SHAKEDOWN ANALYSES OF TWO-SPAN BEAMS AND SYMMETRIC THREE-SPAN BEAMS WITH CONSIDERATION OF BENDING AND SHEAR

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This paper presents an easily comprehensive method of shakedown analysis of two-span beams and symmetric three-span beams with a moving load when taking account of bending and shear. The original problem to be dealt with is a nonlinear mathematical programming due to a nonlinear interaction of bending moment and shearing force on the ultimate strength of cross section. Adoption of Mises yield condition reduces this problem to a quadratic programming. Well-known conception of quadratic inequality is introduced to find the analytical solution of the problem. Numerical results for the beams with various span-ratios, loading conditions and strengths of cross sections clarify the shakedown characteristics, particularly the influence of shearing force on shakedown loads.

Keywords : *shakedown analysis, bending-shear interaction, moving load*

1. INTRODUCTION

Problems of determining the ultimate loading capacity of girder bridges have become increasingly important as the limit state design method is adopted for bridge structures. The perfectly plastic theory may be accepted in practice for steel girder bridges with compact section. However most bridge design codes restrict the method of structural analysis within elastic theory. One of the reasons of this restriction may be attributed to the uncertainty of limit behaviour under complex loading conditions such as combinations of traffic loads, wind loads and own weight loads.

Significance of shakedown problems has been well known for bridge structures subjected to moving loads[1]. However, classical shakedown problems have almost been confined within bending theory. In consequence, it has been said that the shakedown load of statically indeterminate girder bridge is only slightly lower than the static collapse load under the extreme loading condition and the shakedown problem is not so severe for bridge structures. But actual girder bridges are subjected to various kinds of forces such as bending moment, shearing force, axial force and twisting moment. Even at present, shakedown problems considering both the effects of bending and shear remain unclear. In particular, the effect of shear must be severer for reinforced concrete bridges.

This paper is intended for revealing the shakedown characteristics of multi-span girder bridges. It is well known that shakedown problems within the scope of bending theory can be solved by a linear programming[2] where a shakedown load multiplier is objective function, residual moments or reactive

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forces are state variables and plasticity condition gives inequality constraints in a functional type of state variables. On the other hand, consideration of both bending and shear reduces the inequality constraints to nonlinearity, and the problems of determining a shakedown load are reduced to a nonlinear mathematical programming. Literature on the studies of shakedown problem by a nonlinear programming are seen for elastoplastic arches[3], grids[4] and plates[5],[6]. In these studies, nonlinearity due to cross sectional yield conditions were dealt with graphically for arches and piecewise linearly for grids and plates. For structures with moving loads, the shakedown problem becomes further complex, because resultant forces at the cross section are given by a nonlinear function of the position of load. Generally speaking, to solve analytically a nonlinear programming of such a type of structure is impossible. But in the case of two-span beams or symmetric three-span beams, the unknown variables are only two of shakedown load multiplier and one residual reactive force. In this case the adoption of Mises yield condition reduces the problems to a quadratic programming to be able to solve analytically.

2. SHAKEDOWN ANALYSIS

(1) General formulation for multi-span beams

A m-span continuous beam shown in **Fig. 1** is considered here. Since the beam has $m-1$ degrees of statical indeterminancy, elastic bending moment M_{ex} and elastic shearing force S_{ex} at any point of the beam can be expressed with unknown reactive forces X_k ($k=1, 2, 3 \cdots, m-1$) at intermediate supports as follows :

$$\left. \begin{aligned} M_{ex} &= M_{ox} + Px(1-\xi) - H_\xi P(x-\xi L) + \sum_{k=1}^{m-1} X_k \overline{M}_{kx}, \qquad \overline{M}_{kx} = x\left(1-\frac{l_k}{L}\right) - H_k(x-l_k) \\ S_{ex} &= S_{ox} + P(1-\xi) - H_\xi P + \sum_{k=1}^{m-1} X_k \overline{S}_{kx}, \qquad \overline{S}_{kx} = 1 - \frac{l_k}{L} - H_k \end{aligned} \right\} \quad \cdots\cdots\cdots\cdots (1)$$

where M_{ox} and S_{ox} are bending moment and shearing force due to a fixed load q, which are easily obtained by static conditions as a simple beam with span L, and H_ξ and H_k are Heaviside's step functions defined as follows :

$$\left. \begin{aligned} H_\xi &= \begin{cases} 0, & 0 \le x < \xi L \\ 1, & \xi L \le x \le L \end{cases} \\ H_k &= \begin{cases} 0, & 0 \le x < l_k \\ 1, & l_k \le x < L \end{cases} \end{aligned} \right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Fig. 1 m-span beam.

