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A CONSISTENT FORMULATION OF TRUSSES AND NON-WARPING BEAMS IN LINEARIZED FINITE DISPLACEMENTS

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The explicit stiffness equations and the corresponding differential equations are formulated for a truss and a non-warping beam in the framework of the linearized finite displacement theory. The derivation is consistent with the theory of thin-walled members. One main objective is to show the exact correspondence between the stiffness equations and the differential equations with their boundary conditions. An alternative scheme of deriving the stiffness matrices is given as the direct modification of the already obtained matrix of thin-walled members.
Keywords : finite displacement, stiffness matrix, FEM, truss, warping-free

1. INTRODUCTION

There exists the explicit form of the stiffness equations for thin-walled members within the framework of the linearized finite displacement theory, and the corresponding differential equations as well[1),2)]. Since the stiffness equation is formulated in the ordinary manner of finite element technique, the results depend mainly on the choice of the interpolation functions. In this problem, while the highest order of derivative of the axial displacement in virtual work is one, the other displacement components have their second derivatives in it. Therefore, taking the continuity conditions of displacements at nodes into account, the first and third Hermite polynomials are adopted to formulate stiffness equations.

On the other hand, the truss commonly used as a structural member is governed by the differential equations of second order of all displacement components. Hence appear only the first derivatives of displacements in the virtual work equation. This alters the choice of interpolation functions ; i. e. the first Hermite polynomial is enough to formulate the stiffness equations. Similarly without warping, the governing equation for torsion of a beam loses its otherwise highest derivative of the torsional rotation.

This note presents the explicit form of the stiffness equations and the differential equations for a truss and a non-warping beam in linearized finite displacements. In the derivation of the stiffness equations, there will be shown two methods, one of which is the reformulation with new interpolation functions, and another is the modification of the stiffness equation of the thin-walled member obtained in Ref. 1).

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2. DIRECT REFORMULATION OF STIFFNESS EQUATIONS

As is explained above, since the highest derivatives of some displacement components vanish in the truss and the non-warping beam, the interpolation functions can be different from those used in the general thin-walled beam element. Therefore, in this section, we show first the complete reformulation of the stiffness equations from the beginning with the newly chosen interpolation functions.

(1) A Non-Warping Beam

If a beam is relatively long or its cross-section is closed or relatively stocky, the warping freedom becomes negligible, and thus the torsion of such a beam can be considered within the St. Venant torsion theory. In such cases, there drop the terms related to the warping. Hence exists only the first derivative of torsional rotation in the virtual work equation, which is expressed as

$$\int_V \Big[\{(N^0/A)+(M_y^0/I_{yy})y+(M_z^0/I_{zz})z\}\delta[z v_s''\phi - y w_s''\phi + \frac{1}{2}\{(v_s')^2+(w_s')^2\}+(z_s v_s' - y_s w_s')\phi' + \frac{1}{2}\{(y-y_s)^2 + (z-z_s)^2\}(\phi')^2] + E(u_c' - y v_s'' - z w_s'')\delta(u_c' - y v_s'' - z w_s'') + G\Theta\phi'\delta(\Theta\phi')\Big] dv - \boldsymbol{F}^T\delta\boldsymbol{d} = 0 \quad \cdots\cdots (1)$$

where

$$\boldsymbol{F}=[\boldsymbol{F}_x^T, \boldsymbol{F}_y^T, \boldsymbol{F}_z^T, \boldsymbol{T}_S^T]^T, \quad \boldsymbol{d}=[\boldsymbol{U}^T, \boldsymbol{V}^T, \boldsymbol{W}^T, \boldsymbol{\Phi}_S^T]^T \quad \cdots\cdots (2)$$

in which

$$\boldsymbol{T}_S=[C_{Ti}, C_{Tj}]^T, \quad \boldsymbol{\Phi}_S=[\phi_i, \phi_j]^T \quad \cdots\cdots (3)$$

and all other quantities are defined in Ref. 1).

This form of virtual work requires the first Hermite polynomial for the torsional rotation, ϕ, and the ordinary procedure of the FEM technique results in the following stiffness equation:

$$\begin{Bmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \\ \boldsymbol{F}_z \\ \boldsymbol{T}_S \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K}_{11} & & & \text{Sym.} \\ 0 & \boldsymbol{K}_{22} & & \\ 0 & 0 & \boldsymbol{K}_{33} & \\ 0 & \boldsymbol{K}_{42} & \boldsymbol{K}_{43} & \boldsymbol{K}_{44} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \\ \boldsymbol{\Phi}_S \end{Bmatrix} \quad \cdots\cdots (4)$$

where the size of the stiffness matrix is 12×12 and

$$\left.\begin{aligned} &\boldsymbol{K}_{42}=(N^0 z_s/l)\boldsymbol{K}_4+(M_z^0/l)\boldsymbol{K}_5, \quad \boldsymbol{K}_{43}=-(N^0 y_s/l)\boldsymbol{K}_4-(M_y^0/l)\boldsymbol{K}_5 \\ &\boldsymbol{K}_{44}=(GJ/l)\boldsymbol{K}_0+[(N^0/l)r_s^2+(M_y^0/l)\beta_y+(M_z^0/l)\beta_z]\boldsymbol{K}_0 \end{aligned}\right\} \quad \cdots\cdots (5)$$

in which

$$\boldsymbol{K}_4=\begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{K}_5=\begin{bmatrix} -1 & l & 1 & 0 \\ 1 & 0 & -1 & -l \end{bmatrix} \quad \cdots\cdots (6)$$

and the definitions of other symbols are also given in Ref. 1).

