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A NEW FORMULATION OF FINITE DISPLACEMENT THEORY OF CURVED AND TWISTED RODS

By Yoshiaki GOTO*, Sei MATSUURA**, Akio HASEGAWA*** and Fumio NISHINO****

The governing equations for the finite displacement beam theory are often formulated through the principle of virtual work by introducing the pertinent kinematic field with displacement components defined in terms of the coordinates fixed in space. However, this formulation can hardly be applied for the theory of space beam without any restrictions on the magnitude of displacements, since the kinematic field becomes highly nonlinear largely due to the finite rotations in space.

This paper presents a new formulation which considerably simplifies the derivations through the principle of virtual work. By the formulation, the governing equations can be easily obtained even for the exact theory under beam assumptions.

1. INTRODUCTION

The governing equations for the finite displacement beam theory are often formulated through the principle of virtual work by introducing the pertinent kinematic field based on the beam assumptions[1)~6)]. This is mainly because the equilibrium equations as well as the mechanical boundary conditions consistent with the compatibility equations can be easily derived by the purely mathematical manipulations without any complicated considerations on the deformed geometry. However, the customary Lagrangian formulations through the principle of virtual work become much more complicated when applied for the space beams with finite rotations. Therefore, most of the formulations[1),2),4),6)] are approximated by introducing restrictions on the magnitude of displacements in addition to the conditions of small strains[5)].

In the customary Lagrangian formulations through the principle of virtual work, the kinematic field under the beam assumptions is expressed by the independent displacement components on the member axis, that is to say, three translational components in the directions of the coordinates fixed in space and one rotational component around the member axis[1),2),4)~6)]. With these components, the accurate kinematic field for space beams becomes highly nonlinear and complicated[4)~6)], compared with the beams restricted to two-dimension[3)]. Hence, the virtual kinematic field obtained as the result of variation becomes much more complicated and the partial integration[1)] for the derivation of the governing equations can hardly be executed in the equation of virtual work. Consequently, some approximations have to be introduced even when the accurate displacement field is obtained[4),6)]. For these reasons, most of the customary theories are of the second or the third order approximation with restrictions on the magnitude of displacements.

* Member of JSCE, Dr. Eng. Assoc. Prof. of Civil Eng., Nagoya Institute of Technology (Showaku, Nagoya)
** Member of JSCE, Dr. Eng. Prof. of Civil Eng., Nagoya Institute of Technology (Showaku Nagoya)
*** Member of JSCE, Dr. Eng. Assoc. Prof. of Civil Eng., Univ. of Tokyo (Bunkyoku Tokyo)
**** Member of JSCE, Ph. D. Prof. of Civil Eng., Univ. of Tokyo (Bunkyoku Tokyo)

This paper presents a new Lagrangian formulation of the exact finite displacement theory of initially curved and twisted rods under beam assumptions. In the formulation, new deformation components corresponding to curvature, torsional rate and extensional rate on the member axis are introduced as independent unknowns instead of the customarily used displacement components. These deformation components except extensional rate are firstly introduced by Love[7] for the formulation of inextensional straight rods, though Love's formulation is not based on the principle of virtual work. In addition to the deformation components, new components of virtual displacements are also introduced. They are the components of small translational and rotational displacements with separation of rigid body rotations, which are physically interpreted as the small incremental displacements from the deformed state. By usng the new components of deformation and virtual displacement, the kinematic field as well as the virtual kinematic field become much simpler and the exact governing equations can be obtained through the principle of virtual work even for the finite displacement theory with finite strains.

The beam assumptions introduced here are those commonly used for the beams with solid cross section, that is, the assumptions of no change of cross sectional shapes and the Bernoulli-Euler hypothesis where the transverse plane is assumed to remain plane and normal to the beam axis throughout the deformation[6]. It should be noted that the Bernoulli-Euler hypothesis ignores the warping due to torsion.