The unknown reactive forces, X_k, can be determined by the principle of least work well known in the elementary structural mechanics.

On the other hand, residual moments M_{rx} and residual shearing forces S_{rx} at any point of the beam, which are induced by partial plasticization of the beam, can be expressed with residual reactive forces X_{rk} $=1, 2, 3, \cdots, m-1$) at intermediate supports as follows :

$$M_{rx} = \sum_{k=1}^{m-1} X_{rk} \overline{M}_{kx}, \qquad S_{rx} = \sum_{k=1}^{m-1} X_{rk} \overline{S}_{kx} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

Let the beam considered be made of perfectly plastic material. Interaction on plastic strength of the beam section under the actions of both bending moment M_x and shearing force S_x generally depends on the shape of cross section, but a lower bound interaction curve with a good approximation irrespective of any shape of cross section can be given by the following simple expression[7] :

$$\left(\frac{M_x}{M_0}\right)^2 + \left(\frac{S_x}{S_0}\right)^2 = 1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

where M_0 and S_0 mean the fully plastic moment and the fully plastic shearing force, respectively.

Now let the beam be subjected to any combination of a cyclically reapeated moving load P and a fixed load q, which may be a typical problem in girder bridges. Melan's theory says that the problem of determining
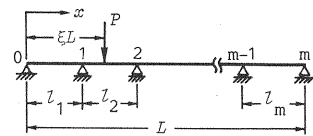
the shakedown moving load P_s is reduced to the following mathematical programming :

$$P_s = \text{maximize } P$$
$$\text{subjected to} \left(\frac{M_{ex} + M_{rx}}{M_0}\right)^2 + \left(\frac{S_{ex} + S_{rx}}{S_0}\right)^2 \leq 1 \quad \Big\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

The above problem is a nonlinear programming with the variables, x, ξ, P and X_{rk}. If we transform this problem into a convex programming, which may be solved numerically, the variables, x and ξ must be replaced by a large number of known constants x_i and $\xi_i (i=1, 2, \cdots\cdots, n)$ which are specified by the divided points with a small spacing L/n for the whole length of beam. In consequence, Prob. (5) becomes a convex programming with quadratic inequality constraints of n^2 in number. Furthermore, if we desire to make a linearization of quadratic inequalities replacing the circular interaction curve by an inscribed polygon with a large number of sides as shown in **Fig. 2** (such a technique of linearization is used in the literature[4)~6)]), we obtain a linear programming with linear constraints of sn^2 in number where s is the number of sides of the inscribed polygon. However, in order to obtain the solution of Prob. (5) with a good accuracy, we need to increase both n and s to a large number and consequently must deal with a linear programming of large size.

Therefore, this study attempts to devise a semi-analytical method for obtaining the direct solution of nonlinear problem (5). For the sake of this, problems dealt with here are confined with only two-span beams and symmetric three-span beams, but they are expected to have an essential characteristics of shakedown of multi-span beams.

(2) Two-span beams

A two-span beam to be dealt with herein is shown in **Fig. 3**. Denoting the elastic bending moments and shearing forces at any cross section i under the unit moving load $P=1$ at the position $x=\xi L$ (L=total span length) and the unit uniform load $q=1$ by $\overline{M}_{pi}(\xi)$, $\overline{S}_{pi}(\xi)$, $\overline{M}_{qi}$, $\overline{S}_{qi}$, respectively, the domain of elastic stress variation can be expressed as follows :

$$R_i = \{M_i, S_i \mid M_i = P\overline{M}_{pi}(\xi) + q\overline{M}_{qi}, \ S_i = P\overline{S}_{pi}(\xi) + q\overline{S}_{qi}\} \cdots\cdots\cdots\cdots\cdots\cdots (6)$$

where $0 \leq \xi \leq 1$.

