Structural Eng. / Earthquake Eng. Vol. 4. No. 1, 221s-224s. April 1987 / Technical Note
Japan Society of Civil Engineers (Proc. of JSCE No. 380/I-7)

AN ELASTIC POST-BUCKLING BEHAVIOUR OF PROPPED-CANTILEVER COLUMN

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This paper presents the elastic post-buckling behaviour of propped-cantilever column predicted within the framework of FEM procedure, and the result is compared with that by the analytical elliptic-integral solution. Discussed in conjunction with the particular boundary conditions is a characteristic of deformation of this structure which is completely different from that of a cantilever column.

Keywords : post-buckling, propped-cantilever column, elliptic integral, instability

1. INTRODUCTION

It is well-known through the elliptic integrals[1)~4)] or the nonlinear FEM analysis that an elastic straight cantilever column of uniform section always has the stable equilibrium path even after the buckling, and the load can monotonically increase as the displacement becomes larger. Although it is also well-known that the buckling load is strongly affected by the boundary conditions, the post-buckling behaviour has not been extensively examined for so many types of boundary conditions. As can be seen later, the column with one end fixed and the other hinged, as called the propped-cantilever column here, shows unstable behaviour after the buckling unlike the cantilever column. Although the stable equilibrium right after the bifurcation has been examined in Ref. 2), the report of the unstable state afterwards has not been found in literatures available. This paper is intended to present the result of an elastic finite displacement FEM analysis of the straight propped-cantilever column with uniform section, and to discuss its characteristics in comparison with the cantilever column. Finally, the FEM result is compared with the analytical solution utilizing the elliptic integrals, which solves one kind of moving boundary problems (contact problems) with an unknown position of the inflection point between two domains of the different sign of curvature in the column.

2. NUMERICAL RESULTS AND DISCUSSIONS OF FEM SOLUTIONS

A propped-cantilever column illustrated in Fig. 1 is analyed by the FEM updated-Lagrangian formulation[5)] of plane been element. It is noted that a disturbing moment of the small magnitude of

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$P_{cr}L/1\,000$ (where $P_{cr}=20.19\,EI/L^2$=an elastic buckling of this column) is applied at the top to avoid bifurcation. By using the non-dimensionalized quantities of load, P/P_{cr}, vertical displacement u/L and slope angle α at the top, and horizontal displacement v/L in the middle of the column, the load-displacement curves of the propped-cantilever column are plotted in Fig. 2. Fig. 3 illustrates the deformed configuration of the column at the various equilibrium states indicated by the corresponding encircled numeral in Fig. 2.

Although the bifurcated path right after the buckling point shows the stable deformation with the positive slopes in Fig. 2[2)], there exists an unstable region between (2) and (4) unlike the ordinary cantilever column. This is because of the particular boundary condition, which forces the column to deform with double curvature from the beginning to the state (4). Hence there exists one inflection point, where the resisting moment vanishes as shown in Fig. 5 (a). This double curvature deformation with a moving inflection point and the horizontal reaction force at the top explain the unstable behaviour of the propped-cantilever column. The ordinary cantilever column resists the increase of the load by the monotonically increasing single curvature. Whether this column is stable (from the beginning to (2)) or not (from (2) to (4)) can be examined by the stability condition of a hinged-hinged column of its length equal to that of the portion of negative curvature in Fig. 5 (a), once the position of the inflection point and the horizontal reaction force at the top, R, are known as in Fig. 4. It is also noted that the equilibrium is stable after the point (4) where the load can increase monotonically due to the continuously increasing single curvature.

3. ANALYTICAL SOLUTIONS

Unlike the elliptic-integral solution of a cantilever column[1)~4)], the analytical procedure for this

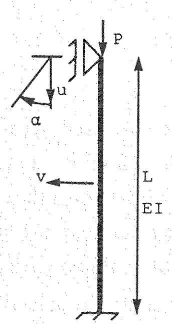


Fig. 1 Propped-cantilever column.

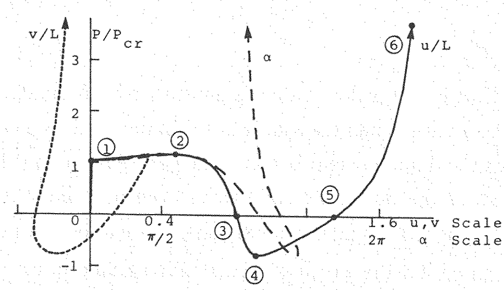


Fig. 2 Post-buckling behaviour of propped-cantilever column.

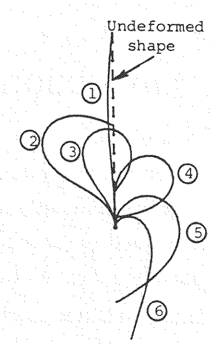


Fig. 3 Deformed configuration.