2. COORDINATES AND INITIAL GEOMETRY

Consider a curved and twisted space member as shown in Fig. 1. Orthogonal curvilinear coordinate system (x, y, z) is introduced at the initial configuration of the member with the coordinate z along the member axis.

The initial configuration of the member is expressed by the components $(\kappa_x, \kappa_y, \tau)$ defined as

$$\frac{d}{dz}\begin{Bmatrix} \boldsymbol{g}_{x0} \\ \boldsymbol{g}_{y0} \\ \boldsymbol{g}_{z0} \end{Bmatrix} = [D]\begin{Bmatrix} \boldsymbol{g}_{x0} \\ \boldsymbol{g}_{y0} \\ \boldsymbol{g}_{z0} \end{Bmatrix}, \quad [D]=\begin{bmatrix} 0 & \tau & -\kappa_y \\ -\tau & 0 & \kappa_x \\ \kappa_y & -\kappa_x & 0 \end{bmatrix} \quad \cdots\cdots (1.\text{a, b})$$

where $(\boldsymbol{g}_{x0}, \boldsymbol{g}_{y0}, \boldsymbol{g}_{z0})$ are the base vectors at the origin of the coordinates (x, y) defined in terms of the coordinate system (x, y, z). Physically, (κ_x, κ_y) and τ correspond to the components of curvature in the directions of the axes (x, y) and the torsional rate, respectively, of the z axis at the origin of the coordinates (x, y). It is noted henceforth that the z axis at the origin of the coordinates (x, y) is specifically referred as a member axis.

The base vectors $(\boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{g}_z)$ at an arbitrary point of the coordinate system (x, y, z) can be expressed by $(\boldsymbol{g}_{x0}, \boldsymbol{g}_{y0}, \boldsymbol{g}_{z0})$ and $(\kappa_x, \kappa_y, \tau)$ defined on the member axis. The position vector $\boldsymbol{R}$ of an arbitrary point (x, y, z) is written as

$$\boldsymbol{R}=\boldsymbol{R}_0+x\boldsymbol{g}_{x0}+y\boldsymbol{g}_{y0} \quad \cdots\cdots (2)$$

where $\boldsymbol{R}_0$ is the position vector of the member axis. With the help of eqs. (1) and (2), the base vectors at an arbitrary point are obtained as follows

$$\boldsymbol{g}_x=\boldsymbol{g}_{x0}, \quad \boldsymbol{g}_y=\boldsymbol{g}_{y0},$$
$$\boldsymbol{g}_z=-y\tau\boldsymbol{g}_{x0}+x\tau\boldsymbol{g}_{y0}+(1-\kappa_y x+\kappa_x y)\boldsymbol{g}_{z0} \cdots (3.\text{a}\sim\text{c})$$

For later convenience, the absolute value of $\boldsymbol{g}_z$ is defined as

$$|\boldsymbol{g}_z|=\sqrt{g}=\sqrt{(1-\kappa_y x+\kappa_x y)^2+(x^2+y^2)\tau^2} \cdots\cdots (4)$$

3. DEFORMED GEOMETRY

Consider a space member deformed under external forces as shown in Fig. 1. The Lagrangian approach is employed here to analyze the deformed state, where the

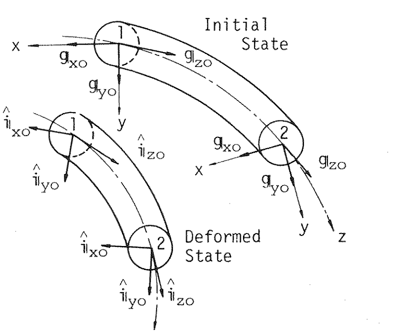

Fig. 1 Geometry of the Initial and the Deformed Rod.