Shakedown of a statically indeterminate beam means the phenomenon that after a finite number of repetition of moving load, the beam behaves elastically and the residual forces remain constant. Since a two-span beam has one degree of statical indeterminancy, the residual moments M_{ri} and the residual shearing force S_{ri} can be expressed by the residual reactive force X at the intermediate support. Namely,

$$M_{ri} = \overline{M}_{ri} X, \qquad S_{ri} = \overline{S}_{ri} X \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

where $\overline{M}_{ri}$ and $\overline{S}_{ri}$ means the bending moment and shear force at cross section i due to unit residual reactive force.

Introducing the nondimensional quantities ; $\lambda = Pl_1/M_0$, $\alpha = q l_1^2/M_0$, $\mu = Xl_1/M_0$, $\overline{m}_{pi} = \overline{M}_{pi}(\xi)/l_1$, $\overline{s}_{pi} = \beta\overline{S}_{pi}(\xi)$, $\overline{m}_{qi} = \overline{M}_{qi}/l_1^2$, $\overline{s}_{qi} = \beta\overline{S}_{qi}/l_1$, $\overline{m}_{ri} = \overline{M}_{ri}/l_1$, $\overline{s}_{ri} = \beta\overline{S}_{ri}$ and $\beta = M_0/(S_0 l_1)$, Prob. (5) for determining the shakedown load multiplier λ_s of moving load is reduced to the following mathematical
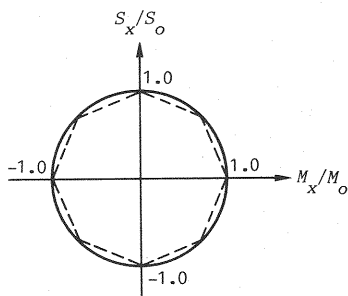

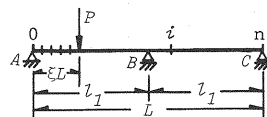Fig. 2   Lower bound interaction curve and piecewise linearization.

Fig. 3   Two-span beam.

programming :

$$\lambda_s = \text{maximize } \lambda, \text{ subject to } (\lambda\overline{m}_{pi} + \alpha\overline{m}_{qi} + \mu\overline{m}_{ri})^2 + (\lambda\overline{s}_{pi} + \alpha\overline{s}_{qi} + \mu\overline{s}_{ri})^2 \leqq 1 \quad\cdots\cdots\cdots\cdots\cdots (8)$$

Since $\overline{m}_{pi}$ and $\overline{s}_{pi}$ are a function of both the position of moving load and the space, it is considerably difficult to find the exact solution of Prob. (8). For this reason, we here try to find an analytical but approximate solution of Prob. (8) with good accuracy. Generally, the existent domain of $(\overline{m}_{pi}, \overline{s}_{pi})$ has a nonconvexity surrounded by the area with soled lines as shown in Fig. 4. However, a domain to be dealt with in Prob. (8) can be substituted by a convex circumscribed polyhedron about the original domain, because Prob. (8) is a kind of convex programming. Then, if such a polyhedron has some vertexes $(\overline{m}_{ij}, \overline{s}_{ij})$, $j=1, 2, \cdots, r_i$, the problem of Prob. (8) can be transformed into

$$\lambda_s = \underset{i,j}{\text{maximize}} \ \lambda, \text{ subject to } (\lambda\overline{m}_{ij} + \alpha\overline{m}_{qi} + \mu\overline{m}_{ri})^2 + (\lambda\overline{s}_{ij} + \alpha\overline{s}_{qi} + \mu\overline{s}_{ri})^2 \leqq 1 \cdots\cdots\cdots\cdots\cdots (9)$$

Further, dividing the whole length of the beam by n points with a small equal spacing and imposing the inequality constraints on only such points, Prob. (9) is reduced to a typical quadratic programming with two unknown variables of λ and μ and $\sum r_i$ inequality constraints.

Now we develop the inequality constraints in Prob. (9) into a quadratic form about μ as follows :

$$(\overline{m}_{ri}^2 + \overline{s}_{ri}^2)\mu^2 + 2[(\overline{m}_{ij}\overline{m}_{ri} + \overline{s}_{ij}\overline{s}_{ri})\lambda + (\overline{m}_{qi}\overline{m}_{ri} + \overline{s}_{qi}\overline{s}_{ri})\alpha]\mu$$
$$+ [(\lambda\overline{m}_{ij} + \alpha\overline{m}_{qi})^2 + (\lambda\overline{s}_{ij} + \alpha\overline{s}_{qi})^2 - 1] \leqq 0 \quad\cdots\cdots\cdots\cdots\cdots (10)$$

