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A TANGENT THEORY OF THIN-WALLED BEAMS AFTER ANY LARGE DISPLACEMENTS

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Upon an existing nonlinear theory of thin-walled beams for truly large displacements, linearized governing relations are rigorously developed on a general equilibrium after any large displacements. Apart from those in that nonlinear theory, another set of parameters are employed for the infinitesimal variations under the same kinematic field to develop a normalized formulation. Any bucklings of thin-walled beams arising after large displacements can be put exactly on the present tangent equations : as an unprecedented example, the flexural buckling of an open-cross-section beam subject to finite torsion and compression is analyzed.

Keywords : *thin-walled beam, large displacement, buckling*

1. INTRODUCTION

As for the thin-walled beams, there have been nonlinear formulations taking account of up to the quadratic or cubic terms of finite displacements around the initial configurations[2)~4),6),9)] Recently, we have a next-stage formulation where within the frame of small strains, no truncations are made upon higher nonlinear terms of displacements[1)].

The formulations of up to the quadratic terms are enough to be linearized around the initial configurations and to analyze those stability problems whose preceding deformations are small enough. At the same time, there are other direct derivations of the similar stability equations of thin-walled beams, by applying the variational principle with taking account of the initial stresses[5),7),8),10),12)]. When governing equations containing the nonlinear terms higher than the second are linearized after the displacements, the resulting stability equations contain to some extent the effects of the preceding deformations. But, in order to obtain an exact solution after large displacements, linearized relations have to be rigorously developed upon that deformed configuration. For instance, by doing so on the large pure bending of a thin-walled beam, Vacharajittiphan and Trahair[11)] presented an exact solution for its lateral buckling.

In this paper, a set of linearized governing equations of thin-walled beams are rigorosly established upon a general equilibrium after any large displacements, not upon a peculiar one such as the finite pure bending. The formulation is to be developed in parallel with that for large displacements themselves to describe the various preceding equilibria : the present linearization is developed on the basis of an existing nonlinear theory of thin-walled beams, which can deal with any large displacements. By reducing the general preceding state of deformations to each particular one, the present formulation is capable to solve exactly any stability problems after large displacements. As an unprecedented example, the flexural buckling of a straight thin-walled beam subject to large torsion and compression is analyzed.

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2. GEOMETRICALLY NONLINEAR THEORY OF THIN-WALLED BEAMS

Our present analysis is made upon an existing finite-displacement theory of thin-walled beams[1], which is valid to any large displacements so far as the resulting strains are small compared with the unity. We here summarize that governing equations before we proceed to our study.

Considering Cartesian coordinates $\{x,\ y,\ z\}$ in the 3-D space, with $\{\boldsymbol{i}_{(xyz)}\}=\{\boldsymbol{i}_x,\ \boldsymbol{i}_y,\ \boldsymbol{i}_z\}$ being their orthonormal base vectors, we represent a thin-walled beam's axial line, namely G-line, by one-parameter curve $\{x_G(\zeta),\ y_G(\zeta),\ z_G(\zeta)\}$ $(0\leqq\zeta\leqq l)$, where ζ is a Lagrangian coordinate defined along natural length of the line, with l being that length between the two ends. Then, unit elongation ε_G, curvature $\varkappa_G$ and torsion γ_G of the G-line with respect to ζ are represented in terms of the $\{x_G,\ y_G,\ z_G\}$, as follows :

$$\varepsilon_G=\sqrt{(x'^2+y'^2+z'^2)_G}-1,\quad \varkappa_G=\frac{1}{1+\varepsilon_G}\sqrt{(x''^2+y''^2+z''^2-\varepsilon'^2)_G},\quad \gamma_G=\frac{1}{(1+\varepsilon_G)^3\varkappa_G^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}_G \qquad \text{(1. a–c)}$$

where prime means the differentiation with respect to ζ. And, when the unit vectors into the bi-normal, principal-normal and tangent directions on the G-line, respectively denoted by $\boldsymbol{i}_{G\tau}$, $\boldsymbol{i}_{G\varkappa}$ and $\boldsymbol{i}_{G\zeta}$, are to be represented in the form of $\{\boldsymbol{i}_{G(\tau\varkappa\zeta)}\}=[T_G(\zeta)]\{\boldsymbol{i}_{(xyz)}\}$, elements of the orthonormal $[T_G]$ are obtained as follows :

$$[T_G(\zeta)]=\begin{bmatrix} \dfrac{1}{(1+\varepsilon)^2\varkappa}<\begin{vmatrix} y'' & z'' \\ y' & z' \end{vmatrix},\ \begin{vmatrix} z'' & x'' \\ z' & x' \end{vmatrix},\ \begin{vmatrix} x'' & y'' \\ x' & y' \end{vmatrix}> \\ \dfrac{1}{(1+\varepsilon)\varkappa}<x'',\ y'',\ z''>-\dfrac{\varepsilon'}{(1+\varepsilon)^2\varkappa}<x',\ y',\ z'> \\ \dfrac{1}{1+\varepsilon}<x',\ y',\ z'> \end{bmatrix}_G \qquad (2)$$

Hereafter, letters enclosed in { }, < > and [ ] indicate a column and a row vector, and a matrix, respectively.

Two other coordinate sets in cross-section, as shown in **Fig. 1** : $\{\xi,\ \eta\}$ are orthonormal coordinates with origin at the G-point (movable as a rigid) ; and $\{s,\ n\}$ are convected such that s is defined by natural length along the thin-wall's middle line with n being perpendicular to the s-direction. Then, the original uniform cross-section is described by shape of the middle line $\{\xi^*(s),\ \eta^*(s)\}$ and thickness of the wall $t(s)$, where asterisk means a quantity on the middle line.

Suppose that a set of unit vectors $\{\boldsymbol{i}_{G\xi},\ \boldsymbol{i}_{G\eta}\}$ are determined by rotating the $\{\boldsymbol{i}_{G\tau},\ \boldsymbol{i}_{G\varkappa}\}$ around $\boldsymbol{i}_{G\zeta}$ by angle $\phi(\zeta)$. By regarding them as the base vectors of the former $\{\xi,\ \eta\}$, we have $\phi(\zeta)$ as the fourth freedom of torsional rotation of cross-section. Denoting this rotation by $\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}=[T_\phi(\zeta)]\{\boldsymbol{i}_{G(\tau\varkappa\zeta)}\}$, we have the orthonormal $[\hat{T}(\zeta)]=[T_\phi][T_G]$ relating the $\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}$ to the spatial $\{\boldsymbol{i}_{(xyz)}\}$. When the rates of change of $\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}$ into ζ are to be represented in the form of $d\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}/d\zeta=[\hat{\Phi}(\zeta)]\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}$, the derivative matrix is represented as

$$[\hat{\Phi}(\zeta)]=\begin{bmatrix} 0, & \hat{\gamma}, & -\varkappa_\xi \\ -\hat{\gamma}, & 0, & -\varkappa_\eta \\ \varkappa_\xi, & \varkappa_\eta, & 0 \end{bmatrix} \qquad (3)$$

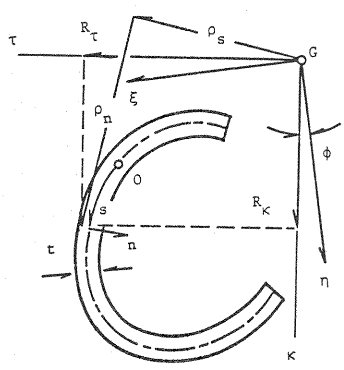

Fig. 1 Cross-Section.

where, by the use of the former $\varkappa_G$ and γ_G

$$\varkappa_\xi=\varkappa_G\sin\phi,\quad \varkappa_\eta=\varkappa_G\cos\phi,\quad \hat{\gamma}=\gamma_G+\phi' \qquad \text{(4. a–c)}$$

and these are now called curvatures into $\{\xi,\ \eta\}$-directions and torsion of cross-section, respectively.