coordinates (x, y, z) representing the initial position of the member are adopted as parameters which specify the material point throughout the deformation. In order to express the geometry of the deformed member, the deformation components of the member axis are introduced instead of the customary displacement components. The deformation components are represented by four unknowns of $\hat{\kappa}_x$, $\hat{\kappa}_y$, $\hat{\tau}$ and $\sqrt{\hat{g}_0}$ defined as

$$\frac{d}{dz}\begin{Bmatrix}\hat{\boldsymbol{i}}_{x0}\\ \hat{\boldsymbol{i}}_{y0}\\ \hat{\boldsymbol{i}}_{z0}\end{Bmatrix}=[\hat{D}]\begin{Bmatrix}\hat{\boldsymbol{i}}_{x0}\\ \hat{\boldsymbol{i}}_{y0}\\ \hat{\boldsymbol{i}}_{z0}\end{Bmatrix},\quad \hat{\boldsymbol{g}}_{z0}=\sqrt{\hat{g}_0}\,\hat{\boldsymbol{i}}_{z0},\quad [\hat{D}]=\begin{bmatrix}0 & \hat{\tau} & -\hat{\kappa}_y\\ -\hat{\tau} & 0 & \hat{\kappa}_x\\ \hat{\kappa}_y & -\hat{\kappa}_x & 0\end{bmatrix} \qquad (5.\text{a}\sim\text{c})$$

where $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ are the unit vectors obtained by normalizing the deformed base vectors $(\hat{\boldsymbol{g}}_{x0}, \hat{\boldsymbol{g}}_{y0}, \hat{\boldsymbol{g}}_{z0})$ on the member axis which are orthogonal due to the beam assumptions cited before. Physically, $(\hat{\kappa}_x/\sqrt{\hat{g}_0}$, $\hat{\kappa}_y/\sqrt{\hat{g}_0})$, $\hat{\tau}/\sqrt{\hat{g}_0}$ and $\sqrt{\hat{g}_0}-1$ correspond to the components of curvature in the directions of the deformed x and y axes, the torsional rate and the extensional rate, respectively, of the deformed member axis.

The deformed base vectors $(\hat{\boldsymbol{g}}_x, \hat{\boldsymbol{g}}_y, \hat{\boldsymbol{g}}_z)$ at an arbitrary point can be expressed by $(\hat{\boldsymbol{g}}_{x0}, \hat{\boldsymbol{g}}_{y0}, \hat{\boldsymbol{g}}_{z0})$ and $(\hat{\kappa}_x, \hat{\kappa}_y, \hat{\tau}, \sqrt{\hat{g}_0})$ defined on the member axis. The deformed position vector $\hat{\boldsymbol{R}}$ of an arbitrary point (x, y, z) is written from the beam assumptions as

$$\hat{\boldsymbol{R}}=\hat{\boldsymbol{R}}_0+x\hat{\boldsymbol{i}}_{x0}+y\hat{\boldsymbol{i}}_{y0} \qquad (6)$$

where $\hat{\boldsymbol{R}}_0$ is the deformed position vector of the member axis. With the help of eqs. (5) and (6), the deformed base vectors at an arbitrary point are given by

$$\hat{\boldsymbol{g}}_x=\hat{\boldsymbol{i}}_{x0},\quad \hat{\boldsymbol{g}}_y=\hat{\boldsymbol{i}}_{y0},\quad \hat{\boldsymbol{g}}_z=-y\hat{\tau}\hat{\boldsymbol{i}}_{x0}+x\hat{\tau}\hat{\boldsymbol{i}}_{y0}+(\sqrt{\hat{g}_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)\,\hat{\boldsymbol{i}}_{z0} \qquad (7.\text{a}\sim\text{c})$$