Considering the characteristics of quadratic inequality about μ, we obtain a necessary condition on λ to have a real solution of μ as

$$\lambda \leqq \frac{\sqrt{\overline{m}_{ri}^2 + \overline{s}_{ri}^2}}{|\overline{m}_{ij}\overline{s}_{ri} - \overline{s}_{ij}\overline{m}_{ri}|} + \frac{\overline{s}_{qi}\overline{m}_{ri} - \overline{m}_{qi}\overline{s}_{ri}}{\overline{m}_{ij}\overline{s}_{ri} - \overline{s}_{ij}\overline{m}_{ri}}\alpha \quad\cdots\cdots\cdots\cdots\cdots\cdots (11)$$

Besides, we must limit λ to be a positive in view of actual situation. While the existent region of μ in Eq. (10) becomes as follows :

max. $\gamma_{ij}^- \leqq \mu \leqq$ min. γ_{ij}^+

$$\begin{matrix} \gamma_{ij}^+ \\ \gamma_{ij}^- \end{matrix} = \frac{1}{\overline{m}_{ri}^2 + \overline{s}_{ri}^2}\left\{ -[(\overline{m}_{ij}\overline{m}_{ri} + \overline{s}_{ij}\overline{s}_{ri})\lambda + (\overline{m}_{qi}\overline{m}_{ri} + \overline{s}_{qi}\overline{s}_{ri})\alpha] \right.$$
$$\left. \pm \sqrt{(\overline{m}_{ri}^2 + \overline{s}_{ri}^2) - [(\overline{m}_{ij}\overline{s}_{ri} - \overline{s}_{ij}\overline{m}_{ri})\lambda + (\overline{m}_{qi}\overline{s}_{ri} - \overline{m}_{ri}\overline{s}_{qi})\alpha]^2} \right\} \quad\cdots\cdots\cdots (12)$$

As mentioned before, a constant value of residual reactive force exists under a shakedown behaviour. Therefore, at the shakedown load, the extreme value of μ must exist as follows :

$$\mu_s = \underset{i,j}{\text{maximize}} \ \gamma_{ij}^- = \underset{i,j}{\text{minimize}} \ \gamma_{ij}^+ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (13)$$

On the other hand, the condition (11) to be satisfied at the whole position of the beam yields the following expression :

$$0 \leqq \lambda_s \leqq \lambda_0, \quad \lambda_0 = \underset{i,j}{\text{minimize}} \left\{ \frac{\sqrt{\overline{m}_{ri}^2 + \overline{s}_{ri}^2}}{|\overline{m}_{ij}\overline{s}_{ri} - \overline{s}_{ij}\overline{m}_{ri}|} + \frac{\overline{s}_{qi}\overline{m}_{ri} - \overline{m}_{qi}\overline{s}_{ri}}{\overline{m}_{ij}\overline{s}_{ri} - \overline{s}_{ij}\overline{m}_{ri}}\alpha \right\} \quad\cdots\cdots\cdots\cdots (14)$$

Simultaneous satisfaction of both Eqs. (13) and (14) induces the shakedown load multiplier λ_s. An iterative method to find λ_s is graphically represented in Fig. 5, in which λ_e is the elastic limit load multiplier.
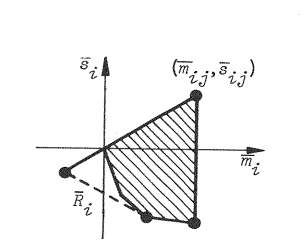

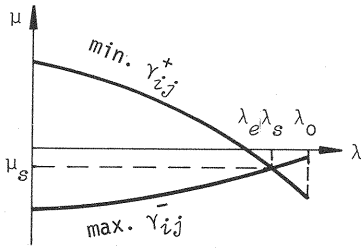Fig. 4   Vertexes and convex region.   Fig. 5   Illustrated iterative procedure.   Fig. 6   Three-span beam.
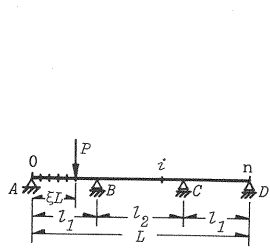
（2） Three-span beam

A three-span beam to be analyzed herein is shown in **Fig. 6**. The beam has a symmetric configuration and symmetric strength of cross section. The domain R_i of elastic stress variation expressed by Eq. (6) is obtained through a familiar elastic analysis. The beam is statically indeterminate of two degrees, but for symmetry of the beam, residual reactive forces at two intermediate supports are equal to each other when the beam shakes down. Therefore, a shakedown problem of the beam has only two unknown variables of λ and μ_1 ($\mu_2 = \mu_1$), where μ_1 and μ_2 are nondimensional residual reactive forces at the intermediate supports. This problem can be easily solved through a similar procedure to Eqs. (6)–(14).