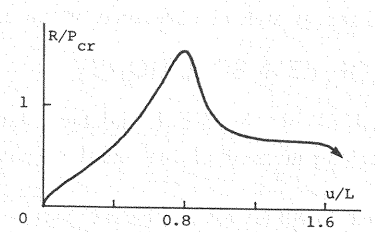
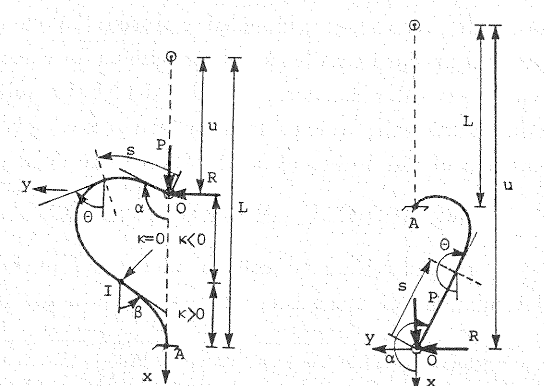


Fig. 4 Relation of horizontal reaction and displacement at the top.



(a) Double curvature (b) Single curvature

Fig. 5 Analytical solutions of propped-cantilever column.

propped-cantilever column has to account for the moving inflection point, and thus becomes one kind of the moving boundary problems with the elliptic integrals. When the column deforms with double curvature (from (1) to (4) in Fig. 2), the domain of integration must be divided into two parts at the unknown inflection point, I in Fig. 5 (a). Then in each domain, the usual procedure of elliptic-integral solution of a column can be applied. The deformation after (4) in Fig. 2 can be analysed by the ordinary procedure, because there exists no inflection point in the domain. The analytical process of solving this problem is summarized below.

(1) Double Curvature Solutions

Fig. 5 (a) defines the coordinate system similar to that in Ref. 1). The equilibrium equation in terms of curvature, $\varkappa \equiv d\theta/ds$, can be given by

$$EI\varkappa \equiv -Py+Rx$$

Differentiating the above equation leads to

$$d^2\theta/ds^2=-P/EI(\sin\theta-R/P\cos\theta) \quad (1)$$

where

$$dy/ds\equiv\sin\theta,\ dx/ds\equiv\cos\theta \quad (2)$$

Integration of Eq. (1) yields

$$(d\theta/ds)^2/2=P/EI(\cos\theta+R/P\sin\theta)+C \quad (3)$$

Let α denote the slope at the top and β that at the inflection point, I in Fig. 5 (a). Then Eq. (3) results in

$$d\theta/ds=\pm\sqrt{2}\,k\{\cos(\theta-\xi)-\cos\eta\}^{1/2}/\sqrt{\cos\xi} \quad (4)$$

and

$$R/P=\tan\xi \quad (5)$$

in which

$$k^2\equiv P/EI,\ \xi\equiv(\alpha+\beta)/2,\ \eta\equiv(\alpha-\beta)/2 \quad (6)$$

If the curvature is assumed negative from the point O to I and positive from I to A in Fig. 5 (a), the condition of inextensibility leads to the following expression of the total length of this column, L:

$$L=\int_0^L ds=\frac{\sqrt{\cos\xi}}{k}\left\{\int_{\phi 1}^{\pi/2}(1-p_1^2\sin^2\phi)^{-1/2}d\phi-\int_{\pi/2}^{-\pi/2}(1-p_1^2\sin^2\phi)^{-1/2}d\phi\right\} \quad (7)$$

where

$$\sin\phi\equiv\sin(\theta/2)/p_1,\ \sin\phi_1\equiv\sin(\xi/2)/p_1,\ p_1\equiv\sin(\eta/2) \quad (8)$$

Using Eqs. (2), (4), (6) and (8), we can express the kinematical boundary conditions at the top of this column as follows:

$$0=\int_0^L\frac{dy}{ds}ds=\frac{\sqrt{\cos\xi}}{k}\Big\{\sqrt{2}\cos\xi(\cos\xi-\cos\eta)^{1/2}$$
$$-\sin\xi\Big\{2\int_{\phi 1}^{\pi/2}(1-p_1^2\sin^2\phi)^{1/2}d\phi-2\int_{\pi/2}^{-\pi/2}(1-p_1^2\sin^2\phi)^{1/2}d\phi$$
$$-\int_{\phi 1}^{\pi/2}(1-p_1^2\sin^2\phi)^{-1/2}d\phi+\int_{\pi/2}^{-\pi/2}(1-p_1^2\sin^2\phi)^{-1/2}d\phi\Big\}\Big\} \quad (9)$$

and

$$\frac{u}{L}=1-\frac{1}{L}\int_0^L\frac{dx}{ds}ds=1-\frac{\sqrt{\cos\xi}}{k}\Big\{\sqrt{2}\sin\xi(\cos\xi-\cos\eta)^{1/2}$$
$$+\cos\xi\Big\{2\int_{\phi 1}^{\pi/2}(1-p_1^2\sin^2\phi)^{1/2}d\phi-2\int_{\pi/2}^{-\pi/2}(1-p_1^2\sin^2\phi)^{1/2}d\phi$$
$$-\int_{\phi 1}^{\pi/2}(1-p_1^2\sin^2\phi)^{-1/2}d\phi+\int_{\pi/2}^{-\pi/2}(1-p_1^2\sin^2\phi)^{-1/2}d\phi\Big\}\Big\} \quad (10)$$