In addition to the former translations and rotations as a rigid of each cross-section, we consider the distortion of cross-section under the following well-known hypotheses : i) no strain components in the plane of cross-section ; ii) no shear strain in (n, ζ)-direction ; and iii) while no shear strain in (s, ζ)-direction on the middle surface of an opened thin-wall, as for thin-wall of a closed cross-section's cell, the shear strain in that surface is to be taken into account by the shear flow theory. Considering inplane components as well as the warping, we resolve the relative displacements on cross-section into $\{s, n, \zeta\}$-directions and denote them by $\{u_s(s, n, \zeta), u_n(s, n, \zeta), w(s, n, \zeta)\}$, respectively. By solving the partial-differential equations related to the above restraints on the strain components with assuming that any strains are small enough compared with the unity, we arrive at the following results : By introducing the warping obtained from ii) and iii) in the usual form of $w=\hat{\gamma}(\zeta)\ W(s, n)$ into the restraints of i), we can see that in a thin-walled beam of open cross-section, which admits larger torsions even within the frame of small strains, the inplane components have to be taken into account in the form of $\{u_s, u_n\}=\hat{\gamma}(\zeta)^2\{U_s(s, n), U_n(s, n)\}$. On the other hand, as for a beam of closed cross-section where the warping itself is small enough, the inplane components become negligible. The modes of the displacements, dependent only upon the shape of cross-section, are eventually obtained as

$$\left.\begin{aligned} W(s, n)&=\int_0^s\left[\rho_s^*-\left(\frac{\oint \rho_s^*\,ds}{\oint 1/t\,ds}\cdot\frac{1}{t}\right)_{\text{closed}}\right]ds+\rho_n n \\ U_s(s, n)&=\int_0^s \rho_s^*(\tau)\left\{\frac{1}{2}\rho_s^*(\tau)+(\theta(s)-\theta(\tau))\,\rho_n(\tau)\right\}d\tau \\ U_n(s, n)&=\int_0^s \rho_s^*(\tau)\,\rho_n(\tau)\,d\tau+\frac{1}{2}\rho_n^2 n \end{aligned}\right\} \quad \cdots\cdots (5.\ \text{a–c})$$

where $\theta(s)$=angle of s- from ξ-direction ; $\rho_n(=\rho_n^*)$ and $\rho_s(=\rho_s^*-n)$ are, as shown in **Fig. 1**, lengths defined by

$$\begin{Bmatrix} \rho_n \\ -\rho_s \end{Bmatrix}=\begin{bmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{bmatrix}\begin{Bmatrix} \xi \\ \eta \end{Bmatrix}; \quad \cdots\cdots (6)$$

and subscript $(\ )_{\text{closed}}$ means to be accounted of only in thin-wall of a closed cross-section's cell.

The present kinematic field is, instead of usually for the displacements, for the beam's spatial configurations themselves which might be either before or after deformations : when a set of four actual functions $\{x_G, y_G, z_G, \phi\}$ are given, there is determined an associated spatial configuration of the beam. In this kinematic field, under the condition of small strains, only the following two of the Green's strain components are allowed to take place :

$$e_{\zeta\zeta}=[\varepsilon_G]-\xi[\kappa_\xi]-\eta[\kappa_\eta]-W[\hat{\gamma}']+\left(\frac{1}{2}(\xi^2+\eta^2)[\hat{\gamma}^2]\right)_{\text{O.B.}},\quad e_{s\zeta}=\frac{1}{2}\Theta\,[\hat{\gamma}] \quad \cdots\cdots (7.\ \text{a, b})$$

where

$$\Theta(s, n)=\left(\frac{\oint \rho_s^*\,ds}{\oint 1/t\,ds}\cdot\frac{1}{t}\right)_{\text{closed}}-2n; \quad \cdots\cdots (8)$$

and notation $[\ \ ]$ means difference between before and after deformation, i. e. $[F]=F-F^0$ with $(\ )^0$ denoting an initial quantity ; and subscript $(\ )_{\text{O.B.}}$ means to be accounted of only for an open cross-section.

We define the following stress resultants :

$$\left.\begin{aligned} &N=\int_A \sigma_{\zeta\zeta}\,d(\text{area}),\quad M_{(\xi)}=\int_A \sigma_{\zeta\zeta}\xi\,d(\text{area}),\quad M_{(\eta)}=\int_A \sigma_{\zeta\zeta}\eta\,d(\text{area}) \\ &M_W=\int_A \sigma_{\zeta\zeta}W\,d(\text{area}),\quad K=\int_A \sigma_{\zeta\zeta}(\xi^2+\eta^2)\,d(\text{area}),\quad T_S=\int_A \sigma_{s\zeta}\Theta\,d(\text{area}) \end{aligned}\right\} \quad \cdots\cdots (9.\ \text{a–f})$$

where $\sigma_{\zeta\zeta}$ and $\sigma_{s\zeta}$ are the stress components conjugate to $e_{\zeta\zeta}$ and $e_{s\zeta}$, respectively. The body forces, and the surface tractions acting on the end cross-sections ($\zeta=0$ and l) are here assumed to be prescribed in terms of the components into spatial $\{x, y, z\}$-directions on the material $\{s, n, \zeta\}$-field : $\{\bar{p}_x^d, \bar{p}_y^d, \bar{p}_z^d\}$ and $\{\bar{q}_x^d, \bar{q}_y^d, \bar{q}_z^d\}$, respectively. Then, we define the following body-force resultants :

$$\left[\bar{p}\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \bar{m}_{(\xi)}\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \bar{m}_{(\eta)}\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \bar{m}_{(U)}\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \bar{m}_{(V)}\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \bar{m}_{(W)}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]=\int_A \begin{Bmatrix} \bar{p}_x^d \\ \bar{p}_y^d \\ \bar{p}_z^d \end{Bmatrix} \langle 1, \xi, \eta, U, V, W\rangle\, d(\text{area}) \cdots\cdots (10)$$

where $\{U, V\}$ =components of the inplane-relative-displacement mode into $\{\xi, \eta\}$-directions, transformed from the $\{U_s, U_n\}$ of (5. b, c). And, by introducing surface-traction components $\{\bar{q}_x^d, \bar{q}_y^d, \bar{q}_z^d\}$, in place of $\{\bar{p}_{(xyz)}^d\}$, into the above (10), we define their resultants, $\{\bar{F}_{(xyz)}\}, \cdots, \{\bar{M}_{(W)(xyz)}\}$, similarly to $\{\bar{p}_{(xyz)}\}, \cdots, \{\bar{m}_{(W)(xyz)}\}$.

