l_{\hat{a}b}] = \begin{bmatrix} l_{\hat{x}x}, & l_{\hat{x}y}, & l_{\hat{x}z} \\ l_{\hat{y}x}, & l_{\hat{y}y}, & l_{\hat{y}z} \\ l_{\hat{z}x}, & l_{\hat{z}y}, & l_{\hat{z}z} \end{bmatrix} \quad \cdots\cdots (7\cdot a, b)$$

Following the procedures explained in Refs. 10) and 11), the above displacement components and the deformation components are related to each other as

$$d[l_{\hat{a}b}]/dz = [\hat{D}][l_{\hat{a}b}] \quad \cdots\cdots (8)$$

$$u'_0 = \sqrt{\hat{g}_0}\, l_{\hat{z}x}, \quad v'_0 = \sqrt{\hat{g}_0}\, l_{\hat{z}y}, \quad w'_0 = \sqrt{\hat{g}_0}\, l_{\hat{z}z} - 1 \quad \cdots\cdots (9\cdot a \sim c)$$

3. FIELD TRANSFER EQUATION

Transfer matrix method is introduced here to solve the nonlinear differential equations in Table 1.

Field transfer equations, i. e. basic discrete equations in this method, are derived here by the Taylor expansion method. For this purpose, the governing equations are firstly transformed into the first order differential equations expressed in terms of the physical quantities. The first order differential equations, so obtained, are summarized in Table 2. The components of mechanical quantities adopted in Table 2 are defined by

$$\boldsymbol{F}=\hat{F}_x\hat{\boldsymbol{i}}_{x0}+\hat{F}_y\hat{\boldsymbol{i}}_{y0}+\hat{F}_z\hat{\boldsymbol{i}}_{z0},\quad \boldsymbol{M}=\tilde{M}_y\hat{\boldsymbol{i}}_{x0}-\tilde{M}_x\hat{\boldsymbol{i}}_{y0}+\tilde{M}_z\hat{\boldsymbol{i}}_{z0} \quad\cdots\cdots (10\cdot a,\ b)$$

where $\boldsymbol{F}$ and $\boldsymbol{M}$ are vectors denoting sectional force and moment respectively. The components of geometrical quantities in Table 2 are already explained in the previous section.

The field transfer equations can be obtained by expanding physical quantities with respect to the element length $l=z_{i+1}-z_i$. These discrete equations are in the form of transferring the physical quantities from node i to $i+1$ of a finite element i as

$$Q_j|_{i+1}=Q_j|_i+\sum_{n=1}^{\infty}\overset{(n)}{Q}_j|_i l^n/n! \quad\cdots\cdots (11)$$

where Q_j is the component of the state vector $\{Q_j\}$ defined later by eq. (13) and $\overset{(n)}{Q}_j|_i$ is the n-th order derivative of Q_j at node i. The derivative $\overset{(n)}{Q}_j|_i$ can be expressed in terms of the components of the state vector $\{Q_j|_i\}$ at node i by successive differentiation and substitution of the first order differential equations in Table 2.

In the usual analysis, external forces are assumed conservative and it is convenient to use the component in the directions fixed in space. Thus, the components in the directions of the coordinates (x, y, z) are defined, as follows, respectively for external distributed force $\boldsymbol{p}$, external distributed moment $\boldsymbol{m}$, sectional force $\boldsymbol{F}$, and sectional moment $\boldsymbol{M}$.

$$\left.\begin{array}{ll}\boldsymbol{p}=p_x\boldsymbol{g}_x+p_y\boldsymbol{g}_y+p_z\boldsymbol{g}_z, & \boldsymbol{m}=m_x\boldsymbol{g}_x+m_y\boldsymbol{g}_y+m_z\boldsymbol{g}_z\\ \boldsymbol{F}=\overline{F}_x\boldsymbol{g}_x+\overline{F}_y\boldsymbol{g}_y+\overline{F}_z\boldsymbol{g}_z, & \boldsymbol{M}=\overline{M}_y\boldsymbol{g}_x-\overline{M}_x\boldsymbol{g}_x+\overline{M}_z\boldsymbol{g}_z\end{array}\right\} \quad\cdots\cdots (12)$$

Using the components fixed in space, the state vector $\{Q_j\}$ is defined as

$$\{Q_j\}=\{u_0,\ v_0,\ w_0,\ [l_{\hat{a}b}],\ \overline{F}_x,\ \overline{F}_y,\ \overline{F}_z,\ \overline{M}_y,\ -\overline{M}_x,\ \overline{M}_z\} \quad\cdots\cdots (13)$$

Although $[l_{\hat{a}b}]$ contains nine components, only three components are independent.

The derivative $\overset{(n)}{Q}_j|_i$ is explicitly expressed up to the 2nd order in terms of the components of the state vector $\{Q_j|_i\}$ defined above and the terms higher than the 2nd order are truncated in Eq. (11). This algorithm is equivalent to Runge-Kutta method of order 2 which is known to be the lowest order algorithm required to obtain convergent solutions in numerical analysis. The derivatives, so derived, are shown in Appendix A.