3. NUMERICAL RESULTS

（1） Two-span beams with equal span length

Fig. 7 demonstrates the variations of λ_s, λ_e, and μ_s about a nondimensional parameter $\beta = M_0/(S_0 l_1)$. The results in the figure are for the beams subjected to only a moving load. Elastic limit load multiplier λ_e and static collapse load multiplier being 5.83 which was determined by the method of limit analysis are also included for comparison. From this figure, it may be found that in the range of β greater than 0.25, λ_s and λ_e take almost identical values, while in the range of β less than 0.03, the effect of shear disappears and then λ_s almost coincides with the shakedown load multiplier considering bending only, which takes 5.72. Therefore, we can conclude for the beams that the shakedown problems considering both the effects of bending and shear are significant only in the range of $0.03 < \beta < 0.25$.
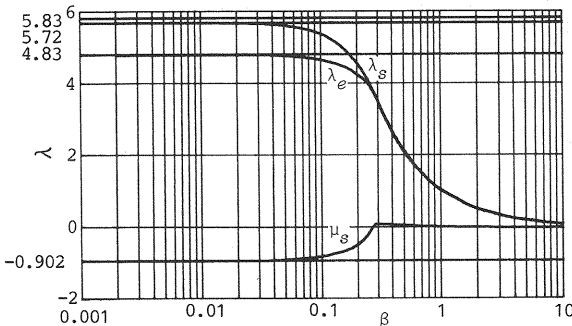

Fig. 7   Bending-shear interacted λ_s, λ_e, μ_s vs. β and corresponding
bending λ_s, λ_e, μ_s for two-equal-span beams together with
static collapse load multiplier.

Fig. 9 P_s vs. l_1 for two-equal-span beams of H-shaped
steels listed in **Table 1**.

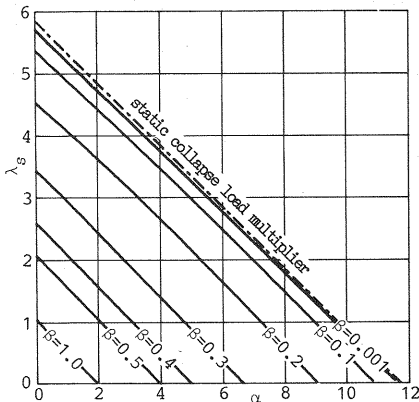Fig. 8 λ_s vs. α for two-equal-span beams with different β
and the variation of static collapse load multiplier.
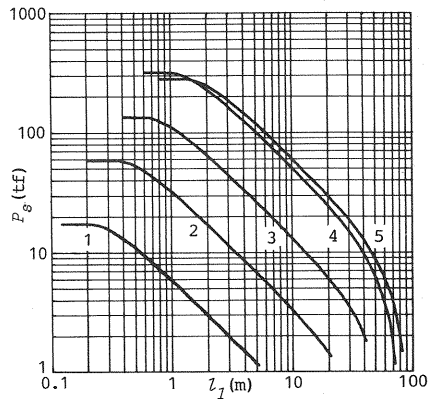
Fig. 10 P_s/P'_s vs. l_1 for two-equal-span beams of H-shaped
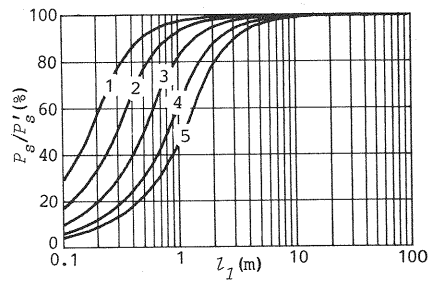steels listed in **Table 1**.

Table 1   Some typical H-shaped steels[9].