4. STRAIN FIELD

From eqs. (3) and (7), the non-zero covariant components of Green strain tensor are expressed by the deformation components as

$$\begin{aligned}e_{zz}&=(\hat{\boldsymbol{g}}_z\cdot\hat{\boldsymbol{g}}_z-\boldsymbol{g}_z\cdot\boldsymbol{g}_z)/2=(\hat{g}_0-1)/2+y\,(\sqrt{\hat{g}_0}\,\hat{\kappa}_x-\kappa_x)-x\,(\sqrt{\hat{g}_0}\,\hat{\kappa}_y-\kappa_y)\\ &\quad+x^2(\hat{\kappa}_y^2-\kappa_y^2)/2+y^2(\hat{\kappa}_x^2-\kappa_x^2)/2-xy\,(\hat{\kappa}_x\hat{\kappa}_y-\kappa_x\kappa_y)+(x^2+y^2)(\hat{\tau}^2-\tau^2)/2 \qquad (8.\text{a}\sim\text{c})\\ e_{zx}&=e_{xz}=(\hat{\boldsymbol{g}}_z\cdot\hat{\boldsymbol{g}}_x-\boldsymbol{g}_z\cdot\boldsymbol{g}_x)/2=-y\,(\hat{\tau}-\tau)/2,\quad e_{zy}=e_{yz}=(\hat{\boldsymbol{g}}_z\cdot\hat{\boldsymbol{g}}_y-\boldsymbol{g}_z\cdot\boldsymbol{g}_y)/2=x\,(\hat{\tau}-\tau)/2\end{aligned}$$

5. EQUILIBRIUM EQUATIONS AND MECHANICAL BOUNDARY CONDITIONS

Consider a member 1, 2 subject to distributed line force $\boldsymbol{p}$ and line moment $\boldsymbol{m}$ on the member axis as well as the distributed surface forces $\sigma_S^\alpha(\alpha=1,\ 2)$ on the end cross sections. These forces and moment are defined in terms of the initial configuration of the member. Equilibrium equations and the associated mechanical boundary conditions are derived through the principle of virtual work due to the small virtual incremental displacement $\delta\boldsymbol{d}$ from the deformed state of equilibrium.

The external force and moment vectors $(\boldsymbol{p}, \boldsymbol{m}, \sigma_S^\alpha)$ and the virtual displacement vector $\delta\boldsymbol{d}$ are expressed by the components in the directions of the vectors $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ as

$$\boldsymbol{p}=\hat{p}_x\hat{\boldsymbol{i}}_{x0}+\hat{p}_y\hat{\boldsymbol{i}}_{y0}+\hat{p}_z\hat{\boldsymbol{i}}_{z0},\quad \boldsymbol{m}=\hat{m}_y\hat{\boldsymbol{i}}_{x0}+\hat{m}_y\hat{\boldsymbol{i}}_{y0}+\hat{m}_z\hat{\boldsymbol{i}}_{z0},\quad \sigma_S^\alpha=\sigma_{Sx}^\alpha\hat{\boldsymbol{i}}_{x0}+\sigma_{Sy}^\alpha\hat{\boldsymbol{i}}_{y0}+\sigma_{Sz}^\alpha\hat{\boldsymbol{i}}_{z0} \qquad (9.\text{a}\sim\text{c})$$

$$\delta\boldsymbol{d}=\delta\hat{u}\,\hat{\boldsymbol{i}}_{x0}+\delta\hat{v}\,\hat{\boldsymbol{i}}_{y0}+\delta\hat{w}\,\hat{\boldsymbol{i}}_{z0} \qquad (10)$$

In the same manner, the components of the virtual translational and rotational displacement vectors $(\delta\boldsymbol{d}_0$, $\delta\boldsymbol{\alpha})$ on the member axis are given by

$$\delta\boldsymbol{d}_0=\delta\hat{u}_0\hat{\boldsymbol{i}}_{x0}+\delta\hat{v}_0\hat{\boldsymbol{i}}_{y0}+\delta\hat{w}_0\hat{\boldsymbol{i}}_{z0},\quad \delta\boldsymbol{\alpha}=\delta\hat{\alpha}_x\hat{\boldsymbol{i}}_{x0}+\delta\hat{\alpha}_y\hat{\boldsymbol{i}}_{y0}+\delta\hat{\alpha}_z\hat{\boldsymbol{i}}_{z0} \qquad (11.\text{a, b})$$

where the components of virtual rotation are small enough to be treated as vector quantities.