Field transfer equations derived here are nonlinear in terms of the physical quantities. Therefore, the basic algebraic equations involved in the present problem also become nonlinear and some iteration is required to solve them. Thus, in view of the application of Newton-Raphson method as well as the analysis of singular points on equilibrium path, the incremental transfer equations are derived from eq. (11). These incremental equations can be obtained as follows, by taking the increment of eq. (11) with respect to the physical quantities and ignoring nonlinear incremental terms.

Table 2 First Order Differential Equations.

$$\hat{F}_x'=\hat{F}_y\hat{\tau}-\hat{F}_z\hat{\kappa}_y-\hat{p}_x$$

$$\hat{F}_y'=-\hat{F}_x\hat{\tau}+\hat{F}_z\hat{\kappa}_x-\hat{p}_y$$

$$\hat{F}_z'=-\hat{F}_y\hat{\kappa}_x+\hat{F}_x\hat{\kappa}_y-\hat{p}_z$$

$$\tilde{M}_x'=\hat{F}_x+\tilde{M}_y\hat{\tau}-\tilde{M}_z\hat{\kappa}_x+\hat{m}_y$$

$$\tilde{M}_y'=\hat{F}_y-\tilde{M}_x\hat{\tau}-\tilde{M}_z\hat{\kappa}_y-\hat{m}_x$$

$$\tilde{M}_z'=\tilde{M}_x\hat{\kappa}_x+\tilde{M}_y\hat{\kappa}_y-\hat{m}_z$$

$$u_0'=\sqrt{g_0}\,l_{\hat{z}x},\quad v_0'=\sqrt{g_0}\,l_{\hat{z}y},\quad w_0'=\sqrt{g_0}\,l_{\hat{z}z}-1$$

$$d[l_{\hat{a}b}]/dz=[\hat{D}][l_{\hat{a}b}]$$

where

$$\hat{\kappa}_x=\tilde{M}_y/EI_y,\quad \hat{\kappa}_y=-\tilde{M}_x/EI_x,\quad \hat{\tau}=\tilde{M}_z/GJ$$

$$\sqrt{g_0}=\hat{F}_z/EA+1-J(\tilde{M}_z/GJ)^2/2A$$

$$\Delta Q_j|_{i+1}=\Delta Q_j|_i+\sum_{n=1}^{\infty}\overset{(n)}{\Delta Q_j}|_i\, l^n/n! \quad \cdots\cdots (14)$$

where ΔQ_j and $\overset{(n)}{\Delta Q_j}$ denote incremental quantities.

For the ease of applying the customary techniques used in the transfer matrix method, incremental rotational angles ($\Delta\alpha_x$, $\Delta\alpha_y$, $\Delta\alpha_z$) respectively around the coordinate axes (x, y, z) fixed in space are introduced in lieu of the incremental directional cosines $[\Delta l_{\hat{a}b}]$. The incremental directional consines in eq. (14) can be easily transformed into the incremental rotational angles by making use of the following relations as derived in Appendix B.

$$[\Delta l_{\hat{a}b}]=[l l_{\hat{a}b}]\begin{bmatrix} 0, & \Delta\alpha_z, & -\Delta\alpha_y \\ -\Delta\alpha_z, & 0, & \Delta\alpha_x \\ \Delta\alpha_y, & -\Delta\alpha_x, & 0 \end{bmatrix} \quad \cdots\cdots (15)$$

$$[l l_{\hat{a}b}]=\begin{bmatrix} l_{\hat{y}y}l_{\hat{z}z}-l_{\hat{z}y}l_{\hat{y}z}, & l_{\hat{y}z}l_{\hat{z}x}-l_{\hat{z}z}l_{\hat{y}x}, & l_{\hat{y}x}l_{\hat{z}y}-l_{\hat{z}x}l_{\hat{y}y} \\ l_{\hat{z}y}l_{\hat{x}z}-l_{\hat{x}y}l_{\hat{z}z}, & l_{\hat{z}z}l_{\hat{x}x}-l_{\hat{x}z}l_{\hat{z}x}, & l_{\hat{z}x}l_{\hat{x}y}-l_{\hat{x}x}l_{\hat{z}y} \\ l_{\hat{x}y}l_{\hat{y}z}-l_{\hat{y}y}l_{\hat{x}z}, & l_{\hat{x}z}l_{\hat{y}x}-l_{\hat{y}z}l_{\hat{x}x}, & l_{\hat{x}x}l_{\hat{y}y}-l_{\hat{y}x}l_{\hat{x}y} \end{bmatrix} \quad \cdots\cdots (16)$$

The components of incremental state vector $\{\Delta Q_j^*\}$ can be expressed as

$$^t\{\Delta Q_j^*\}=\{\Delta u_0,\ \Delta v_0,\ \Delta w_0,\ \Delta\alpha_x,\ \Delta\alpha_y,\ \Delta\alpha_z,\ \Delta\overline{F}_x,\ \Delta\overline{F}_y,\ \Delta\overline{F}_z,\ \Delta\overline{M}_y,\ -\Delta\overline{M}_x,\ \Delta\overline{M}_z,\ 1\} \quad \cdots\cdots (17)$$

Hereinafter, the starred notation, as used above, is adopted to distinguish the incremental state vector of eq. (17) from the increment of eq. (13).