No.	H (mm)	B (mm)	t_w (mm)	t_f (mm)	A (cm^2)	W (kg/m)	σ_y (kg/cm^2)	M_0 (10^6kg cm)	S_0 (10^6kg)	M_0/S_0 (cm)
1	100	50	5	7	11.85	9.3	2400	0.100308	0.0164198	6.11
2	150	150	7	10	40.14	31.5	2400	0.574980	0.0556196	10.34
3	250	250	9	14	92.18	72.4	2400	2.248534	0.1275898	17.62
4	400	400	13	21	218.7	172	2400	8.640319	0.3030396	28.51
5	588	300	12	20	192.5	151	2400	10.341389	0.2667358	38.77

In the case of beams subjected to both a moving load and a uniform dead load, the changes of λ_s with combinations of β and α ($=q\,l_1^2/M_0$) are graphed as Fig. 8. As α becomes larger, the uniform dead load increasingly plays an important role for the shakedown problems. As mentioned before, the effect of shear disappears when β is less then 0.03. In this range of β, though in Fig. 8 only the case of $\beta=0.001$ is shown, the shakedown load becomes closer and closer to the static collapse load as α increases. While, in the range of β greater than 0.03 a similar tendency is found in the relation between λ_s and α.

In order to explain an application on actual girder bridges, five typical kinds of H-shaped steels[9] listed in Table 1 are taken into computation. Fig. 9 exhibits the variations of shakedown load P_s with respect to span length for each H-shaped steel with consideration of dead load. It may be realized that the magnitudes of shakedown loads of H-shaped steel girders depend mainly upon their ratios of fully plastic moment to fully plastic shearing force of cross sections. In Fig. 10, we also give the relationship between span length and the ratio of bending-shear interacted shakedown load, P_s, to bending shakedown load, P'_s. From this figure, we learn how much effect of shear will bring on in the shakedown problems of girder bridges.

(2)   Three-span beams

First, we performed computations on three-span beams without uniform dead load in four cases of l_2/l_1 =1.0, 1.2, 1.5 and 1.8, using an analogous procedure to two-span beams. The results obtained are shown in Fig. 11. This figure corresponds to Fig. 7 for two-span beams, but the effect of the ratio of mid span length to side span length is included in the figure. From this figure we know that at the same value of β the beam with a smaller ratio of l_2/l_1 has larger λ_s but such difference of λ_s about l_2/l_1 disappears as β increases.

On the other hand, we also applied the five kinds of H-shaped steels as listed in Table 1 on three-span beams with equal span length. Fig. 12 and 13 show the results obtained. From the comparison of Fig. 9 and
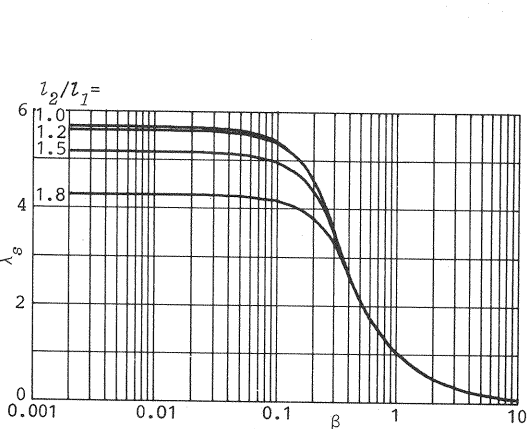

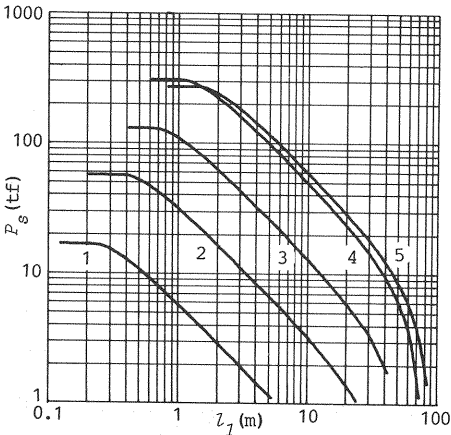Fig. 11 λ_s vs. β for symmetric three-span beams with different span ratio.

Fig. 12 P_s vs. l_1 for three-equal-span beams of H-shaped steels listed in Table 1.

Fig. 12, we can conclude that two-span beam and three-span beam have almost the same magnitude of shakedown load if they have the same span length. Further Fig. 14 shows the variation of P_s/P_s' for the three-span beams with unequal span ratios of $l_2/l_1 = 1.0$, 1.2, 1.5 and 1.8. Appreciable differences in the shear effect are seen among the different span ratios.