When applying the principle of virtual work in a kinematic field, variationals of its independent functions, $\{x_G, y_G, z_G, \phi\}$ in the present case, are usually employed also as the independents of virtual displacements. However, it is possible to employ other independent variationals as long as they are mathematically in the one-to-one to the former direct ones. In our reference paper[1], the rate of change of the G-line's variational position vector into ζ is resolved into the foregoing $\{i_{G(\tau\kappa\zeta)}(\zeta)\}$-directions :

$$\frac{d}{d\zeta}\delta r_G(\zeta)=<\delta\alpha_{(\tau\kappa\zeta)}(\zeta)>\{i_{G(\tau\kappa\zeta)}\} \quad \cdots\cdots (11)$$

And, noting that these $\{\delta\alpha_{(\tau\kappa\zeta)}(\zeta)\}$ together with translations at $\zeta=0$, $\{\delta x_G(0), \delta y_G(0), \delta z_G(0)\}$, are in one-to-one to the direct $\{\delta x_G(\zeta), \delta y_G(\zeta), \delta z_G(\zeta)\}$, a set of $\{(\delta\alpha_\tau(\zeta), \delta\alpha_\kappa(\zeta), \delta\alpha_\zeta(\zeta), \delta\phi(\zeta)) ; (\delta x_G(0), \delta y_G(0), \delta z_G(0))\}$ are there employed as the independents to avoid complicated variational calcules. For the sake of physical understanding, we rewrite the resulting equilibrium equations as follows :

$$\left.\begin{aligned} &N=i_{G\zeta}\cdot<F_{(xyz)}(\zeta)>\{i_{(xyz)}\} \\ &\frac{d}{d\zeta}(<M_{(\eta)}, -M_{(\xi)}, T>[\hat{T}])\{i_{(xyz)}\}+\begin{vmatrix} i_x & i_y & i_z \\ x_G' & y_G' & z_G' \\ F_x & F_y & F_z \end{vmatrix}+<\overset{*}{m}_x, \overset{*}{m}_y, \overset{*}{m}_z>\{i_{(xyz)}\}=\mathbf{0} \\ &T=M_W'+T_S+(\hat{\gamma}K)_{\text{O.B.}}+\overset{*}{m}_W \end{aligned}\right\} \quad \cdots\cdots (12.\text{a–c})$$

where N=axial force; M_W=bi-moment; T_S=St. Venant's torsional moment; $\{M_{(\eta)}, -M_{(\xi)}, T\}$ are understood as components of cross-sectional moment around $\{i_{G(\xi\eta\zeta)}\}$-directions; K represents an additional torsional moment per unit of torsion $\hat{\gamma}$, produced by those $\sigma_{\zeta\zeta}$ in other fibers than in the G-line whose directions are deviated due to the large torsion from the G's tangent direction; $\{F_{(xyz)}(\zeta)\}$ are components of cross-section force into $\{i_{(xyz)}\}$-directions, represented by

$$\{F_{(xyz)}(\zeta)\}=\{\bar{F}_{(xyz)}\}_0-\int_0^\zeta\{\bar{p}_{(xyz)}(\zeta)\}\,d\zeta\,; \quad \cdots\cdots (13)$$

and $\{\overset{*}{m}_{(xyz)}\}$ and $\overset{*}{m}_W$ are components of distributed external moment around $\{i_{(xyz)}\}$-directions and external bi-moment, respectively. Here, superimposed asterisk means the dependence upon a reference frame movable with displacements. Indeed, the $\{\overset{*}{m}_{(xyz)}\}$ and $\overset{*}{m}_W$ are related to the prescribed body-force resultants of (10), as follows :

$$<\overset{*}{m}_{(xyz)}>\{i_{(xyz)}\}=\begin{vmatrix} i_x & i_y & i_z \\ \{\hat{T}\}_x & \{\hat{T}\}_y & \{\hat{T}\}_z \\ \{\hat{m}\}_x & \{\hat{m}\}_y & \{\hat{m}\}_z \end{vmatrix}, \quad \overset{*}{m}_W=\{\hat{T}\}_\zeta\cdot\{\bar{m}_{(W)(xyz)}\}+[2\,\hat{\gamma}(\{\hat{T}\}_\xi\cdot\{\bar{m}_{(U)(xyz)}\}+\{\hat{T}\}_\eta\cdot\{\bar{m}_{(V)(xyz)}\})]_{\text{O.B.}}$$

$$\cdots\cdots (14.\text{a, b})$$

where

$$\begin{bmatrix} \hat{m}_{\xi x} & \hat{m}_{\xi y} & \hat{m}_{\xi z} \\ \hat{m}_{\eta x} & \hat{m}_{\eta y} & \hat{m}_{\eta z} \\ \hat{m}_{\zeta x} & \hat{m}_{\zeta y} & \hat{m}_{\zeta z} \end{bmatrix}=\begin{bmatrix} <\bar{m}_{(\xi)(xyz)}>-(\hat{\gamma}^2<\bar{m}_{(U)(xyz)}>)_{\text{O.B.}} \\ <\bar{m}_{(\eta)(xyz)}>-(\hat{\gamma}^2<\bar{m}_{(V)(xyz)}>)_{\text{O.B.}} \\ -\hat{\gamma}<\bar{m}_{(W)(xyz)}> \end{bmatrix}; \quad \cdots\cdots (15)$$

and $\{\hat{T}\}_x$, $\{\hat{T}\}_\xi$, $\cdots$ and $\{\hat{m}\}_x$, $\cdots$ indicate the relevant row and/or column vectors in $[\hat{T}]$ and $[\hat{m}]$. In (14. b) and (15), it is to be mentioned that the linear and quadratic terms of torsion $\hat{\gamma}$ are additional ones related to the distortion of cross-section, and small enough compared with $<\bar{m}_{(\xi)(xyz)}>$ or $<\bar{m}_{(\eta)(xyz)}>$.

The mechanical boundary conditions are obtained as

$$\left.\begin{aligned} &\{F_{(xyz)}\}|_{\zeta=l}=\{\bar{F}_{(xyz)}\}_l \\ &[\hat{T}]^T\{M_{(\eta)}, -M_{(\xi)}, T\}|_{\zeta=0,\text{or }l}=\{\overset{*}{M}_x, \overset{*}{M}_y, \overset{*}{M}_z\}_{0,\text{or }l} \\ &M_W|_{\zeta=0,\text{or }l}=\overset{*}{M}_{W\,0,\text{or }l} \end{aligned}\right\} \quad \cdots\cdots (16.\text{a–c})$$

where $\{\overset{*}{M}_{(xyz)}\}$ and $\overset{*}{M}_W$ are external moment components around $\{\boldsymbol{i}_{(xyz)}\}$ -directions and bi-moment acting on the end cross-sections, which are related to the surface-traction resultants, $\{\bar{M}_{(\xi)(xyz)}\}$, …, $\{\bar{M}_{(W)(xyz)}\}$, in the same manner as $\{\overset{*}{m}_{(xyz)}\}$ and $\overset{*}{m}_W$ to the body-force resultants by Eqs. (14. a, b) and (15).

