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ON THE TRACING CALCULATION OF THE EQUILIBRIUM PATH FOR IMPERFECT SYSTEMS

By Fumihito ITOH and Kuniei NOGAMI***

In this report, discussions on the numerical calculation of the non-linear equilibrium equation are stated, Especially, the problems arising in the tracing calculation of the path that exists the conjugate path in the vicinity of the branch point and that changes the direction due to yielding of the material are made clear and a more generalized incremental method elaborated for their disposal is proposed.

Consequently, by adopting a spherical search surface in which the arc-length of the path is used as restrictive parameter, the equilibrium points can be obtained automatically with good intervals for the system in which the path makes an abrupt change, and furthermore, it is made clear that the elimination method proposal by the authors is highly effective for the systems that have small initial imperfection.

1. INTRODUCTION

For getting precise knowledge on the load-carrying capacity of an imperfect structural systems, the numerical tracing of equilibrium path of the system is often necessary. In such problems, the governing equations frequently have rather complicated non-linear type.

Various methods of analysis have been proposed for such purpose and put into practical use. In any cases, the equilibrium path should be traced step by step discretely with rather small intervals. Each point is determined as an intersection of the path and arbitrarily chosen surface. Then, it becomes an important what kind of restrictive surface should be chosen in the calculation.

In most cases in the past, the methods called the load or the displacement incremental method in which the load or one of the components of displacement is properly chosen as independent variable have often been employed. The restrictive surface have constant value of the independent variable and its increment is also chosen rather arbitrarily[1].

These methods have simple algorithm and are greatly effective ones in most cases. But in either case, when the equilibrium path does not come into intersect with the restrictive surface, it becomes difficult to get the solution with satisfactory accuracy. So, it is necessary to exchange the independent variable chosen as restrictive surface. Therefore, as the method originated to avoid such trouble, so-called the arc-length incremental method[2)~4)] are shown. If the arc-length as incremental value should be chosen to become sufficiently small in comparison with the minimum radius of the curvature of equilibrium curve, this method is well-known to be generally very effective.

* Member of JSCE, Dr. Eng., Professor, Department of Civil Engineering
** Member of JSCE, Research associate, Department of Civil Engineering, Tokyo Metropolitan University (2-1-1 Fukazawa, Setagaya-ku, Tokyo)

But if equilibrium path of the imperfect system with very small initial imperfection came near the vicinity of the bifurcation point of the so-called "corresponding perfect system", the required arc-length becomes often too far short. Further if the arc-length is too long, there remains possibility that the solution might get into the conjugate path[5] and since then is a unfavorable one.

Nevertheless, it is very uneconomic that the whole equilibrium path of the system is consistantly traced with too small arc-length. So, it is extremely convenient that there is the analytical method in which the path is started to calculate with the rough arc-length and as the need arises, the arc-length can be reduced or enlarged, which can be automatically gotten out of the unfavorable path.

From the above-mentioned point of view, the authors newly developed the method in which the equilibrium path is calculated with detecting the unfavorable solution. Although this method can not be declared as almighty in any problem, but we could make sure by some illustrative examples that must gives good results in many problems. Following is the theoretical explanation about the method[6].

2. FUNDAMENTAL METHOD

Now, it is assumed that a set of m equilibrium equations

$$A_i(x_j)=0 \quad (i=1,2,\cdots,m) \qquad (1)$$

are given for system of variables $x_j\,(j=1,\ 2,\ \cdots,\ m+1)$ consisting of m components of displacement and one component of load.

In most circumstance, it is supposed that the so-called arc-length incremental method would be adequate as the numerical calculation for tracing the equilibrium path. Now, denoting the arc-length used with ρ, the known point P on the given equilibrium path with x_{pj} and the solution vector on the spherical surface with x_j, the additional conditions according to the arc-length incremental method are given by the following:

$$\sum_{j=1}^{n}(x_j-x_{pj})^2-\rho^2=0 \quad (n=m+1) \qquad (2)$$

(This represents a spacial surface of spherical shape with the radius ρ having its center at the known point x_{pj} as shown in **Fig. 1**.) Therefore, all the variables x_j for the given arc-length ρ are obtained by solving Eqs. (1) and (2) simultaneously. This is the concept of the arc-length incremental method.