In actual three-span girder bridges, it may be often seen that side span and mid span have different plastic strengths of cross section. Here, we carried out computations for the beams with the span ratio of 1.5 and $v = 0.5$, 0.7 and 1.0 where v means the ratio of fully plastic moment of side span to that of mid span. Such a ratio on fully plastic shearing force is assumed to be equal to the ratio on fully plastic moment for the convenience of numerical computation. Fig. 15 show the results obtained, where P_s/P_u means the
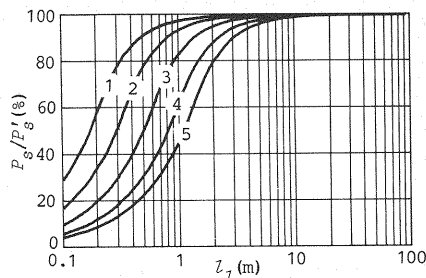

Fig. 13 P_s/P_s' vs. l_1 for three-equal-span beams of H-shaped steels listed in Table 1.
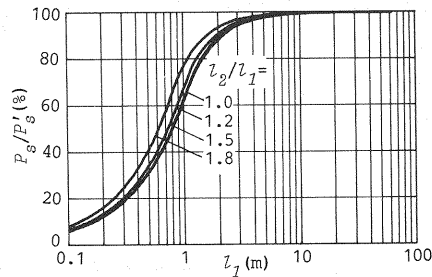

Fig. 14 P_s/P_s' vs. l_1 for symmetric three-span beams of H-shaped steel 3 from Table 1, with different span ratio.
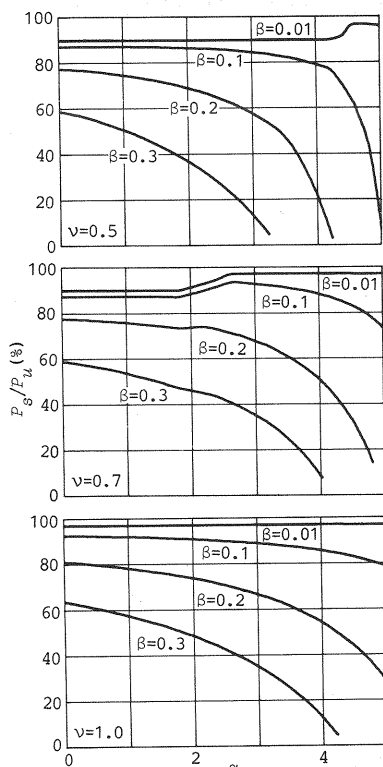

Fig. 15 P_s/P_u vs. α of different β for symmetric three-span beams with span ratio 1.5 and $v = 0.5$, 0.7 and 1.0, respectively.
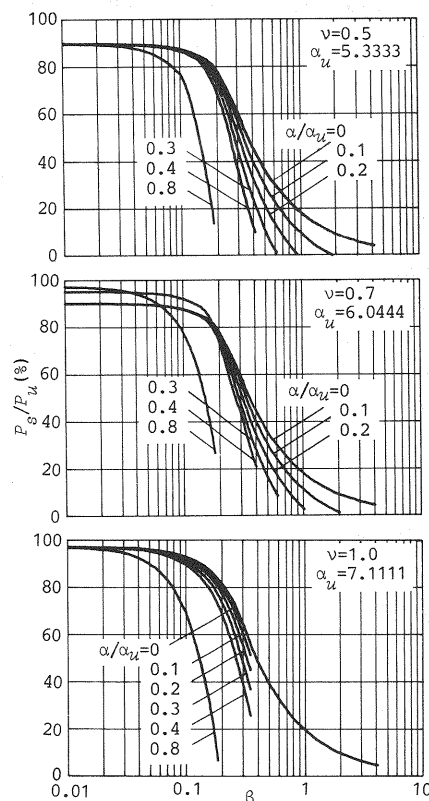

Fig. 16 P_s/P_u vs. β of different α for symmetric three-span beams with span ratio 1.5 and $v = 0.5$, 0.7 and 1.0, respectively.

Table 2   Comparative results between Reference[1] and present method for three-span beams with equal span length considering bending only.