We can write down the geometrical boundary conditions as follows :

$$\left.\begin{aligned}&\{x_G,\ y_G,\ z_G\}|_{\zeta=0,\text{or } l}=\{\bar{x}_G,\ \bar{y}_G,\ \bar{z}_G\}_{0,\text{or } l}\\&\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}|_{\zeta=0,\text{or } l}(=[\hat{T}(\zeta)]|_{\zeta=0,\text{or } l}\{\boldsymbol{i}_{(xyz)}\})=\{\bar{\boldsymbol{i}}_{G(\xi\eta\zeta)}\}_{0,\text{or } l}\\&\hat{\gamma}|_{\zeta=0,\text{or } l}=\bar{\gamma}_{0,\text{or } l}\end{aligned}\right\}\cdots\cdots (17.\text{a–c})$$

Let the $\{\xi,\ \eta\}$ be chosen into the principal axes of cross-section with the G-point taken at the gravity center, and, as well, the origin of the s-coordinate be chosen to satisfy $\int_A W(s,\ n)\ d(\text{area})=0$. Then, we define the following cross-section constants :

$$\begin{aligned}&A=\int_A d(\text{area}), && I_{\xi\xi}=\int_A \xi^2\, d(\text{area}), && I_{\eta\eta}=\int_A \eta^2\, d(\text{area})\\&I_{WW}=\int_A W^2\, d(\text{area}), && I_{W\xi}=\int_A W\xi\, d(\text{area}), && I_{W\eta}=\int_A W\eta\, d(\text{area})\\&I_G=\int_A(\xi^2+\eta^2)\, d(\text{area}), && I_{GG}=\int_A(\xi^2+\eta^2)^2\, d(\text{area}), && I_{G\xi}=\int_A(\xi^2+\eta^2)\xi\, d(\text{area})\\&I_{G\eta}=\int_A(\xi^2+\eta^2)\eta\, d(\text{area}), && I_{GW}=\int_A(\xi^2+\eta^2)W\, d(\text{area}), && J_S=\int_A \Theta^2\, d(\text{area})\end{aligned}\cdots\cdots (18.\text{a–1})$$

Assuming that the strain components, $e_{\zeta\zeta}$ and $e_{s\zeta}$ of Eqs. (7. a, b), make their each relevant stress components arise proportionally : $\sigma_{\zeta\zeta}=Ee_{\zeta\zeta}$ and $\sigma_{s\zeta}=2\,Ge_{s\zeta}$, and executing the integrations of (9. a–f) with the use of the above cross-section constants, we obtain the constitutive relations, as follows :

$$\begin{Bmatrix}N\\M_{(\xi)}\\M_{(\eta)}\\M_W\\(K)_{\text{O.B.}}\end{Bmatrix}=E\begin{bmatrix}A, & & & & \begin{pmatrix}I_G\\I_{G\xi}\\I_{G\eta}\\I_{GW}\\I_{GG}\end{pmatrix}_{\text{O.B.}}\\ & I_{\xi\xi}, & & I_{W\xi}, & \\ & & I_{\eta\eta}, & I_{W\eta}, & \\ & & & I_{WW}, & \\ & \text{sym.} & & & \end{bmatrix}\begin{Bmatrix}[\varepsilon_G]\\ [-\kappa_\xi]\\ [-\kappa_\eta]\\ [-\hat{\gamma}']\\ \left(\frac{1}{2}[\hat{\gamma}^2]\right)_{\text{O.B.}}\end{Bmatrix},\quad T_S=GJ_S[\hat{\gamma}]\cdots\cdots (19.\text{a, b})$$

3. LINEARIZATION AFTER ARBITRARY FINITE DISPLACEMENTS

Around an arbitrary state of equilibrium governed by the nonlinear equations of Sec. 2, we consider its infinitesimal variations under the same kinematic field. While a set of linearized equations can be derived by literally linearizing those nonlinear equations regarding the independent $\{x_G,\ y_G,\ z_G,\ \phi\}$, we here employ another set of independent variationals upon the foregoing equilibrium :

Let the displacement of the G-line, $\delta\boldsymbol{r}_G$, be decomposed into the foregoing $\{\boldsymbol{i}_{G(\tau\kappa\zeta)}\}$ -directions :

$$\delta\boldsymbol{r}_G(\zeta)=\langle\delta u_\tau(\zeta),\ \delta u_\kappa(\zeta),\ \delta u_\zeta(\zeta)\rangle\{\boldsymbol{i}_{G(\tau\kappa\zeta)}(\zeta)\}\cdots\cdots (20)$$

Hereafter, we denote a variational quantity by prefixing δ. When the rate of change of $\delta\boldsymbol{r}_G$ into ζ is to be represented by the form of (11), by differentiating (20) with the use of (3), we have

$$\begin{Bmatrix}\delta\alpha_\tau\\ \delta\alpha_\kappa\\ \delta\alpha_\zeta\end{Bmatrix}=\begin{Bmatrix}\delta u'_\tau\\ \delta u'_\kappa\\ \delta u'_\zeta\end{Bmatrix}+\begin{bmatrix} & -\gamma_G & \\ \gamma_G & & \kappa_G\\ & -\kappa_G & \end{bmatrix}\begin{Bmatrix}\delta u_\tau\\ \delta u_\kappa\\ \delta u_\zeta\end{Bmatrix}\cdots\cdots (21)$$

And, we can develop the relevant variations of ε_G and $\boldsymbol{i}_{G\zeta}$ as follows :

$$\delta\varepsilon_G=\delta\sqrt{\boldsymbol{r}'_G\cdot\boldsymbol{r}'_G}=\delta\alpha_\zeta,\qquad \delta\boldsymbol{i}_{G\zeta}=\delta\left(\frac{\boldsymbol{r}'_G}{|\boldsymbol{r}'_G|}\right)=\frac{\delta\alpha_\tau}{1+\varepsilon_G}\boldsymbol{i}_{G\tau}+\frac{\delta\alpha_\kappa}{1+\varepsilon_G}\boldsymbol{i}_{G\kappa}\cdots\cdots (22.\text{a, b})$$

After the variations of (20), we consider such an angular position of each cross-section that its $\{\xi,\ \eta\}$ -axes are not rotated around the (foregoing) $\boldsymbol{i}_{G\zeta}$. When denoting the unit vectors into those $\{\xi,\ \eta\}$ -directions as $\{\boldsymbol{i}_{G\xi}+\delta\boldsymbol{i}^l_{G\xi},\ \boldsymbol{i}_{G\eta}+\delta\boldsymbol{i}^l_{G\eta}\}$, we have the $\{\delta\boldsymbol{i}^l_{G\xi},\ \delta\boldsymbol{i}^l_{G\eta}\}$ determined as :

$$\delta i^{l}_{G\xi}=\frac{-1}{1+\varepsilon_G}(\cos\phi\delta\alpha_\tau+\sin\phi\delta\alpha_x)\, i_{G\zeta},\quad \delta i^{l}_{G\eta}=\frac{-1}{1+\varepsilon_G}(-\sin\phi\delta\alpha_\tau+\cos\phi\delta\alpha_x)\, i_{G\zeta} \quad \cdots\cdots\cdots\cdots (23.\text{a, b})$$

As the fourth freedom of variation, we now introduce infinitesimal rotation $\delta\psi(\zeta)$ of cross-section around $i_{G\zeta}$ from the former $\{\xi, \eta\}$-directions. To be mentioned, this $\delta\psi(\zeta)$ is related to the direct variation of $\phi(\zeta)$ as $\delta\psi=\delta\phi-\{\delta\alpha'_\tau-\gamma_G\delta\alpha_x\}/\varkappa_G$. The change of $\{\xi, \eta\}$-directions due to $\delta\psi$ are given by $\{\delta i^{T}_{G\xi}, \delta i^{T}_{G\eta}\}=\delta\psi\{i_{G\eta}, -i_{G\xi}\}$. Thus, the total variations of $\{i_{G(\xi\eta\zeta)}\}$ after $\{\delta u_{(\tau x\zeta)}, \delta\psi\}$ (fundamental unknowns in the variations) are represented as

$$\{\delta i_{G(\xi\eta\zeta)}\}=[\delta\hat{\Psi}(\zeta)]\{i_{G(\xi\eta\zeta)}\},\quad [\delta\hat{\Psi}(\zeta)]=\begin{bmatrix} 0, & \delta\psi, & \frac{-1}{1+\varepsilon_G}(\cos\phi\delta\alpha_\tau+\sin\phi\delta\alpha_x) \\ & 0, & \frac{-1}{1+\varepsilon_G}(-\sin\phi\delta\alpha_\tau+\cos\phi\delta\alpha_x) \\ \text{anti-sym.} & & 0 \end{bmatrix} \cdots (24.\text{a, b})$$

By the use of this $[\delta\hat{\Psi}]$, the change of $[\hat{T}]$ is given by $[\delta\hat{T}]=[\delta\hat{\Psi}][\hat{T}]$.