It should be noted that the incremental rotational angles in eq. (17), which are infinitesimally small, can be treated as vector quantities. Considering that the incremental equations of eq. (14) are linear in terms of the components of incremental state vector, eq. (14) can be rewritten in a matrix form as

$$\{\Delta Q_j^*|_{i+1}\}=\begin{bmatrix} [\Delta T(\{Q_j|_i\})] & \{\Delta f_j\} \\ ^t\{0\} & 1 \end{bmatrix}\{\Delta Q_j^*|_i\} \quad \cdots\cdots (18)$$

where $[\Delta T(\{Q_j|_i\})]$ is 12×12 matrix expressed by the components of state vector $\{Q_j|_i\}$ and $\{\Delta f_j\}$ is a vector resulting from the incremental distributed forces. For later convenience, eq. (18) is expressed in a simpler form.

$$\{\Delta Q_{i+1}^*\}=[\Delta T_i^*]\{\Delta Q_i^*\} \quad \cdots\cdots (19)$$

The form of the above equation is the same as that used in the customary Transfer Matrix Method for small displacement theory. Thus, so far as the incremental quantities are concerned, all the solution techniques developed for the customary Transfer Matrix Method are applicable to the present analysis.

4. POINT TRANSFER EQUATION

If the centroidal axes of adjacent elements are not parallel to each other, or if concentrated loads are applied at the nodes of the elements, it is necessary to transfer the state vector from the left to the right at the node. Thus, the point transfer equations used for the above purpose are derived in this section.

Suppose elements i and $i+1$ are rigidly jointed to each other at node $i+1$, as illustrated in Fig. 2. The directional cosines $[L_{ab}]_{i+1}$ of the element coordinbtes (x^{i+1}, y^{i+1}, z^{i+1}) are defined in terms of the adjacent element coordinates (x^i, y^i, z^i) as

$$\begin{Bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{Bmatrix}^{i+1}=[L_{ab}]_{i+1}\begin{Bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{Bmatrix}^{i} \quad \cdots\cdots (20)$$

$\boldsymbol{F}_{i+1}^c$ and $\boldsymbol{M}_{i+1}^c$ in Fig. 2 are the concentrated force and moment, respectively applied at node $i+1$, and the (x^{i+1}, y^{i+1}, z^{i+1})-components of these concentrated loads are defined by

$$\left.\begin{aligned} \boldsymbol{F}_{i+1}^c &= F_{xi+1}^c\boldsymbol{g}_x^{i+1}+F_{yi+1}^c\boldsymbol{g}_y^{i+1}+F_{zi+1}^c\boldsymbol{g}_z^{i+1} \\ \boldsymbol{M}_{i+1}^c &= M_{xi+1}^c\boldsymbol{g}_x^{i+1}+M_{yi+1}^c\boldsymbol{g}_y^{i+1}+M_{zi+1}^c\boldsymbol{g}_z^{i+1} \end{aligned}\right\} \quad \cdots\cdots (21\cdot a,\ b)$$

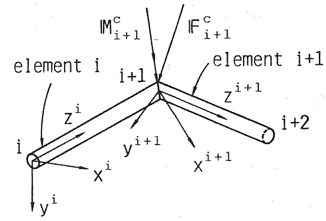


Fig. 2 Joint of Elements in Original State.

Considering the equilibrium and the compatibility at node $i+1$, the point transfer equations can be easily derived. With the directional cosines as well as the components of concentrated loads defined above, the point transfer equations can be expressed as follows for each physical quantities.

$$\left.\begin{array}{l}\begin{Bmatrix} u_{i+1} \\ v_{i+1} \\ w_{i+1}\end{Bmatrix}^{i+1} = [L_{ab}]_{i+1}\begin{Bmatrix} u_{i+1} \\ v_{i+1} \\ w_{i+1}\end{Bmatrix}^{i}, \quad [\hat{l}_{ab}|_{i+1}]^{i+1}=[L_{ab}]_{i+1}[\hat{l}_{ab}|_{i+1}]^{i}\,{}^{t}[L_{ab}]_{i+1} \\ \begin{Bmatrix} \overline{F}_{xi+1} \\ \overline{F}_{yi+1} \\ \overline{F}_{zi+1}\end{Bmatrix}^{i+1} = [L_{ab}]_{i+1}\begin{Bmatrix} \overline{F}_{xi+1} \\ \overline{F}_{yi+1} \\ \overline{F}_{zi+1}\end{Bmatrix}^{i} + \begin{Bmatrix} F^{c}_{xi+1} \\ F^{c}_{yi+1} \\ F^{c}_{zi+1}\end{Bmatrix}, \quad \begin{Bmatrix} \overline{M}_{yi+1} \\ -\overline{M}_{xi+1} \\ \overline{M}_{zi+1}\end{Bmatrix}^{i+1} = [L_{ab}]_{i+1}\begin{Bmatrix} \overline{M}_{yi+1} \\ -\overline{M}_{xi+1} \\ \overline{M}_{zi+1}\end{Bmatrix}^{i} + \begin{Bmatrix} M^{c}_{xi+1} \\ M^{c}_{yi+1} \\ M^{c}_{zi+1}\end{Bmatrix}\end{array}\right\} \quad \cdots\cdots (22\cdot a\sim d)$$