$\alpha = q l_1^2 / M_0$	0.0	1.16569	2.33137	3.49706	4.66274	5.82843	6.99411	8.15980	9.32548	10.49117
ULT.LOAD =5.82843 −0.5α	5.82843	5.24558	4.66274	4.07990	3.49706	2.91421	2.33137	1.74853	1.16569	0.58284
*S.D.LOAD(1)	5.70747	5.13785	4.56797	3.99784	3.42746	2.85683	2.28596	1.71483	1.14346	0.57185
*S.D.LOAD(2)	5.71149	5.14082	4.57015	3.99948	3.42882	2.85815	2.28748	1.71681	1.14614	0.57547
$\frac{[(2)-(1)]}{(1)}$ %	0.070	0.058	0.048	0.041	0.039	0.046	0.067	0.012	0.234	0.633

*S.D.LOAD$= P_s l_1 / M_0$, (1):Yoshida etc. (2):present method

ratio of bending-shear interacted shakedown load to static collapse load. It is recognizable from these figures that the uniform dead load much contributes into the reduction of shakedown loads of three-span beams. We also note some fluctuations in the curves of P_s/P_u versus α in Figs. of $\upsilon=0.5$ and 0.7. Such fluctuations may be caused by the variation of static collapse load which is controlled by a collapse mechanism occurring at whether side span or mid span depending on both υ and α. Namely, in the ranges of smaller values of β and of larger values of α and in the cases of $\upsilon=0.5$ and 0.7, the collapse mechanism occurs at side span, but in the other ranges at mid span.

In Figs. 16, we also investigate the variations of P_s/P_u from another angle. Here β is taken as abscissa and α is set as 0, 0.1, 0.2, 0.3, 0.4 and 0.8 times of the ultimate dead load multiplier which is determined by the limit analysis considering bending only. From these figures, we also conclude that the uniform dead load enlarges the effect of shear on the shakedown load.

4. CONCLUDING REMARKS

We have attempted to present a comprehensive analytical method on shakedown problems of two-span beams or symmetric three-span beams. There is no literature in hand to learn how good accuracy the present method has. However, we made a comparison on bending shakedown analysis by the present method (see Appendix) and that of Reference[1]. Their results are tabulated in Table 2. In order to discuss a general shakedown characteristics of girder bridge, the effect of distributed live load which was included in Reference[1] should be also considered. Such an effect can be easily included in the present method, but it was excluded in this paper for the convenience of numerical computation. For more actual structures such as reinforced concrete and prestressed concrete beams, the present method is supposed to have an applicability, if we have reliable informations on the interaction curves between bending and shear on plastic strengths of cross section.

APPENDIX
SHAKEDOWN ANALYSIS CONSIDERING BENDING ONLY

For this problem, we drop out the terms including shearing part from Eqs. (1) through (14) and follow the same analytic procedure developed in section 2. However, the number of convex vertexes in the elastic stress domain of cross section in such a case is reduced to two, namely, maximum and minimum bending moments. Their constraints and analytic procedure corresponding to Eqs. (9), (10) and (12) become as follows :

$$\lambda_s = \text{maximize } \lambda, \text{ subject to } (\lambda \overline{m}_{ij} + \alpha \overline{m}_{qi} + \mu \overline{m}_{ri})^2 \leq 1 \quad \cdots\cdots\cdots (9')$$

$$\overline{m}_{ri}^2 \mu^2 + 2 \overline{m}_{ri} (\lambda \overline{m}_{ij} + \alpha \overline{m}_{qi}) \mu + [(\lambda \overline{m}_{ij} + \alpha \overline{m}_{qi})^2 - 1] \leq 0 \quad \cdots\cdots\cdots (10')$$

$$\text{max. } \gamma_{ij}^- \leq \mu \leq \text{min. } \gamma_{ij}^+, \quad \begin{matrix} \gamma_{ij}^+ \\ \gamma_{ij}^- \end{matrix} = \frac{-(\lambda \overline{m}_{ij} + \alpha \overline{m}_{qi})}{\overline{m}_{ri}} \pm \frac{1}{|\overline{m}_{ri}|} \quad \cdots\cdots\cdots (12')$$

In inequality (10′), μ is constantly a real value for any given real value of λ. Therefore, instead of the use of the range of constraints (11) and (14), we execute iterative computations with appropriate increment of λ on Eq. (12′) until a cross points of curves of maximum γ_{ij}^- and minimum γ_{ij}^+ appears. However, there exists the case that there is no cross point of the curves, in another words, the beam collapses by its own dead load. If so, we may interrupt the computation at a beforehand decided value of λ.

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