Generally, Eqs. (1) can be supposed to be non-linear. Then, if k-th approximation $x_j^{(k)}$ satisfies the Eq. (2) denoting errors by Φ_i, Eqs. (1) and Eq. (2) become

$$\left.\begin{aligned} &A_i(x_j^{(k)})=\Phi_i,\\ &\sum_{j=1}^{n}(x_j^{(k)}-x_{pj})^2-\rho^2=0. \end{aligned}\right\} \qquad (3)$$

Using Newton-Raphson method, the incremental vector dx_j for the next approximation can be obtained through

$$\left.\begin{aligned} &\sum_{j=1}^{n}A_{i,j}(x_j^{(k)})dx_j=-\Phi_i,\\ &\sum_{j=1}^{n}(x_j^{(k)}-x_{pj})dx_j=0. \end{aligned}\right\} \qquad (4)$$

where

$$A_{i,j}=\partial A_i/\partial x_j.$$

Above Eqs. (4) may be represented simply as

$$K_{ij}dx_j+\Phi_i=0 \quad (i,j=1,2,\cdots,n) \qquad (5)$$

where the elements of square coefficient matrix K and Φ_n are given by

$$\left.\begin{aligned} &K_{ij}=A_{ij}(x_j^{(k)}) \quad \begin{pmatrix} i=1,2,\cdots,m\\ j=1,2,\cdots,n \end{pmatrix}\\ &K_{nj}=x_j^{(k)}-x_{pj},\\ &\Phi_n=0. \end{aligned}\right\} \qquad (6)$$

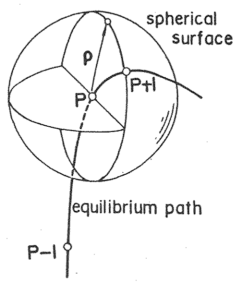

Fig. 1 Ideal incremental method.

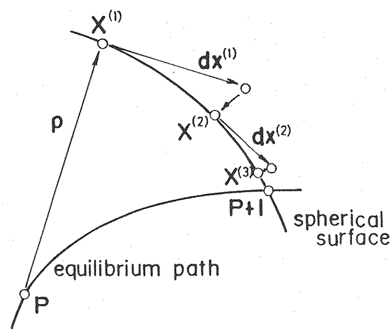

Fig. 2 Basic procedure.

In case of usual application of Newton-Raphson method, next approximation is estimated with the obtained solution dx_j by

$$x_j^{(k+1)}=x_j^{(k)}+dx_j \quad \cdots\cdots (7)$$

but the vector given by above Eq. (7) does not satisfy Eq. (2). So, as shown in Fig. 2, we should modify the next approximate vector onto the spherical surface by the following formula,

$$x_j^{(k+1)}=\frac{(x_j-x_{pj}+dx_j^{(k)})\rho}{\left\{\sum_{s=1}^{n}(x_s^{(k)}-x_{ps}+dx_s)^2\right\}^{1/2}}+x_{pj}. \quad \cdots\cdots (8)$$

Of course it may be said more theoretical to use normal method of Newton-Raphson taking account the error in Eq. (2), but sometimes in case of imperfect system, absolute value of incremental vector dx_j can become too large to regard the vector given by Eq. (7) as sufficient approximation.

3. PROBLEMS IN ESTIMATION OF EQUILIBRIUM PATH AND THE SIGN OF DET.K

The method of non-linear calculation mentioned in the previous section would cause practically no trouble so far as it is applied to the system that does not involve any singular point in the vicinity of or just on the equilibrium path. However, in case with very small imperfection and the point of problem is in the vicinity of the bifurcation point of the corresponding perfect system, the path of the imperfect system will changes the direction abruptly, and following troubles have chance to occur.