In a similar way, point transfer equations for incremental quantities defined by eq. (17) can be given in a matrix form as

$$\{\Delta Q^{*}_{j}|_{i+1}\}^{i+1}=[\Delta P^{*}_{i+1}]\{\Delta Q^{*}_{j}|_{i+1}\}^{i} \quad \cdots\cdots (23)$$

$$\left.\begin{array}{l}[\Delta P^{*}_{i+1}]=\begin{bmatrix} [L_{ab}]_{i+1} & & & & \{0\} \\ & [L_{ab}]_{i+1} & & 0 & \{0\} \\ & & [L_{ab}]_{i+1} & & \{\Delta F^{c}_{j}|_{i+1}\} \\ 0 & & & [L_{ab}]_{i+1} & \{\Delta M^{c}_{j}|_{i+1}\} \\ & & & & 1 \end{bmatrix} \\ {}^{t}\{\Delta F^{c}_{j}|_{i+1}\}=\{\Delta F^{c}_{xi+1},\ \Delta F^{c}_{yi+1},\ \Delta F^{c}_{zi+1}\}, \quad {}^{t}\{\Delta M^{c}_{j}|_{i+1}\}=\{\Delta M^{c}_{xi+1},\ \Delta M^{c}_{yi+1},\ \Delta M^{c}_{zi+1}\}\end{array}\right\} \quad \cdots\cdots (24\cdot a\sim c)$$

where the superscript i in eq. (23) denotes the quantities in member coordinates i, and $[\Delta P^{*}_{i+1}]$ is a point transfer matrix which exactly coincides with that for small displacement theory.

5. SOLUTION PROCEDURE

For simplicity, the solution procedure is explained primarily for the case when the present method is applied for a two-point boundary value problem as shown in Fig. 3, where boundary conditions are given at both ends of the structure.

If the components of state vector at node 1 are known, the state vector at node n can be calculated numerically by transferring the state vector at node 1 with the successive use of eqs. (11) and (22). The result of the above procedure can be symbolically expressed in the form

$$\{Q_j|_n\}^{n-1}=F(\{Q_j|_1\}^{1}) \quad \cdots\cdots (25)$$

In a usual case, it is a rarity that all the components of the state vector are given at node 1, and, hence, some iteration is required to obtain solutions which satisfy the boundary conditions at node n. Herein, Newton-Raphson method, as explained below, is introduced as an iterative procedure.

Let $\{Q_j|_1\}^{1}_{(0)}$ be a trial value of the state vector at node 1. With this trial value, it is possible to calculate the state vector at node n, following the procedure expressed symbolically in eq. (25). However, the components of the calculated state vector at node n do not necessarily coincide with the prescribed values given as boundary conditions, unless the trial value is correct, thus, resulting in the following error.

$$\{\Delta Q^{E}_{j}|_n\}^{n-1}=\{Q_j|_n\}^{n-1}-F(\{Q_j|_1\}^{1}_{(0)}) \quad \cdots\cdots (26)$$

In a rod model, six independent components of the state vector are given as boundary conditions at node n and the corresponding six components of $\{\Delta Q^{E}_{j}|_n\}^{n-1}$ can be calculated. If an appropriate correction $\{\Delta Q^{E}_{j}|_1\}^{1}$ is made to the trial value, the following relation holds

$$\{Q_j|_n\}^{n-1}=F(\{Q_j|_1\}^{1}_{(0)}+\{\Delta Q^{E}_{j}|_1\}^{1}) \quad \cdots\cdots (27)$$

Herein, Newton-Raphson Method is employed to estimate the

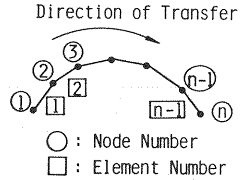


Fig. 3 Finite Elements for Two-Point Boundary Value Problem.

correction $\{\Delta Q_j^E|_1\}^1$. Supposing that the error $\{\Delta Q_j^E|_n\}^{n-1}$ and the correction $\{\Delta Q_j^E|_1\}^1$ are small, the relation given by eq. (15) is applicable, and these vectors can be respectively transformed to $\{\Delta Q_j^{E*}|_n\}^{n-1}$ and $\{\Delta Q_j^{E*}|_1\}^1$ expressed in terms of the components shown in eq. (17).

With the successive use of the incremental transfer equations given by eqs. (19) and (23), $\{\Delta Q_j^{E*}|_1\}^1$ can be related to $\{\Delta Q_j^{E*}|_n\}^{n-1}$ approximately as

$$\{\Delta Q_j^{E*}|_n\}^{n-1} \doteqdot [\Delta T_{n-1}^*(\{Q_j|_{n-1}\}_{(0)}^{n-1})][\Delta P_n^*][\Delta T_{n-2}^*(\{Q_j|_{n-2}\}_{(0)}^{n-2})][\Delta P_{n-1}^*]\cdots$$
$$\cdots[\Delta T_2^*(\{Q_j|_2\}_{(0)}^2)][\Delta P_2^*][\Delta T_1^*(\{Q_j|_1\}_{(0)}^1)][\Delta P_1^*]\{\Delta Q_j^{E*}|_1\}^1 \quad (28)$$

where $\{Q_j|_k\}_{(0)}^k$ ($k=1, 2, \cdots, n-1$) is the state vector at node k which is calculated, using the trial value $\{Q_j|_1\}_{(0)}^1$. It should be noted in eq. (28) that the increments of external forces, both concentrated and distributed, are zero, because there is no increment in external forces in the iterative procedure. For simplicity, eq. (28) is rewritten in the form

$$\{\Delta Q_j^{E*}|_n\}^{n-1} \doteqdot [\Delta F^*]\{\Delta Q_j^{E*}|_1\}^1 \quad (29)$$