Further, we consider to represent, similarly to (3), the rates of change of those $\{i_{G(\xi\eta\zeta)}+\delta i_{G(\xi\eta\zeta)}\}$ into ζ as $\{i_{G(\xi\eta\zeta)}+\delta i_{G(\xi\eta\zeta)}\}'=([\hat{\Phi}(\zeta)]+[\delta\hat{\Phi}(\zeta)])\{i_{G(\xi\eta\zeta)}+\delta i_{G(\xi\eta\zeta)}\}$. By differentiating (24. a) with the use of (3), we have

$$\frac{d}{d\zeta}\{i_{G(\xi\eta\zeta)}+\delta i_{G(\xi\eta\zeta)}\}=([\hat{\Phi}]+[\delta\hat{\Psi}']+[\delta\hat{\Psi}][\hat{\Phi}])\{i_{G(\xi\eta\zeta)}\}$$

In the relation, $\{i_{G(\xi\eta\zeta)}+\delta i_{G(\xi\eta\zeta)}\}=([I]+[\delta\hat{\Psi}])\{i_{G(\xi\eta\zeta)}\}$ with $[I]$ being the unit matrix, $[\delta\hat{\Psi}]$ is a matrix of variationals, infinitesimals compared with the unity, and therefore we can invert that relation as $\{i_{G(\xi\eta\zeta)}\}=([I]-[\delta\hat{\Psi}])\{i_{G(\xi\eta\zeta)}+\delta i_{G(\xi\eta\zeta)}\}$. By the use of this inversion in the former relation, we have the $[\delta\hat{\Phi}]$ represented as

$$[\delta\hat{\Phi}(\zeta)]=[\delta\hat{\Psi}']+[\delta\hat{\Psi}][\hat{\Phi}]-[\hat{\Phi}][\delta\hat{\Psi}] \quad\cdots\cdots (25)$$

Introducing the $[\hat{\Phi}]$ of (3) and the $[\delta\hat{\Psi}]$ of (24. b) into this expression, we obtain our present $[\delta\hat{\Phi}(\zeta)]$ as

$$[\delta\hat{\Phi}(\zeta)]=$$

$$\begin{bmatrix} 0, & \delta\psi'-\frac{\varkappa_G}{1+\varepsilon_G}\delta\alpha_\tau, & -\left\{\frac{1}{1+\varepsilon_G}(\cos\phi\delta\alpha_\tau+\sin\phi\delta\alpha_x)\right\}'+\frac{\hat{\gamma}}{1+\varepsilon_G}(-\sin\phi\delta\alpha_\tau+\cos\phi\delta\alpha_x)-\varkappa_\eta\delta\psi \\ & 0, & -\left\{\frac{1}{1+\varepsilon_G}(-\sin\phi\delta\alpha_\tau+\cos\phi\delta\alpha_x)\right\}'-\frac{\hat{\gamma}}{1+\varepsilon_G}(\cos\phi\delta\alpha_\tau+\sin\phi\delta\alpha_x)+\varkappa_\xi\delta\psi \\ & \text{anti-sym.} & 0 \end{bmatrix}$$

$$\cdots\cdots (26)$$

In this $[\delta\hat{\Phi}(\zeta)]$, under the condition of small strains, we neglect ε_G in comparison with the unity, and, according to the definition of $\varkappa_\xi$, $\varkappa_\eta$ and $\hat{\gamma}$ by (3), we have the variations of those strain parameters as

$$\left.\begin{aligned} &\delta\varkappa_\xi=\cos\phi(\delta\alpha'_\tau-\gamma_G\delta\alpha_x)+\sin\phi(\delta\alpha'_x+\gamma_G\delta\alpha_\tau)+\varkappa_\eta\delta\psi, \\ &\delta\varkappa_\eta=-\sin\phi(\delta\alpha'_\tau-\gamma_G\delta\alpha_x)+\cos\phi(\delta\alpha'_x+\gamma_G\delta\alpha_\tau)-\varkappa_\xi\delta\psi, \\ &\delta\hat{\gamma}=-\varkappa_G\delta\alpha_\tau+\delta\psi' \end{aligned}\right\} \cdots\cdots (27.\text{a–c})$$

By the use of the former variations, we linearize the equilibrium equations of (12. a–c) and (13), as follows :

$$\left.\begin{aligned} &\delta N=\langle F_{(xyz)}\rangle[T_G]^T\{\delta\alpha_\tau, \delta\alpha_x, 0\}+\langle x'_G, y'_G, z'_G\rangle\{\delta F_{(xyz)}\} \\ &\frac{d}{d\zeta}(\langle\delta M_{(\eta)}, -\delta M_{(\xi)}, \delta T\rangle[\hat{T}]+\langle M_{(\eta)}, -M_{(\xi)}, T\rangle[\delta\hat{\Psi}][\hat{T}])\{i_{(xyz)}\} \\ &\quad+\begin{vmatrix} i_x & i_y & i_z \\ \multicolumn{3}{c}{\langle\delta\alpha_{(\tau x\zeta)}\rangle[T_G]} \\ F_x & F_y & F_z \end{vmatrix}+\begin{vmatrix} i_x & i_y & i_z \\ x'_G & y'_G & z'_G \\ \delta F_x & \delta F_y & \delta F_z \end{vmatrix}+\langle\delta\overset{*}{m}_{(xyz)}\rangle\{i_{(xyz)}\}=\mathbf{0} \\ &\delta T=\delta M'_w+\delta T_s+(\hat{\gamma}\delta K+K\delta\hat{\gamma})_{\text{O.B.}}+\delta\overset{*}{m}_w \end{aligned}\right\} \cdots\cdots (28.\text{a–c})$$

where

$$\{\delta F_{(xyz)}(\zeta)\}=\{\delta\bar{F}_{(xyz)}\}_0-\int_0^{\zeta}\{\delta\bar{p}_{(xyz)}\}\,d\zeta\ ; \quad\cdots\cdots (29)$$

quantities in the foregoing equilibrium are assumed known; and variations of the distributed external moment components and external bi-moment, $\{\delta\overset{*}{m}_{(xyz)}\}$ and $\delta\overset{*}{m}_{W}$, are to be expanded in terms of variations of the body-force resultants, $\{\delta\bar{m}_{(\xi)(xyz)}\}$, $\cdots$, $\{\delta\bar{m}_{W(xyz)}\}$, by linearizing Eqs. (14. a, b) and (15) with the use of $[\delta\hat{T}]=[\delta\hat{\Psi}][\hat{T}]$ and $\delta\hat{\gamma}$.