The corresponding differential equations can be obtained from the same virtual work equation (1) and are expressed as

$$\left.\begin{aligned} &EAu_c''=0 \\ &EI_{yy}v_s''''-N^0 v_s''+(-z_sN^0+M_z^0)\phi''=0 \\ &EI_{zz}w_s''''-N^0 w_s''+(y_sN^0-M_y^0)\phi''=0 \\ &GJ\phi''-(-z_sN^0+M_z^0)v_s''-(y_sN^0-M_y^0)w_s''+(r_s^2N^0+\beta_yM_y^0+\beta_zM_z^0)\phi''=0 \end{aligned}\right\} \quad \cdots\cdots (7)$$

with the boundary conditions at $k=i$ and j as

$$\left.\begin{aligned} &u_c=u_{ck} && \text{or} \quad n_x EAu_c'=F_{xk} \\ &v_s=v_{sk} && \text{or} \quad n_x[-EI_{yy}v_s'''+N^0v_s'+(z_sN^0-M_z^0)\phi']=F_{yk} \\ &-v_s'=-v_{sk}' && \text{or} \quad n_x[-EI_{yy}v_s''-M_z^0\phi]=D_{yk} \\ &w_s=w_{sk} && \text{or} \quad n_x[-EI_{zz}w_s'''+N^0w_s'+(-y_sN^0+M_y^0)\phi']=F_{zk} \\ &-w_s'=-w_{sk}' && \text{or} \quad n_x[-EI_{zz}w_s''+M_y^0\phi]=D_{zk} \\ &\phi=\phi_k && \text{or} \quad n_x[GJ\phi'+z_sN^0v_s'-y_sN^0w_s'+(r_s^2N^0+\beta_yM_y^0+\beta_zM_z^0)\phi']=C_{Tk} \end{aligned}\right\} \quad \cdots\cdots (8)$$

where $n_x=-1$ at $k=i$ and $n_x=1$ at $k=j$.

(2) A Truss

Since the truss does have neither bending nor torsional freedom, it can not transfer any moment load. Hence remains only the axial initial force in the member, and at the boundary the moment forces vanish. Therefore the virtual work equation becomes

$$\int_V\left[(N^0/A)\delta\left\{\frac{1}{2}[(v')^2+(w')^2]\right\}+Eu'\delta(u')\right]dv-\boldsymbol{F}^T\delta\boldsymbol{d}=0 \quad\cdots\cdots (9)$$

where

$$\boldsymbol{F}=[\overline{\boldsymbol{F}}_x^T, \overline{\boldsymbol{F}}_y^T, \overline{\boldsymbol{F}}_z^T]^T,\quad \boldsymbol{d}=[\overline{\boldsymbol{U}}^T, \overline{\boldsymbol{V}}^T, \overline{\boldsymbol{W}}^T]^T \quad\cdots\cdots (10)$$

in which

$$\left.\begin{array}{l}\overline{\boldsymbol{F}}_x=[F_{xi}, F_{xj}]^T,\quad \overline{\boldsymbol{F}}_y=[F_{yi}, F_{yj}]^T,\quad \overline{\boldsymbol{F}}_z=[F_{zi}, F_{zj}]^T\\ \overline{\boldsymbol{U}}=[u_i, u_j]^T,\quad \overline{\boldsymbol{V}}=[v_i, v_j]^T,\quad \overline{\boldsymbol{W}}=[w_i, w_j]^T\end{array}\right\} \quad\cdots\cdots (11)$$

As only the first derivatives of all displacement components appear in the integrand of Eq. (9), the first Hermite polynomial is used as the interpolation function, to obtain the (6×6) stiffness equation in the form as

$$\begin{Bmatrix}\overline{\boldsymbol{F}}_x\\ \overline{\boldsymbol{F}}_y\\ \overline{\boldsymbol{F}}_z\end{Bmatrix}=\begin{bmatrix}\boldsymbol{K}_{11} & & \text{Sym.}\\ \boldsymbol{0} & \boldsymbol{K}_{22} & \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{33}\end{bmatrix}\begin{Bmatrix}\overline{\boldsymbol{U}}\\ \overline{\boldsymbol{V}}\\ \overline{\boldsymbol{W}}\end{Bmatrix} \quad\cdots\cdots (12)$$