As seen from eq. (17), $\{\Delta Q_j^{E*}|_1\}^1$ consists of twelve components. Among the twelve independent components of $\{Q_j|_1\}^1$, six components are given as boundary conditions. Therefore, the six components of $\{\Delta Q_j^{E*}|_1\}^1$ are always zero, and the rest components are obtained from eq. (29), utilizing the six known components of $\{\Delta Q_j^{E*}|_n\}^{n-1}$. All the components of $\{\Delta Q_j^{E*}|_1\}^1$, so obtained, are again transformed to $\{\Delta Q_j^E|_1\}^1$ by eq. (15) in order to calculate a new trial value $(\{Q_j|_1\}_{(0)}^1+\{\Delta Q_j^E|_1\}^1)$. However, the new trial value derived from the above procedure is still approximate due to the approximation in Eq. (28) and this procedure has to be repeated until the error $\{\Delta Q_j^E|_n\}^{n-1}$ given by eq. (26) becomes within some prescribed tolerance.

6. NUMERICAL EXAMPLES

Several structures are analyzed in order to demonstrate the accuracy and the validity of the present numerical method.

(1) Analysis of plane cantilevers

Herein, the accuracy of the present method is examined in comparison with the closed-form solutions. Since no closed-form solutions are available for space rods, plane rods are used for comparison.

The accuracy of the numerical methods for plane rods has customarily been evaluated, making use of the elliptic integral solutions for inextensional elastica where the elongation of member axis is ignored. However, the use of the solutions for inextensional elastica is not adequate in an exact sense because the numerical methods usually consider the elongation of member axis. In view of the fact that the present numerical method is based on the theory of finite displacements with small strains, the closed-form solutions for the same theory are utilized here. These closed-form solutions derived by us[15] make use of the elliptic integrals and hence, the solutions are highly accurate.

Cantilevers used for this analysis are illustrated in **Fig. 4**, where an axial force a little higher than the buckling load is applied at free end together with a small end moment. The slenderness ratios ($\lambda = l/\sqrt{I/A}$) of the structure are of two kinds, that is, 4 and 100. The structure with $\lambda=4$ is added in order to show more clearly that the present numerical method is convergent to the closed-form solutions for the theory of finite displacements with small strains, though such a stocky column is impractical. This is because not so much quantitative difference can be found in the solutions for $\lambda=100$, whether the elongation of member axis is considered or not. Convergence of the numerical solutions according to the number of elements is summarized for end-displacements in **Table 3**. It can be confirmed from the table that the present numerical solutions surely converge to the closed-form solutions for the theory of finite displacements with small strains.

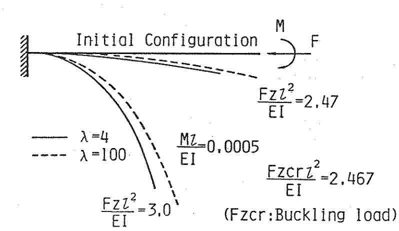


Fig. 4 Cantilever.

(2) Three-dimensional analysis of a 45-degree bend

A cantilever 45-degree bend subjected to a concentrated end

Table 3 Convergence of the Present Method (Cantilever).

λ	$F_z l^2 / EI$	No. of Elements	$v_0 / w_0 fs$	$w_0 / w_0 fs$	$\alpha / \alpha fs$	$\frac{v_0 fs}{v_0 fi}$, $\frac{w_0 fs}{w_0 fi}$, $\frac{\alpha fs}{\alpha fi}$
100	2.47	10	1.733	2.926	1.747	$v_0 fs/v_0 fi$ = 1.000
		25	1.136	1.282	1.137	
		50	1.035	1.068	1.035	
		100	1.009	1.017	1.009	$w_0 fs/w_0 fi$ = 0.9661
		250	1.001	1.003	1.001	
		500	1.000	1.001	1.000	$\alpha fs/\alpha fi$ = 1.000
		1000	1.000	1.000	1.000	
	3.0	10	1.014	1.043	1.022	$v_0 fs/v_0 fi$ = 1.000
		25	1.006	1.006	1.003	$w_0 fs/w_0 fi$ = 1.000
		50	1.000	1.001	1.001	$\alpha fs/\alpha fi$ = 1.000
		100	1.000	1.000	1.000	
4	2.47	10	1.739	1.062	1.747	$v_0 fs/v_0 fi$ = 0.8471
		25	1.136	1.009	1.137	
		50	1.035	1.002	1.035	$w_0 fs/w_0 fi$ = 22.90
		100	1.009	1.001	1.009	
		250	1.001	1.000	1.001	$\alpha fs/\alpha fi$ = 1.000
		500	1.000	1.000	1.000	
	3.0	10	1.017	1.028	1.022	$v_0 fs/v_0 fi$ = 0.8952
		25	1.002	1.004	1.003	$w_0 fs/w_0 fi$ = 1.259
		50	1.000	1.001	1.001	$\alpha fs/\alpha fi$ = 1.000
		100	1.000	1.000	1.000	