The first is the solution may arrives at just on the equilibrium path but backward position of last equilibrium point. Suppose the spherical restrictive surface with its center at the last equilibrium point is "skewered" by the path line, the required point must be just the exit of the sphere but not the entrance as shown in Fig. 3.

The second problem is to jump into the conjugate equilibrium path which may exist just opposite side of bifurcation point of corresponding perfect system, as well known (Fig. 4). The solution on such conjugate

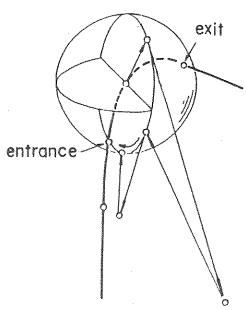

Fig. 3 Problem type-1.

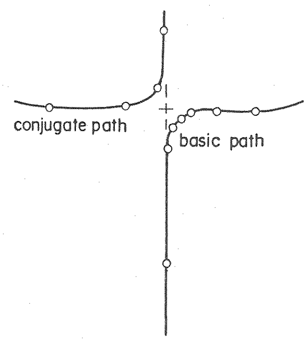

Fig. 4 Problem type-2.

path is never desirable one, and we must come back on the original path which is perhaps near the "post-buckling path" of the corresponding perfect system. By the way, Eqs. (5) necessarily has real solutions on the continuous equilibrium path and the value of det. K never reaches to zero while point on the path continuously progress forward except at the bifurcation point. That means det. K never change the sign throughout on the path of imperfect system without any bifurcation point unless the point reverse the direction of progression. If the point comes backward, all of K_{nj} expressed by Eqs. (6) change the sign and also the sign of det. K.

Besides, if we may suppose det. K is continuous function of not only of x_j but also of the amount of imperfection, det. K of the system with small imperfection will have almost the same value with the one of the nearest point on corresponding perfect system. Det. K of a perfect system has zero value just at a bifurcation point and will changes its value continuously as the point in consideration goes through the bifurcation, and then the sign may be reversed unless the bifurcation has double of multiple singularities. Then, if it is allowed to limit our problem in the single singularity, we may regard as det. K on the basic path of corresponding perfect system after go through the bifurcation point has reversed its sign, and perhaps the same to the one of system with small imperfection.

Of course, the reverse is not always true. Even on the conjugate path, if the direction of progression is backward, the sign of det. K may be the same as at the forward progressing point on the basic path before arrival to bifurcation. Such case may be occur, but must be rather rare. The programmes to detect and conduct such occasion must become too complex for our usual objects.

As a conclusion on these problems, it is sure that a solution with det. K having reversed sign is certainly the undesirable one.

4. PRACTICAL METHOD AGAINST ABOVE-MENTIONED TROUBLES

(1) Elimination procedure in general case

As mentioned in section 2, when Newton-Raphson method is to be applied, the non-linear equilibrium equations to be solved by Eqs. (4). Generally, upper expression of Eqs. (4) can be transformed to

$$\sum_{j=1}^{n} A_{i,j}dx_j = -A_{i,r}dx_r - \Phi_i \quad (j \neq r\,;\ i=1,2,\cdots,m) \cdots\cdots (9)$$

where dx_r denotes an arbitrary component of the vector increment. Then, unless determinant of the coefficients on the left side of these equations (det. A) is equal to zero, a straight line in the n-dimensional space can be obtained as solution. But occasionally, if the choice of independent variable dx_r is not adequate, it is possible that det. A becomes zero in spite of det. $K \neq 0$. That means we have a chance to fail to get correct existing solution because of mischoice of dx_r.