Making use of the contravariant component of 2 nd Piola-Kirchhoff stress tensor and the covariant component of Green strain tensor defined in terms of the coordinates (x, y, z) in addition to the components of external forces and virtual displacements defined in eqs. (9) ~ (11), the equation of virtual work for the member is given by[1)]

$$\int_{z_1}^{z_2}\int_A(\sigma^{zz}\delta e_{zz}+2\sigma^{zx}\delta e_{zx}+2\sigma^{zy}\delta e_{zy})\sqrt{g}\,dAdz-\int_{z1}^{z_2}(\hat{p}_x\delta\hat{u}_0+\hat{p}_y\delta\hat{v}_0+\hat{p}_z\delta\hat{w}_0+\hat{m}_x\delta\hat{\alpha}_x+\hat{m}_y\delta\hat{\alpha}_y+\hat{m}_z\delta\hat{\alpha}_z)\,dz+\left[n_z\int_A(\sigma^{\alpha}_{sx}\delta\hat{u}+\sigma^{\alpha}_{sy}\delta\hat{v}+\sigma^{\alpha}_{sz}\delta\hat{w})\,dA\right]_1^2=0 \qquad (12)$$

where $\int_A dA$ indicates the integration over the cross sectional area, z_1 and z_2 are the z coordinates of the end sections 1 and 2, and n_z has the values of -1 and 1, respectively, at z_1 and z_2.

From the beam assumptions, virtual strains and virtual displacements can be expressed by the four components of the virtual displacements on the member axis, that is, the three translational components $(\delta\hat{u}_0,\ \delta\hat{v}_0,\ \delta\hat{w}_0)$ and the one rotational component $\delta\hat{\alpha}_z$. Hence, the virtual strains $(\delta e_{zz},\ \delta e_{zx},\ \delta e_{zy})$ and the virtual displacements $(\delta\hat{u},\ \delta\hat{v},\ \delta\hat{w})$ in eq. (12) are expressed by the independent virtual displacements $(\delta\hat{u}_0,\ \delta\hat{v}_0,\ \delta\hat{w}_0,\ \delta\hat{\alpha}_z)$.

The variation of eq. (8) yields the virtual strains as

$$\delta e_{zz}=(\sqrt{\hat{g}_0}+y\hat{\kappa}_x-x\hat{\kappa}_y)(\delta\sqrt{\hat{g}_0}+y\delta\hat{\kappa}_x-x\delta\hat{\kappa}_y)+(x^2+y^2)\,\hat{\tau}\delta\hat{\tau},\quad \delta e_{zx}=-y\delta\hat{\tau}/2,\quad \delta e_{zy}=x\delta\hat{\tau}/2 \qquad (13.\,a\sim c)$$

The virtual deformation components in eq. (13) are further to be expressed by the independent virtual displacements. As shown in Appendix A, the virtual deformation components are given by

$$\delta\hat{\kappa}_x=\delta\hat{\alpha}'_x-\hat{\tau}\delta\hat{\alpha}_y+\hat{\kappa}_y\delta\hat{\alpha}_z,\quad \delta\hat{\kappa}_y=\delta\hat{\alpha}'_y+\hat{\tau}\delta\hat{\alpha}_x-\hat{\kappa}_x\delta\hat{\alpha}_z,\quad \delta\hat{\tau}=\delta\hat{\alpha}'_z+\hat{\kappa}_x\delta\hat{\alpha}_y-\hat{\kappa}_y\delta\hat{\alpha}_x,\quad \delta\sqrt{\hat{g}_0}=\delta\hat{w}'_0-\hat{\kappa}_y\delta\hat{u}_0+\hat{\kappa}_x\delta\hat{v}_0 \qquad (14.\,a\sim d)$$