After some manipulation, using the commonly used notations for elliptic integrals, one can express the integral equations (7), (9) and (10) as

$$\sqrt{2}\cos\xi(\cos\xi-\cos\eta)^{1/2}+\sin\xi\{3F(p_1)-F(p_1,\phi_1)-6E(p_1)+2E(p_1,\phi_1)\}=0 \quad (11)$$

$$PL^2/EI=(kL)^2=\cos\xi\{3F(p_1)-F(p_1,\phi_1)\}^2 \quad (12)$$

$$\frac{u}{L}=1-\frac{\sqrt{\cos\xi}}{kL}\{\sqrt{2}\sin\xi(\cos\xi-\cos\eta)^{1/2}-\cos\xi\{3F(p_1)-F(p_1,\phi_1)-6E(p_1)+2E(p_1,\phi_1)\}\} \quad \cdots\cdots (13)$$

where $F(p)$ and $E(p)$ are the complete elliptic integrals of the first and second kind, and $F(p, \phi)$ and $E(p, \phi)$ are the incomplete elliptic integrals of the first and second kind, respectively.

(2) Single Curvature Solutions

When there is no inflection point, the curvature is always negative as in Fig. 5 (b). Hence the ordinary procedure to solve an elastica applies to obtain in place of Eqs. (11)-(13)

$$\sqrt{2}\cos\gamma(\cos\gamma-\cos\lambda)^{1/2}-\sin\gamma\{F(p_2)-F(p_2,\phi_2)-2E(p_2)+2E(p_2,\phi_2)\}=0 \quad \cdots\cdots (14)$$

$$PL^2/EI=(kL)^2=\cos\gamma\{F(p_2)-F(p_2,\phi_2)\}^2 \quad \cdots\cdots (15)$$

$$\frac{u}{L}=1-\frac{\sqrt{\cos\gamma}}{kL}\{-\sqrt{2}\sin\gamma(\cos\gamma-\cos\lambda)^{1/2}-\cos\gamma\{F(p_2)-F(p_2,\phi_2)-2E(p_2)+2E(p_2,\phi_2)\}\} \quad \cdots\cdots (16)$$

where

$$\sin\phi_2\equiv\sin(\gamma/2)/p_2,\ p_2\equiv\sin(\lambda/2),\ \tan\gamma\equiv R/P,\ \lambda\equiv\alpha-\gamma \quad \cdots\cdots (17)$$

Numerical results can be obtained from Eqs. (6), (8), and (11)-(13) [or Eqs. (14)-(17)] by assigning a value for α. The manipulation includes the iterative solution of the nonlinear equations for β [or γ].

In Fig. 6, these analytical results are compared with the FEM solutions for a column with large slenderness ratio. Although a good agreement is obtained in the range of this figure, those two solutions become apart for the larger load, because the inextensibility of an axis is not rigorously taken into account in the FEM solution, but is approximately satisfied within the error of the order of magnitude of strains by using relatively large slenderness ratio.

4. CONCLUDING REMARKS

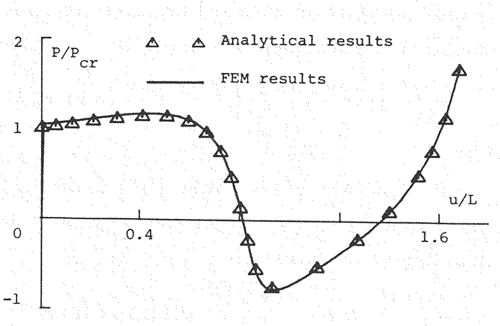


Fig. 6 Comparison of analytical and FEM results of propped-cantilever column.

An investigation of an elastic post-buckling behaviour of the straight propped-cantilever column of uniform section is presented. The obtained characteristics of deformation are significantly different from those of the ordinary cantilever column. It exhibits the snap-through type of behaviour similar to that of an arch, although their boundary conditions and initial geometry are not the same. In the analysis, both the FEM and the analytical approaches are employed, and the results agree with each other. The latter approach forms one kind of moving boundary problems with nonlinear equations in terms of the elliptic integrals. approach forms one kind of moving boundary problems with nonlinear equations in terms of the elliptic integrals.

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(Received August 13 1986)