In the eliminating procedure for solution, it would be better at first to pick out the term with coefficient having the maximum absolute value among the all $A_{i,j}$ as pivot of elimination. If the coefficient is $A_{r,s}$, all coefficients in any row in Eqs. (5) except r-th one can be eliminated by r-th row as to make all $K_{i,s}$ $(i \neq r)$ become zero. After such elimination, the term with coefficient of the maximum absolute value among the all remained $A_{i,j}(i \neq r,\ j \neq s)$ is searched again and used as pivot of elimination for next time. Repeating this, only one column (say No. s) will remains at last. Then, all of expressions in Eqs. (5) have been transformed to the form

$$D_{ij}dx_j + B_{is}dx_s = -C_i \quad (j \neq s) \cdots\cdots (10)$$

where D_{ij} are zero except one used as a pivot of elimination, and in each equation $|B_{is}|$ may be smaller than absolute value of non-zero D_{ij}. The remained component dx_s must be the most proper choice as independent dx_r in Eq. (9) [as shown in Fig. 5].

It may be better to remark that all of $D_{nj}=0$ and det. K can also be calculated easily through above procedure. First m equations represent a tangential line of estimated path in n-dimensional space and the last n-th equation gives the plane perpendicular to the component dx_s which through the last approximation point. Then crossing of the line and the plane can be supposed as next estimated point of equilibrium while it is outside of the restrictive sphere. If the last approximation point is just on the equilibrium path, all of C_i in the Eqs. (10) must be zero and the line expressed by the first m equations represent the tangential line of the path at the point.

In these calculation, the value of det. K depends on arc-length ρ, and this behaviour may be not so desirable if we want to care about the value itself of det. K not only the sign of it for detect the distance from the position of the point under consideration to the singular point of the space, If it is the case, it would be better to change the 2nd of Eqs. (6) to the form

$$K_{nj}=\frac{(x_j^{(k)}-x_{pj})}{\rho} \quad \cdots\cdots (11)$$

Then the value of det. K becomes independent to the amount of arc-length ρ.

Another procedure having desirable characters may able to be developed, but our method mentioned above has shown sufficient results.

$$\left|\begin{array}{ccccc:c} A_{1,1} & A_{1,2} & \cdots & A_{1,m} & A_{1,n} & -\Phi_1 \\ A_{2,1} & (A_{2,2}) & \cdots & A_{2,m} & A_{2,n} & -\Phi_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} & A_{m,n} & -\Phi_m \\ \hdashline A_{n,1} & A_{n,2} & \cdots & A_{n,m} & A_{n,n} & 0 \end{array}\right|$$

$$\left|\begin{array}{ccccc:c} A^1_{1,1} & D_1 & \cdots & A^1_{1,m} & A^1_{1,n} & -\Phi^1_1 \\ A^1_{2,1} & 0 & \cdots & A^1_{2,m} & A^1_{2,n} & -\Phi^1_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ (A^1_{m,1}) & 0 & \cdots & A^1_{m,m} & A^1_{m,n} & -\Phi^1_m \\ \hdashline A^1_{n,1} & 0 & \cdots & A^1_{n,m} & A^1_{n,n} & -\Phi^1_n \end{array}\right|$$

$$\vdots$$

$$\left|\begin{array}{ccccccccc:c} 0 & D_1 & \cdots & 0 & B_1 & 0 & \cdots & 0 & & -C_1 \\ D_2 & 0 & \cdots & 0 & B_2 & 0 & \cdots & 0 & & -C_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & B_i & 0 & \cdots & D_i & & -C_i \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & D_m & B_m & 0 & \cdots & 0 & & -C_m \\ \hdashline 0 & 0 & \cdots & 0 & B_n & 0 & \cdots & 0 & & -C_n \end{array}\right|$$

Fig. 5 Elimination procedure.

(2) Elimination near the singular point

When the solution satisfying the Eqs. (1) and Eq. (2) exists but very near or just on the bifurcation point of corresponding perfect system, the value of det. K must be almost or just equal to zero. In such cases, one of Eqs. (1) would be dependent to remaining equations and all of coefficients belong a certain row become to have zero value in the above mentioned procedure. Then it becomes impossible to find the last term of pivot, and the final form of the equations would be

$$D_{ij}dx_j+B_{ir}dx_r+B_{is}dx_s=-C_i \quad \cdots\cdots (12)$$

These formulae including n-th equation represent not a point but a straight line and one of Eqs. (12) having all zero coefficients is meaningless. That is to say the solution is not unique but distributed on a line. Then we must select and put a solution out of the line.