By similarly linearizing (16. a–c), we obtain the associated mechanical boundary conditions as

$$\left.\begin{aligned}
&\{\delta F_{(xyz)}\}|_{\zeta=l}=\{\delta\bar{F}_{(xyz)}\}_l\\
&[\hat{T}]^T([\delta\hat{\Psi}]^T\{M_{(\eta)},\ -M_{(\xi)},\ T\}+\{\delta M_{(\eta)},\ -\delta M_{(\xi)},\ \delta T\})|_{\zeta=0,\text{or }l}=\{\delta\overset{*}{M}_x,\ \delta\overset{*}{M}_y,\ \delta\overset{*}{M}_z\}_{0,\text{or }l}\\
&\delta M_W|_{\zeta=0,\text{or }l}=\delta\overset{*}{M}_{W\,0,\text{or }l}
\end{aligned}\right\}\quad\cdots\cdots (30.\ \text{a–c})$$

in which the expansions for $\{\delta\overset{*}{M}_{(xyz)}\}$ and $\delta\overset{*}{M}_W$ are similar to the $\{\delta\overset{*}{m}_{(xyz)}\}$ and $\delta\overset{*}{m}_W$. If, in the $[\hat{M}]$ and $\overset{*}{M}_W$ defined similarly to (14. a, b) and (15), we can neglect the before-mentioned additional terms related to the cross-section's distortion, the linearizations are then developed as follows:

$$\left.\begin{aligned}
&<\delta\overset{*}{M}_{(xyz)}>\{\boldsymbol{i}_{(xyz)}\}=\begin{vmatrix}\boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z\\ \{\delta\hat{T}\}_x & \{\delta\hat{T}\}_y & \{\delta\hat{T}\}_z\\ \{\hat{M}\}_x & \{\hat{M}\}_y & \{\hat{M}\}_z\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z\\ \{\hat{T}\}_x & \{\hat{T}\}_y & \{\hat{T}\}_z\\ \{\delta\hat{M}\}_x & \{\delta\hat{M}\}_y & \{\delta\hat{M}\}_z\end{vmatrix}\\
&\delta\overset{*}{M}_W=\{\hat{T}\}_{\zeta}\cdot\{\delta\bar{M}_{(W)(xyz)}\}+\{\bar{M}_{(W)(xyz)}\}\cdot\{\delta\hat{T}\}_{\zeta}\\
&\delta\begin{bmatrix}\hat{M}_{\xi x} & \hat{M}_{\xi y} & \hat{M}_{\xi z}\\ \hat{M}_{\eta x} & \hat{M}_{\eta y} & \hat{M}_{\eta z}\\ \hat{M}_{\zeta x} & \hat{M}_{\zeta y} & \hat{M}_{\zeta z}\end{bmatrix}=\begin{bmatrix}<\delta\bar{M}_{(\xi)(xyz)}>\\ <\delta\bar{M}_{(\eta)(xyz)}>\\ <0,0,0>\end{bmatrix}
\end{aligned}\right\}\quad\cdots\cdots (31.\ \text{a–c})$$

We can represent the linearized geometrical boundary conditions as follows:

$$\left.\begin{aligned}
&\{\delta u_{(xyz)}\}|_{\zeta=0,\text{or }l}=[T_G]^T\{\delta u_{(\tau\kappa\zeta)}\}|_{\zeta=0,\text{or }l}=\{\delta\bar{u}_{(xyz)}\}_{0,\text{or }l}\\
&\left.\begin{Bmatrix}-(-\sin\phi\delta\alpha_\tau+\cos\phi\delta\alpha_x)\\ (\cos\phi\delta\alpha_\tau+\sin\phi\delta\alpha_x)\\ \delta\psi\end{Bmatrix}\right|_{\zeta=0,\text{or }l}=\begin{Bmatrix}\delta\bar{\theta}_\xi\\ \delta\bar{\theta}_\eta\\ \delta\bar{\theta}_\zeta\end{Bmatrix}_{0,\text{or }l}\\
&\delta\hat{\gamma}|_{\zeta=0,\text{or }l}=\delta\bar{\gamma}_{0,\text{or }l}
\end{aligned}\right\}\quad\cdots\cdots (32.\ \text{a–c})$$

where $\{\delta\bar{\theta}_{(\xi\eta\zeta)}\}$ are components of a prescribed infinitesimal rotation around the foregoing $\{\boldsymbol{i}_{G(\xi\eta\zeta)}\}$ -directions under the right-hand screw rule.

By a simple linearization of (19. a, b), we have the constitutive relations between variations of the stress resultants and those of the strain parameters.

4. FLEXURAL BUCKLING OF STRAIGHT BEAM SUBJECT TO FINITE TORSION AND COMPRESSION

Consider a simply-supported thin-walled beam of biaxially symmetric open cross-section: as shown in Fig. 2, at $\zeta=0$, the rotation of cross-section is restricted such that the η-direction is kept in the spatial $(y,\ z)$-plane, with any translations at the G-point being constrained; and, at $\zeta=l$, the translations into the x- and y-directions are constrained. When the beam is subject to uniform compression $\bar{P}$ and torsion $\bar{M}_z$, we examine its flexural buckling.

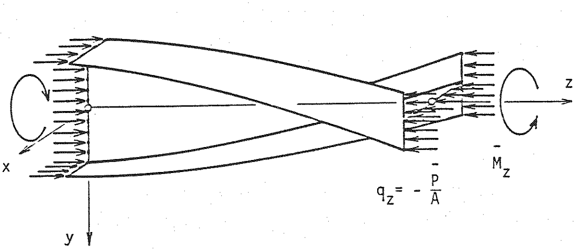

Fig. 2 Pre-Buckling Deformation.

Foregoing Equilibrium. -The beam subject to $\bar{P}$ and $\bar{M}_z$ is in such a uniform deformation that $\varepsilon_G(\zeta)=\bar{\varepsilon}$: $const.$ and $\hat{\gamma}(\zeta)=\bar{\gamma}$:

const. with other strain parameters being zero. While the bi-normal and principal-normal directions on the G-line are indefinite in the cross-section due to no curvature, we can give an additional definition for them. We proceed by setting them to coincide with the principal directions of cross-section, $\{\xi, \eta\}$:

$$\left.\begin{aligned}&\{x_G(\zeta),\ y_G(\zeta),\ z_G(\zeta),\ \phi(\zeta)\}=\{0,\ 0,\ (1+\bar{\varepsilon})\zeta,\ 0\}\\&[T_G(\zeta)]=\begin{bmatrix}\cos(\bar{\gamma}\zeta), & \sin(\bar{\gamma}\zeta) & \\ -\sin(\bar{\gamma}\zeta), & \cos(\bar{\gamma}\zeta) & \\ & & 1\end{bmatrix}\end{aligned}\right\}\cdots\cdots (33.\text{a, b})$$