Remarks: $F_z l^2/EI = 2.467$: Buckling load from the theory of Finite Displacements with Small Strains

v_0, w_0, α_0: Present numerical solutions based on the theory of Finite Displacements with Small Strains

$v_0 fs$, $w_0 fs$, αfs : Elliptic integral solutions for Finite Displacements with Small Strains

$v_0 fi$, $w_0 fi$, αfi : Elliptic integral solutions for Inextensional Finite Displacements

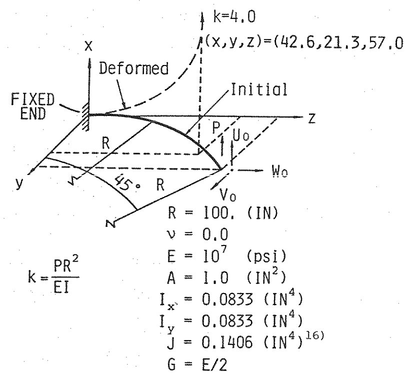


Fig. 5 45-degree Circular Bend.

Table 4 Convergence of the Present Method (45-degree Circular Bend).

PR^2/EI	No. of Elements for One Member	u_0/R	v_0/R	w_0/R
5	1	0.4682	0.1006	-0.1792
	5	0.4718	0.1013	-0.1743
	10	0.4719	0.1013	-0.1741
	15	0.4720	0.1013	-0.1741
	50	0.4720	0.1013	-0.1741
	250	0.4720	0.1013	-0.1741
10	1	0.5668	0.1633	-0.3041
	5	0.5803	0.1673	-0.2962
	10	0.5809	0.1674	-0.2959
	15	0.5810	0.1675	-0.2590
	50	0.5810	0.1675	-0.2590
	250	0.5810	0.1675	-0.2590

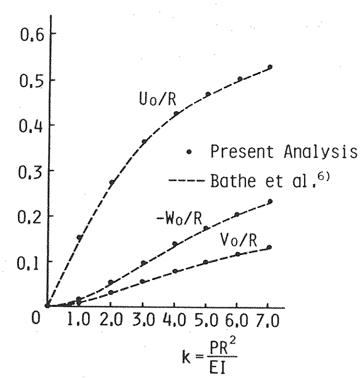


Fig. 6 Tip Displacements of 45-degree Circular Bend.

load as shown in Fig. 5 is calculated. The bend with a radius of 100 in. lies in the y-z plane. The concentrated tip load is applied into the x-direction. The above model is exactly the same as was used by Bathe et al. in his three dimensional finite displacement analysis[6]. Following the idealization by Bathe, the shape of the bend is approximated by the assemblage of 8 equal straight members. In the present analysis, each straight member is further divided into 15 equal finite elements, in view of the convergence of the numerical solutions as shown in Table 4. The material is assumed linear elastic. The torsional rigidity is calculated by the formula described in Ref. 16), since only the size of cross section of the bend is given in Ref. 6).

The results of the analysis are shown in Fig. 6 in terms of end displacements, comparing with the results given by Bathe. In this analysis, Bathe adopted up-dated Lagrangian approach combined with moving coordinate method, where the small displacement theory was used to describe the deformation after the separation of rigid body rotations. Although the present approach and the Bathe's approach are different, a good agreement is observed in the calculated results.

7. CONCLUDING REMARKS

A numerical method is developed to obtain solutions directly from the highly nonlinear governing equations for space rods with Lagrangian expressions. The accuracy and the convergence of the solutions are confirmed by numerical examples.

The present method is based on the theory of finite displacements with small strains and have no restrictions on the magnitude of displacements. This method becomes possible chiefly because the adoption of new deformation components considerably simplifies the governing differential equations without introducing any additional approximations.

Appendix A Derivatives of Physical Quantities

(1) First order derivatives

a) Displacement and Directional Cosine

$$d\,{}^t\{u_0,\ v_0,\ w_0\}/dz={}^t\{\sqrt{\hat{g}_0}\,l_{\hat{z}x},\ \sqrt{\hat{g}_0}\,l_{\hat{z}y},\ \sqrt{\hat{g}_0}\,l_{\hat{z}z}-1\} \qquad \text{(A·1)}$$

$$d[l_{\hat{a}b}]/dz=[\hat{D}][l_{\hat{a}b}] \qquad \text{(A·2)}$$

b) Force and Moment

$$\overline{F}'_x=-p_x,\quad \overline{F}'_y=-p_y,\quad \overline{F}'_z=-p_z \qquad \text{(A·3 a}\sim\text{c)}$$

$$\left.\begin{aligned}\overline{M}'_x&=(l_{\hat{x}x}l_{\hat{y}y}-l_{\hat{x}y}l_{\hat{y}x})\overline{F}_x+(l_{\hat{y}y}l_{\hat{x}z}-l_{\hat{x}y}l_{\hat{y}z})\overline{F}_z+m_y\\ \overline{M}'_y&=(l_{\hat{x}x}l_{\hat{y}y}-l_{\hat{x}y}l_{\hat{y}x})\overline{F}_y+(l_{\hat{x}x}l_{\hat{y}z}-l_{\hat{y}x}l_{\hat{x}z})\overline{F}_z-m_x\\ \overline{M}'_z&=(l_{\hat{x}z}l_{\hat{y}x}-l_{\hat{x}x}l_{\hat{y}z})\overline{F}_x+(l_{\hat{x}z}l_{\hat{y}y}-l_{\hat{y}z}l_{\hat{x}y})\overline{F}_y-m_z\end{aligned}\right\} \qquad \text{(A·4 a}\sim\text{c)}$$