Perhaps the most reasonable way is to add an equation giving new condition that forces the absolute value of incremental vector to make minimum, that is

$$\sum_{j=1}^{n}(dx_j)^2 \rightarrow \text{minimum}$$

But it would be far easier to set the value of dx_s to zero, if $|B_{nr}|>|B_{ns}|$. The solution can be also supposed as a next approximation of the equilibrium point.

In any case, if approximated point be just on the equilibrium path, all of C_i of above formulae must be zero and above procedure gives the exact solution. And at the instance, Eqs. (12) represent the tangential line of post-buckling path of the perfect system.

Even in the case of det. K given by computer is not exactly equal to zero, it will be better to regard it as of zero value and omit the m-th elimination procedure if there is sufficient reason to do so. We shall return to the illustrative case in later section.

(3) Setting of initial value

In general, as the position vector x_{ij} for the first equilibrium point P=1, the no load condition is chosen and so far as an excessively large value of the arc-length is not taken, the solution by the linear analysis may be used as approximation of the first time.

When the approximated solution of each point for the second times and after, the method most reasonably done is to set the initial value on the tangential line to the path in a equilibrium state.

Now, the point obtained x_{pi} satisfies Eqs. (1). Then, it is assumed that the next solution vector must be given by $(x_{pj}+dx_j)$. As this also should satisfy Eqs. (1), dx_j should satisfy

$$\sum_{j=1}^{n} A_{i,j}(x_{pj})dx_j=0 \quad (i=1,2,\cdots,m) \qquad (13)$$

$$\sum_{j=1}^{n}(dx_j)^2-\rho^2=0 \qquad (14)$$

Since Eq. (14) shows that the length of the straight line coincide with ρ, the calculation will be made by the length of the principal variable dx_s to be unity, for instance, and can be so adjusted that the length (absolute value) satisfies Eq. (14) afterward.

Since the solution of the previous time is to be right on the equilibrium path, by Eq. (10) or the result of elimination of dx_r from Eqs. (12)

$$dx_i+B_{is}dx_s=0 \quad (i\neq s) \qquad (15)$$

and dx_i are all given by dx_s. In other word, this principal variable is given by the formula below, solving in Eqs. (14) and (15):

$$dx_s=\pm\rho/\left\{1+\sum_{i=1}^{n}B_{is}^2\right\}^{1/2} \quad (B_{ss}=0) \qquad (16)$$

Thus, all the values of dx_i, accordingly, the next 0-th approximation can be determined. Out of the double sign, it would be natural to use the same sign as that of the vector component that is gained by deduction the previous solution from the present solution. If this should be wrong, the sign of the resulting det. K would become opposite. If so, the sign should be reversed.

(4) Arc-length and the sign of det. K.

Generally, the proper length of a step in arc-length method relates to the curvature of equilibrium path. If adopted arc-length ρ is too long against radius of curvature, necessary number of repetition required to get next equilibrium point must become large. If ρ is too short, number of repetition is certainly small but total number of calculation to get entire line of equilibrium path must become large. So it must be advisable to use a method which can change the arc-length automatically according to the state of the path.

Though it may be many ways to make such adjustment automatically, authors decided to use following formula

$$\rho=\frac{\alpha+N_sN_c}{\alpha+N_c^2}\rho^* \qquad (17)$$

for next arc-length, where N_s is the number of repetition aimed at, N_c is the number of repetition required to get the previous solution using arc-length ρ^*, and α is a parameter related to changing rate of the arc-length which must be decided by technical judgment. Above formula have been made rather intuitively without theoretical bases, but has characters as follows:

① $\rho=\rho^*$, if $N_c=N_s$

② $\rho/\rho^*\doteqdot N_s/N_c$, if $N_c\gg N_s$

③ $\rho/\rho^*\geqq N_s/N_c$, if $N_c>N_s$

④ $\rho/\rho^*\leqq N_s/N_c$, if $N_c<N_s$

and showed rather effective function using $\alpha=3$ in our case of $N_s=5$.