We have $I_{W\xi}=I_{W\eta}=I_{G\xi}=I_{G\eta}=I_{GW}=0$ by the biaxial symmetry of cross-section. Introducing those and $\varkappa_\xi=\varkappa_\eta=0$ into (19. a, b), we have the following non-zero stress resultants :

$$N=EA\bar{\varepsilon}+EI_G\cdot\frac{1}{2}(\bar{\gamma}^2-\gamma^{02}),\quad K=EI_G\bar{\varepsilon}+EI_{GG}\cdot\frac{1}{2}(\bar{\gamma}^2-\gamma^{02}),\quad T_S=GJ_S(\bar{\gamma}-\gamma^0)\cdots\cdots (34.\text{a–c})$$

in which γ^0=natural uniform torsion. Then, by relating those to the $\bar{P}$ and $\bar{M}_z$ as $N=-\bar{P}$ and $(T=)\ T_S+K\bar{\gamma}=\bar{M}_z$, we have the equation to relate $\bar{\gamma}$ to the $\bar{P}$ and $\bar{M}_z$:

$$\frac{E}{2}\left(I_{GG}-\frac{I_G^2}{A}\right)\bar{\gamma}^3+\left\{GJ_S-\frac{I_G}{A}\bar{P}-\frac{E}{2}\left(I_{GG}-\frac{I_G^2}{2}\right)\gamma^{02}\right\}\bar{\gamma}-(GJ_S\gamma^0+\bar{M}_z)=0\cdots\cdots (35)$$

In the below, while taking the geometrical effects due to the large torsion into the equilibrium equations, however, we neglect those by elongation $\bar{\varepsilon}$ under the condition of small strains.

Buckling Equations. -By the use of the former $\{\tau, \varkappa\}$-directions and the $\bar{\gamma}$, the variational quantities defined geometrically in Sec. 3 are, in the present case, written as follows :

$$\{\delta\alpha_{(\tau\varkappa\xi)}\}=\{\delta u_\tau'-\bar{\gamma}\delta u_\varkappa,\ \delta u_\varkappa'+\bar{\gamma}\delta u_\tau,\ \delta u_\xi'\}\cdots\cdots (36)$$

$$\{\delta\varepsilon_G,\ \delta\varkappa_\xi,\ \delta\varkappa_\eta,\ \delta\bar{\gamma}\}=\{\delta u_\xi',\ \delta u_\tau''-2\bar{\gamma}\delta u_\varkappa'-\bar{\gamma}^2\delta u_\tau,\ \delta u_\varkappa''+2\bar{\gamma}\delta u_\tau'-\bar{\gamma}^2\delta u_\varkappa,\ \delta\psi'\}\cdots\cdots (37)$$

Since $\{F_{(xyz)}\}=\{0,\ 0,\ -\bar{P}\}$, and the stress resultants other than N, K and T_S are zero, with no distributed external forces, we can reduce the linearized equilibrium equations, (28. a–c) and (29), as follows :

$$\left.\begin{aligned}&\delta N=\delta F_z\\&\frac{d}{d\zeta}\left(\begin{bmatrix}\cos(\bar{\gamma}\zeta), & -\sin(\bar{\gamma}\zeta), & \\ \sin(\bar{\gamma}\zeta), & \cos(\bar{\gamma}\zeta), & \\ & & 1\end{bmatrix}\left(\begin{Bmatrix}\delta M_{(\eta)}\\-\delta M_{(\xi)}\\\delta T\end{Bmatrix}+\bar{M}_z\begin{Bmatrix}\delta u_\tau'-\bar{\gamma}\delta u_\varkappa\\\delta u_\varkappa'+\bar{\gamma}\delta u_\tau\\0\end{Bmatrix}\right)\right)\\&\quad-\bar{P}\begin{bmatrix}\cos(\bar{\gamma}\zeta), & -\sin(\bar{\gamma}\zeta), & \\ \sin(\bar{\gamma}\zeta), & \cos(\bar{\gamma}\zeta), & \\ & & 1\end{bmatrix}\begin{Bmatrix}\delta u_\varkappa'+\bar{\gamma}\delta u_\tau\\-\delta u_\tau'+\bar{\gamma}\delta u_\varkappa\\0\end{Bmatrix}+\begin{Bmatrix}-\delta F_y\\\delta F_x\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}\\&\delta T=\delta M_w'+\delta T_s+(\bar{\gamma}\delta K+K\delta\Psi')\end{aligned}\right\}\cdots\cdots (38.\text{a–c})$$

in which $\{\delta F_{(xyz)}(\zeta)\}=\{\delta\bar{F}_{(xyz)}\}_0$: *const*. Here, we assume that the $\bar{P}$ and $\bar{M}_z$ are conservative such that their acting directions are spatially unchanged during the variations : chord compression on z-axis and moment around z-direction, respectively. In this case, we can write down the boundary conditions as

$$\left.\begin{aligned}&\delta u_\tau=0,\quad \delta u_\varkappa=0,\quad \delta u_\xi=0\\&\begin{Bmatrix}\delta M_{(\eta)}\\-\delta M_{(\xi)}\end{Bmatrix}+\bar{M}_z\begin{Bmatrix}\delta u_\tau'-\bar{\gamma}\delta u_\varkappa\\\delta u_\varkappa'+\bar{\gamma}\delta u_\tau\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}\\&\delta\psi=0,\quad \text{at } \zeta=0\end{aligned}\right.\cdots\cdots (39.\text{a–e})$$

$$\left.\begin{aligned}&\delta u_\tau=0,\quad \delta u_\varkappa=0,\quad \delta F_z=0\\&\begin{Bmatrix}\delta M_{(\eta)}\\-\delta M_{(\xi)}\\\delta T\end{Bmatrix}+\bar{M}_z\begin{Bmatrix}\delta u_\tau'-\bar{\gamma}\delta u_\varkappa\\\delta u_\varkappa'+\bar{\gamma}\delta u_\tau\\0\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}\\&\text{at } \zeta=l\end{aligned}\right.\cdots\cdots (40.\text{a–d})$$

By introducing (40. c) into (38. a), we have $\delta N=0$. After integrating (38. b) where $\{\delta F_x,\ \delta F_y\}=$ *const*., by the use of boundary conditions (39. d) and (40. d), we have $\{\delta F_x,\ \delta F_y\}=\{0,\ 0\}$ and the integration constants determined. Into the resulting differential equations and the remaining boundary conditions, we introduce the associated constitutive relations, and we can see that displacements $\{\delta u_\tau,\ \delta u_\varkappa\}$ are independent of $\{\delta u_\xi,\ \delta\psi\}$. Focusing only on the former displacements, we have the following eigen-value problem to determine the buckling :

$$\begin{bmatrix} EI_{\xi\xi}D^2+(\bar{P}+\bar{\gamma}\bar{M}_z-\bar{\gamma}^2EI_{\xi\xi}), & (\bar{M}_z-2\bar{\gamma}EI_{\xi\xi})D \\ (\bar{M}_z-2\bar{\gamma}EI_{\eta\eta})D, & -EI_{\eta\eta}D^2-(\bar{P}+\bar{\gamma}\bar{M}_z-\bar{\gamma}^2EI_{\eta\eta}) \end{bmatrix} \begin{Bmatrix} \delta u_\tau \\ \delta u_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \cdots (41)$$

subject to

$$\{\delta u_\tau,\ \delta u_x\}=\{0,\ 0\} \quad \text{at } \zeta=0 \text{ and } l \quad \cdots (42)$$

where D=differential operator $d/d\zeta$.