(2) Second order derivatives

a) Displacement

$$\left.\begin{aligned}u''_0&=\sqrt{\hat{g}'_0}\,l_{\hat{z}x}-\sqrt{\hat{g}_0}(\tilde{M}_x l_{\hat{x}x}/EI_x+\tilde{M}_y l_{\hat{y}x}/EI_y)\\ v''_0&=\sqrt{\hat{g}'_0}\,l_{\hat{z}y}-\sqrt{\hat{g}_0}(\tilde{M}_x l_{\hat{x}y}/EI_x+\tilde{M}_y l_{\hat{y}y}/EI_y)\\ w''_0&=\sqrt{\hat{g}'_0}\,l_{\hat{z}z}-\sqrt{\hat{g}_0}(\tilde{M}_x l_{\hat{x}z}/EI_x+\tilde{M}_y l_{\hat{y}z}/EI_y)\end{aligned}\right\} \qquad \text{(A·5 a}\sim\text{c)}$$

where

$$\begin{aligned}\sqrt{\hat{g}'_0}=&=-\{(l_{\hat{y}x}\overline{F}_x+l_{\hat{y}y}\overline{F}_y+l_{\hat{y}z}\overline{F}_z)\tilde{M}_y/EI_y+(l_{\hat{x}x}\overline{F}_x+l_{\hat{x}y}\overline{F}_y+l_{\hat{x}z}\overline{F}_z)\tilde{M}_z/EI_x\\ &+(l_{\hat{z}x}p_z+l_{\hat{z}y}p_y+l_{\hat{z}z}p_z)\}/EA-J\tilde{M}_z\{(1/EI_y-1/EI_x)\tilde{M}_x\tilde{M}_y\\ &-l_{\hat{z}x}m_x-l_{\hat{z}y}m_y-l_{\hat{z}z}m_z\}/\{A(GJ)^2\}\end{aligned} \qquad \text{(A·6)}$$

$${}^t\{\tilde{M}_y,\ -\tilde{M}_x,\ \tilde{M}_z\}=[l_{\hat{a}b}]^t\{\overline{M}_y,\ -\overline{M}_x,\ \overline{M}_z\} \qquad \text{(A·7)}$$

b) Directional Cosine

$$[l_{\hat{a}b}]''=[\hat{D}]'[l_{\hat{a}b}]+[\hat{D}][\hat{D}][l_{\hat{a}b}] \qquad \text{(A·8)}$$

Non-zero components of $[D]'$ are given by

$$\left.\begin{aligned}\hat{D}'_{12}&=-\hat{D}'_{21}=\{(1/EI_y-1/EI_x)\tilde{M}_x\tilde{M}_y-l_{\hat{z}x}m_x-l_{\hat{z}y}m_y-l_{\hat{z}z}m_z\}/GJ\\ \hat{D}'_{13}&=-\hat{D}'_{31}=\{l_{\hat{x}x}\overline{F}_x+l_{\hat{x}y}\overline{F}_y+l_{\hat{x}z}\overline{F}_z+(1/GJ-1/EI_y)\tilde{M}_y\tilde{M}_z\\ &\qquad+l_{\hat{y}x}m_x+l_{\hat{y}y}m_y+l_{\hat{y}z}m_z\}/EI_x\\ \hat{D}'_{23}&=-\hat{D}'_{32}=\{l_{\hat{y}x}\overline{F}_x+l_{\hat{y}y}\overline{F}_y+l_{\hat{y}z}\overline{F}_z-(1/GJ-1/EI_x)\tilde{M}_x\tilde{M}_z\\ &\qquad-l_{\hat{x}x}m_x-l_{\hat{x}y}m_y-l_{\hat{x}z}m_z\}/EI_y\end{aligned}\right\} \qquad \text{(A·9 a}\sim\text{c)}$$

c) Force and Moment

$$\overline{F}''_x=-p'_x,\quad \overline{F}''_y=-p'_y,\quad \overline{F}''_z=-p'_z \qquad \text{(A·10 a}\sim\text{c)}$$