However, it is neseccary to bound the value of arc-length in proper extent. If too small amount of arc-length is used, the accuracy may worsens because of the limitation by effective number of figures of digital computer. If arc-length is too long, we can not get detail of equilibrium path. The decision of their bounds is one of problems of technical judgment.

Moreover, in case of too long arc-length, it can happens to fall into the undesirable solutions. They must be (1) backward solution on the basic path, or (2) solution on the conjugate path. In both of these cases, it can be detected by reversed sign of det. K. If we find the sign of det. K reversed, we must try again with initial approximated value on the opposite side of the last equilibrium point. Then we may get desirable solution, if the last one is backward point. This situation may happens sometimes in case of yielding of materials.

If above trial gives no improvement, the last solution may be on the conjugate path. Unless the value of ρ is just smallest one, try again with halved ρ. Repeat this to the minimum value of ρ, if proper sign of det. K can not obtained. In many cases, we may arrive at desirable state before trial with minimum ρ must be done.

If the sign stand still in undesirable state even with minimum ρ and with both of initial values opposite side of the last equilibrium point, the last point must be very near the singular point, and we have to regard as det. $K \fallingdotseq 0$. Imperfection of present system must be too small to analyse the system as an imperfect one, but we can treat the system using above-mentioned method for the case of det. $K \fallingdotseq 0$.

(5) On scaling

Generally in the problem of structures, the numerical values representing the strains or displacements are very small compared with the one of load or stress. Therefore, the amount of spherical radius proper to the change of the load variable must be too large for the displacements, and the reverse must be true.

It is necessary the radius has numerical amount proper not only to the load but also to at least the one of displacements or rotations to establish the effective criteria for convergence. It is highly effective to non-dimensionalize the all of variables by "scaling" so that the load and the principal variables of displacement have the numerical size of nearly "same order".

That is to say, denoting the scaling constant β_j for the variable x_j, the scaling can readily be done by introducing new variables $\overline{x}_j$ that have the relation of

$$\overline{x}_j = x_j / \beta_j$$

As these scaling constants β_j, the expected maximum value or one of analytically impotant state may be used. In a buckling problem, it must be reasonable to use buckling load as the scaling constant for load, as an example. It will be needless to select individual scaling constant for each displacement variables but may apply the one for the most principal variable to all of another displacement component.

At the same time, proper scaling will results to make certain distance numerically between bifurcation point and equilibrium path of imperfect system. Thus the generation of troubles in the vicinity of the bifurcation point can be prevented in advance.

5. EXAMPLES OF NUMERICAL CALCULATION

Investigation on this method is going to be undertaken hereafter, by applying it to actual examples. By the way, the computer applied to this calculation is the personal computer PC-9801 E by NEC.

In order to perform the calculation dealing with the structural system that includes the critical points (limit point and bifurcation point), a model of two hinged arch type is chosen. Especially, a system that has a bifurcation point after limit point is reached (the slenderness ratio for the length of element $L/r=32$) is assumed and the tracing of the equilibrium path in the case that has the small load imperfection of $c=0.999$. In this calculation, the scaling constant β is determined so as to given the values of the same order to the horizontal reaction H, load P and the end incremental rotational angle θ_1. Furthermore, the spherical radius ρ is changed by putting $N_s=5$ and $\alpha=3$ in Eq. (17). The results are illustrated in Fig. 6[7]. The diagram presents the curve for the equilibrium path between PL/K_b (K_b is rotational spring constant) and θ_1. The figures and letters at equilibrium point indicates the number of repetition for convergence and the main variable respectively. As clearly seen in this result, the ideal equilibrium point are automatically calculationed in the vicinity of the critical point of the corresponding perfect system without any problem and they can be obtained with wider intervals in the range of linear path and with closer intervals around the critical point.