Here we consider the lateral buckling caused only by pure torsion $\bar{M}_z$. When $\bar{P}=0$, the characteristic equation of (41) is expanded as

$$(D^2+\bar{\gamma}^2)\left\{D^2+\left(\bar{\gamma}-\frac{\bar{M}_z}{EI_{\xi\xi}}\right)\left(\bar{\gamma}-\frac{\bar{M}_z}{EI_{\eta\eta}}\right)\right\}=0 \quad \cdots (43)$$

i) In case of $\bar{M}_z/EI_{\eta\eta}<\bar{M}_z/EI_{\xi\xi}<\bar{\gamma}$, the roots of (43) are written as $\pm D_1=i\bar{\gamma}$ and $\pm D_2=i\alpha$, where i denotes the imaginary unit; and

$$\alpha=\sqrt{\left|\left(\bar{\gamma}-\frac{\bar{M}_z}{EI_{\xi\xi}}\right)\left(\bar{\gamma}-\frac{\bar{M}_z}{EI_{\eta\eta}}\right)\right|} \quad \cdots (44)$$

Then, the general solution in the real number space is given by

$$\begin{Bmatrix} \delta u_\tau \\ \delta u_x \end{Bmatrix}=\delta A_1\begin{Bmatrix} \cos(\bar{\gamma}\zeta) \\ -\sin(\bar{\gamma}\zeta) \end{Bmatrix}+\delta A_2\begin{Bmatrix} \sin(\bar{\gamma}\zeta) \\ \cos(\bar{\gamma}\zeta) \end{Bmatrix}+\delta A_3\begin{Bmatrix} \nu_\eta\cos(\alpha\zeta) \\ -\nu_\xi\sin(\alpha\zeta) \end{Bmatrix}+\delta A_4\begin{Bmatrix} \nu_\eta\sin(\alpha\zeta) \\ \nu_\xi\cos(\alpha\zeta) \end{Bmatrix} \quad \cdots (45)$$

in which δA_1-δA_4 are arbitrary constants ; and

$$\nu_\xi=\sqrt{\left|\bar{\gamma}-\frac{\bar{M}_z}{EI_{\xi\xi}}\right|}\Big/\left(2\bar{\gamma}-\frac{\bar{M}_z}{EI_{\xi\xi}}\right), \qquad \nu_\eta=\sqrt{\left|\bar{\gamma}-\frac{\bar{M}_z}{EI_{\eta\eta}}\right|}\Big/\left(2\bar{\gamma}-\frac{\bar{M}_z}{EI_{\eta\eta}}\right) \quad \cdots (46.\text{a, b})$$

By boundary conditions $\{\delta u_\tau,\ \delta u_x\}=\{0,\ 0\}$ at $\zeta=0$, we have $\delta A_1=-\nu_\eta\delta A_3$ and $\delta A_2=-\nu_\xi\delta A_4$. And, to exist a non-trivial solution satisfying the other $\{\delta u_\tau,\ \delta u_x\}=\{0,\ 0\}$ at $\zeta=l$, we have the following buckling equation :

$$2\nu_\xi\nu_\eta\{1-\cos(\bar{\gamma}l)\cos(\alpha l)\}=(\nu_\xi^2+\nu_\eta^2)\sin(\bar{\gamma}l)\sin(\alpha l) \quad \cdots (47)$$

By solving this equation and (35) together with (44) and (46. a, b), numerically, we obtain the critical $\bar{M}_z$ and $\bar{\gamma}$.

ii) In case of $\bar{M}_z/EI_{\eta\eta}<\bar{\gamma}<\bar{M}_z/EI_{\xi\xi}$, by the use of the former α, the roots of the characteristic equation are written as $\pm D_1=i\bar{\gamma}$ and $\pm D_2=\alpha$. Then, by borrowing constants ν_ξ and ν_η of (46. a, b), we have the following general solution :

$$\begin{Bmatrix} \delta u_\tau \\ \delta u_x \end{Bmatrix}=\delta A_1\begin{Bmatrix} \cos(\bar{\gamma}\zeta) \\ -\sin(\bar{\gamma}\zeta) \end{Bmatrix}+\delta A_2\begin{Bmatrix} \sin(\bar{\gamma}\zeta) \\ \cos(\bar{\gamma}\zeta) \end{Bmatrix}+\delta A_3\begin{Bmatrix} \nu_\eta \\ \nu_\xi \end{Bmatrix}\exp(\alpha\zeta)+\delta A_4\begin{Bmatrix} \nu_\eta \\ -\nu_\xi \end{Bmatrix}\exp(-\alpha\zeta) \quad \cdots (48)$$

By boundary conditions $\{\delta u_\tau,\ \delta u_x\}=\{0,\ 0\}$ at $\zeta=0$, then $\delta A_1=-\nu_\eta(\delta A_3+\delta A_4)$ and $\delta A_2=-\nu_\xi(\delta A_3-\delta A_4)$, and by the same constraints at $\zeta=l$, we have the following buckling equation :

$$-4\nu_\xi\nu_\eta+\{2\nu_\xi\nu_\eta\cos(\bar{\gamma}l)-(\nu_\xi^2-\nu_\eta^2)\sin(\bar{\gamma}l)\}\exp(\alpha l)$$
$$+\{2\nu_\xi\nu_\eta\cos(\bar{\gamma}l)+(\nu_\xi^2-\nu_\eta^2)\sin(\bar{\gamma}l)\}\exp(-\alpha l)=0 \quad \cdots (49)$$

5. CONCLUDING REMARKS

When formulating, for instance, the small displacements of a curved two-dimensional beam, it is conventional and helpful for our understanding to have its displacements decomposed into tangent and normal directions on the axial line, depending upon the initial configuration. If we call the resulting governing equations as formulated in a normalized form, we can suppose that for any linear elasticity problem there exists its normalized formulation. Even for finite displacements, when only their lower nonlinear terms are considered such as up to the quadratic or cubic terms upon the initial configurations, we can find many existing governing equations developed on the same basis. While, in the study[1)] which presented us the nonlinear theory of thin-walled beams in Sec. 2, it is asserted that when dealing with truly large displacemtnts, we can not expect that normalization; for, the displacements change the geometry, entirely. Thus, the unkowns $\{x_G(\zeta), y_G(\zeta), z_G(\zeta), \phi(\zeta)\}$ are there employed as a set of parametric functions to describe directly the spatial configurations, instead of the displacements, with reference to the basic Cartesian coordinates.

As one method, we can derive tangent relations by literally, or mathematically, linearizing the governing equations of Sec. 2 regarding the fundamental unkowns $\{x_G, y_G, z_G, \phi\}$. But, with the following intention, another set of independent variationals, $\{\delta u_{(\tau\chi\zeta)}, \delta\psi\}$, are here employed upon the preceding equilibrium : It is, even after large displacements, still infinitesimal increments that we are here dealing with. As a converse acceptance of the former assertion for large displacements, we can expect a normalized formulation for our variations on the preceding state of equilibrium, under the same kinematic field. The expansions made on those $\{\delta u_{(\tau\chi\zeta)}, \delta\psi\}$ in Sec. 3 are far favorable for understanding in comparison with the direct linearization regarding the $\{x_G, y_G, z_G, \phi\}$.

While, in Sec. 4, the flexural buckling of a straight beam subject to finite torsion and compression is analyzed as an unprecedented example, the present tangent equations can deal with any bucklings of thin-walled beams arising after large displacements : e. g., by reducing the general preceding deformation to the finite pure bending of a circular beam of uniaxially symmetric cross-section, we derive the differential equations for its lateral buckling, and by solving them for the case of biaxially symmetric cross-section, we can obtain the same result by Vacharajittiphan and Trahair[11].

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