$$\left.\begin{aligned}\overline{M}''_x=&\{(l_{\hat{z}x}l_{\hat{y}y}-l_{\hat{z}y}l_{\hat{y}x})\tilde{M}_x/EI_x+(l_{\hat{z}y}l_{\hat{x}x}-l_{\hat{z}x}l_{\hat{x}y})\tilde{M}_y/EI_y\}\overline{F}_x+\{(l_{\hat{y}y}l_{\hat{z}z}-l_{\hat{y}z}l_{\hat{z}y})\tilde{M}_x/EI_x\\ &+(l_{\hat{z}y}l_{\hat{x}z}-l_{\hat{x}y}l_{\hat{z}z})\tilde{M}_y/EI_y\}\overline{F}_z-(l_{\hat{x}x}l_{\hat{y}y}-l_{\hat{x}y}l_{\hat{y}x})p_x-(l_{\hat{y}y}l_{\hat{x}z}-l_{\hat{x}y}l_{\hat{y}z})p_z+m'_x\\ \overline{M}''_y=&\{(l_{\hat{y}y}l_{\hat{z}x}-l_{\hat{z}y}l_{\hat{y}x})\tilde{M}_x/EI_x+(l_{\hat{x}x}l_{\hat{z}y}-l_{\hat{z}x}l_{\hat{x}y})\tilde{M}_y/EI_y\}\overline{F}_y+\{(l_{\hat{z}x}l_{\hat{y}z}-l_{\hat{z}z}l_{\hat{y}x})\tilde{M}_x/EI_x\\ &+(l_{\hat{z}z}l_{\hat{x}x}-l_{\hat{z}x}l_{\hat{x}z})\tilde{M}_y/EI_y\}\overline{F}_z-(l_{\hat{x}x}l_{\hat{y}y}-l_{\hat{x}y}l_{\hat{y}x})p_y-(l_{\hat{x}x}l_{\hat{y}z}-l_{\hat{y}x}l_{\hat{x}z})p_z-m'_x\\ \overline{M}''_z=&\{(l_{\hat{z}z}l_{\hat{y}x}-l_{\hat{z}x}l_{\hat{y}z})\tilde{M}_x/EI_x+(l_{\hat{z}x}l_{\hat{x}z}-l_{\hat{z}z}l_{\hat{x}x})\tilde{M}_y/EI_y\}\overline{F}_x+\{(l_{\hat{z}z}l_{\hat{y}y}-l_{\hat{z}y}l_{\hat{y}z})\tilde{M}_x/EI_x\\ &+(l_{\hat{z}y}l_{\hat{x}z}-l_{\hat{z}z}l_{\hat{x}y})\tilde{M}_y/EI_y\}\overline{F}_y-(l_{\hat{x}z}l_{\hat{y}x}-l_{\hat{x}x}l_{\hat{y}z})p_x-(l_{\hat{x}z}l_{\hat{y}y}-l_{\hat{y}z}l_{\hat{z}y})p_y-m'_z\end{aligned}\right\} \cdots \text{(A·11 a}\sim\text{c)}$$

Appendix B Derivation of Eq. (15)

Infinitesimally small increment $(\Delta\hat{i}_{x0},\ \Delta\hat{i}_{y0},\ \Delta\hat{i}_{z0})$ can be expressed as follows, making use of the incremental rotational angles $(\Delta\hat{\alpha}_x,\ \Delta\hat{\alpha}_y,\ \Delta\hat{\alpha}_z)$ around the vectors $(\hat{i}_{x0},\ \hat{i}_{y0},\ \hat{i}_{z0})$.

$$\begin{Bmatrix} \Delta\hat{\boldsymbol{i}}_{x0} \\ \Delta\hat{\boldsymbol{i}}_{y0} \\ \Delta\hat{\boldsymbol{i}}_{z0} \end{Bmatrix} = \begin{bmatrix} 0, & \Delta\hat{\alpha}_z, & -\Delta\hat{\alpha}_y \\ -\Delta\hat{\alpha}_z, & 0, & \Delta\hat{\alpha}_x \\ \Delta\hat{\alpha}_y, & -\Delta\hat{\alpha}_x, & 0 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i}}_{x0} \\ \hat{\boldsymbol{i}}_{y0} \\ \hat{\boldsymbol{i}}_{z0} \end{Bmatrix} \cdots\cdots (\mathrm{B}\cdot 1)$$

$(\Delta\hat{\boldsymbol{i}}_{x0}, \Delta\hat{\boldsymbol{i}}_{y0}, \Delta\hat{\boldsymbol{i}}_{z0})$ can also be obtained by taking the increment of eq. (7·a) as

$$\begin{Bmatrix} \Delta\hat{\boldsymbol{i}}_{x0} \\ \Delta\hat{\boldsymbol{i}}_{y0} \\ \Delta\hat{\boldsymbol{i}}_{z0} \end{Bmatrix} = [\Delta l_{\hat{a}b}] \begin{Bmatrix} \boldsymbol{g}_x \\ \boldsymbol{g}_y \\ \boldsymbol{g}_z \end{Bmatrix} \cdots\cdots (\mathrm{B}\cdot 2)$$

Considering eqs. (B·1), (B·2), and (7·a), $[\Delta l_{\hat{a}b}]$ is given by

$$[\Delta l_{\hat{a}b}] = \begin{bmatrix} 0, & \Delta\hat{\alpha}_z, & -\Delta\hat{\alpha}_y \\ -\Delta\hat{\alpha}_z, & 0, & \Delta\hat{\alpha}_x \\ \Delta\hat{\alpha}_y, & -\Delta\hat{\alpha}_x, & 0 \end{bmatrix} [l_{\hat{a}b}] \cdots\cdots (\mathrm{B}\cdot 3)$$

Equation (15) is derived by substituting into eq. (B·3) the following relation

$${}^t\{\Delta\hat{\alpha}_x, \Delta\hat{\alpha}_y, \Delta\hat{\alpha}_z\} = [l_{\hat{a}b}]^t \{\Delta\alpha_x, \Delta\alpha_y, \Delta\alpha_z\} \cdots\cdots (\mathrm{B}\cdot 4))$$

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