Next, a column that includes the bifurcation point in the perfect system is picked up , and the elasto-plastic analysis is performed in this case when it has the displacement imperfection $\theta^0=0.001$. The

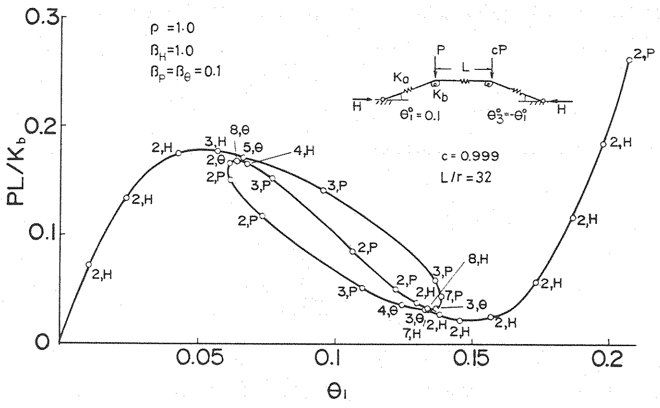

Fig. 6 Equilibrium path of 2-hinged arch model.

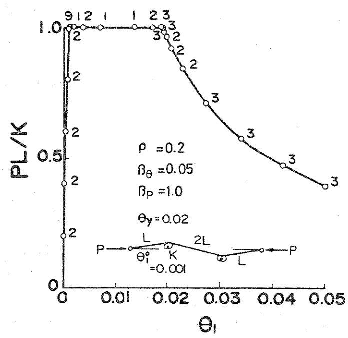

Fig. 7 Elasto-plastic analysis of column.

material is assumed to be a perfect elasto-plastic body and have the yielding angle $\theta_y = 0.02$. The result of this analysis is given in Fig. 7. On the ordinate of the diagram, the load that is made dimensionless with the rotational spring constant K is taken and on the abscissa, displacement θ_1 is taken. By the way the figure written at each point indicates the number of repetition for convergence and as the scaling constant β, the largest value in the range shown in the diagram is applied. The tracing has been fully made at the equilibrium points in the vicinity of the bifurcation point of the corresponding perfect system and over the horizontal path that follows them and the fall of the load-carrying capacity after yielding. For guidance, the judgement for convergence made in the above by norm criterion putting $\varepsilon = 10^{-4}$.

6. CONCLUSION

In this report, the authors proposed the generalized method of solution based on the modified arc-length incremental method as the numerical calculation with which the equilibrium path of the structure is traced. The conclusion can be summarized as stated in the following :

(1) By adopting the modified arc-length incremental method in which the spherical search surface is used, the choice of the parameter for the restrictive surface is made free and the calculation can be made with the equilibrium points having good distances each other. However, since the spherical surface intersect at least at two points with the equilibrium path, there is the possibility of converging to the solution that may transfer to the opposite direction or the conjugate path which necessitates the disposal mentioned in chapter 4.

(2) This elimination method has such various merits that high accuracy of the solution is always maintained, that the principal variable can be automatically chosen and so forth. Accordingly, by introducing this elimination procedure to the problems of the imperfect system that has small initial imperfection, the accuracy of calculation in the vicinity of the bifurcation is up-graded. And also, by calculating the value of the determinant along with applying this elimination method, the bifurcation point can be got through automatically.

(3) When the tracing of the equilibrium path is performed, it is important to make scaling so that the components of the variable to be noticed have the numerical values of the same order. And in doing so, generation of troubles particularly in the vicinity of the bifurcation point is prevented.

(4) This incremetal method can be applied not only to the geometrically non-linear problems, but also to the problems in which non-linearity of the material is considered and furthermore is applicable enough to the analysis that deals with the matrix of very high dimensions.

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