where

$$\boldsymbol{K}_{11}=(EA/l)\boldsymbol{K}_0,\quad \boldsymbol{K}_{22}=\boldsymbol{K}_{33}=(N^0/l)\boldsymbol{K}_0 \quad\cdots\cdots (13)$$

in which $\boldsymbol{K}_0$ is defined in Ref. 1). Also the corresponding diffrential equations are obtained as

$$EAu''=0,\quad N^0v''=0,\quad N^0w''=0 \quad\cdots\cdots (14)$$

with the boundary conditions as

$$\left.\begin{array}{lll}u=u_k & \text{or} & n_xEAu'=F_{xk}\\ v=v_k & \text{or} & n_xN^0v'=F_{yk}\\ w=w_k & \text{or} & n_xN^0w'=F_{zk}\end{array}\right\} \quad\cdots\cdots (15)$$

3. AN ALTERNATIVE TO DIRECT REFORMULATION

In the preceding section, a truss and a non-warping beam are treated as different elements from the ordinary thin-walled beam element derived in Ref. 1). In general, however, the truss and the non-warping beam are often used together with the usual beam in structures, and they are considered substantially the same as the ordinary beam but can be treated as the particular cases that have less nodal degrees of freedom. Therefore it seems to be possible to derive the stiffness matrices of those members directly from the original matrix of the ordinary beam element.

Furthermore, in some structures, there may be inserted "hinges" where rotations and/or warping become discontinuous[3]. Then unlike the truss or the non-warping beam, only one nodal point of two may not transfer the bending or warping moment. In order to derive the stiffness matrices of such members, the procedure described below becomes very useful. Since this method is the pre-processing of the element stiffness matrix, it is more advantageous than that described in Ref. 3) from the viewpoint of the size of memory in computer.

The truss is an idealized member that does not transfer moment at both ends. Therefore at the very end of this member, no moment force is acting, and the rotational freedom in an element is no longer independent but is related to other nodal displacements. This dependence is expressed by the condition that the moment forces at nodes must be zero. For simplicity, consider the bending problem in $x-y$ plane without any torsion. The stiffness equation is expressed by

$$\boldsymbol{F}_y=\overline{\boldsymbol{K}}\boldsymbol{V},\quad \overline{\boldsymbol{K}}=(EI_{yy}/l^3)\boldsymbol{K}_1+(N^0/l)\boldsymbol{K}_2 \quad\cdots\cdots (16)$$

in which the second and fourth equations are those for moment resistance. In the truss, the left hand side of these two equations must be identically zero. This condition yields the geometrically reasonable expression of rotation in terms of lateral displacement as

$$-v'_{si}=-v'_{sj}=(v_{si}-v_{sj})/l \quad \cdots\cdots (17)$$

Some manipulation after substitution of Eq. (17) into Eq. (16) results in the same (2×2) stiffness equation as the second equation of Eq. (12), where all terms related to the bending stiffness, EI_{yy}, disappear.

Now consider the structure with a hinge inserted at the i-th nodal point only. Then the rotation at the i-th end of $i-j$ element becomes dependent freedom, and thus the corresponding moment load is identically zero. Therefore the technique stated above modifies the element stiffness matrix as

$$\overline{K}_{mn}:=\overline{K}_{mn}-(\overline{K}_{m2}\overline{K}_{2n})/\overline{K}_{22},\ (m, n=1, 3, 4),\ \overline{K}_{mn}:=0,\ (m \text{ or } n=2) \quad \cdots\cdots (18)$$

where subscript indicates element of matrix. Since this modified matrix has the same size as the original one, $\overline{\boldsymbol{K}}$, it is easy to handle in a computer program. Note, however, that the original stiffness matrix must be used to finally calculate this dependent rotational displacement at the i-th nodal point. The similar process as Eq. (18) applied simultaneously onto the second and fourth rows and columns of the matrix inserts two hinges at both ends, and yields the same results as substitution of Eq. (17) into Eq. (16).

A similar procedure is applicable to obtain the stiffness equation of a non-warping beam. One can consider it as the beam with the "warping free hinges" inserted at both ends. In this case, the warping deformations, $-\phi'_i$ and $-\phi'_j$, are expressed in terms of $\boldsymbol{V}$, $\boldsymbol{W}$, and $\boldsymbol{\Phi}_S$ by setting the nodal warping moments identically zero. Eliminate these warping deformations to get rid of all the terms relating to the warping rigidity, $EI_{\omega\omega}$, and eventually Eq. (4) is obtained.

4. CONCLUDING REMARKS

The explicit stiffness matrices and the corresponding differential equations have been derived for a truss and a non-warping beam, based on the general formulation of thin-walled members in linearized finite displacements. The same matrices can also be obtained by the direct modification of the original matrix for the thin-walled members by the insertion of some "hinges" at both ends of an element. The latter method is useful to derive the stiffness equations for members with many kinds of hinges arbitrarily inserted.

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