where

$$\delta\hat{\alpha}_x=-(\delta\hat{v}'_0+\hat{\tau}\delta\hat{u}_0-\hat{\kappa}_x\delta\hat{w}_0)/\sqrt{\hat{g}_0},\quad \delta\hat{\alpha}_y=(\delta\hat{u}'_0-\hat{\tau}\delta\hat{v}_0+\hat{\kappa}_y\delta\hat{w}_0)/\sqrt{\hat{g}_0} \qquad (15.\,a,\,b)$$

in which $(\cdot)'$ denotes the differentiation with respect to z and this notation is used henceforth.

The virtual displacement field for $(\delta\hat{u},\ \delta\hat{v},\ \delta\hat{w})$ is obtained as follows by first taking variation of eq. (6) and substituting eqs. (10), (11, a) and (A-1) in Appendix A into the resulting equation.

$$\delta\hat{u}=\delta\hat{u}_0-y\delta\hat{\alpha}_z,\quad \delta\hat{v}=\delta\hat{v}_0+x\delta\hat{\alpha}_z,\quad \delta\hat{w}=\delta\hat{w}_0+y\delta\hat{\alpha}_x-x\delta\hat{\alpha}_y \qquad (16.\,a\sim c)$$

where $\delta\hat{\alpha}_x$ and $\delta\hat{\alpha}_y$ are given by eq. (15).

Substituting eqs. (13)～(16) into eq. (12) of virtual work and integrating by part lead to

$$\Big[\{(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)/\sqrt{\hat{g}_0}-n_z\hat{F}^c_x\}\,\delta\hat{u}_0+\{(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)/\sqrt{\hat{g}_0}-n_zF^c_y\}\,\delta\hat{v}_0+(\tilde{N}-n_z\hat{F}^c_z)\,\delta\hat{w}_0+(\tilde{M}_y-n_z\hat{M}^c_x)\,\delta\hat{\alpha}_x+(-\tilde{M}_x-n_z\hat{M}^c_y)\,\delta\hat{\alpha}_y+(\tilde{M}_z-n_z\hat{M}^c_z)\,\delta\hat{\alpha}_z\Big]_{z1}^{z_2}$$
$$-\int_{z_1}^{z_2}\Big(\Big[\{(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)/\sqrt{\hat{g}_0}\}'-(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)\,\hat{\tau}/\sqrt{\hat{g}_0}+\tilde{N}\hat{\kappa}_y+\hat{p}_x\Big]\,\delta\hat{u}_0+\Big[\{(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)/\sqrt{\hat{g}_0}\}'+(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)\,\hat{\tau}/\sqrt{\hat{g}}-\tilde{N}\hat{\kappa}_x+\hat{p}_y\Big]\,\delta\hat{v}_0$$
$$+\{\tilde{N}'+(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)\,\hat{\kappa}_x/\sqrt{\hat{g}_0}-(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)\,\hat{\kappa}_y/\sqrt{\hat{g}_0}+\hat{p}_z\}\,\delta\hat{w}_0+(\tilde{M}'_z-\tilde{M}_y\hat{\kappa}_y-\tilde{M}_x\hat{\kappa}_x+\hat{m}_z)\,\delta\hat{\alpha}_z\Big)\,dz=0 \qquad (17)$$

where the following notations are used for simplicity.

$$\left.\begin{array}{l}\tilde{N}=N\sqrt{\hat{g}_0}+M_y\hat{\kappa}_x-M_x\hat{\kappa}_y,\quad \tilde{M}_x=M_x\sqrt{\hat{g}_0}+M_{xy}\hat{\kappa}_x-M_{xx}\hat{\kappa}_y\\ \tilde{M}_y=M_y\sqrt{\hat{g}_0}+M_{yy}\hat{\kappa}_x-M_{yx}\hat{\kappa}_y,\quad \tilde{M}_z=T_s+K\hat{\tau}\end{array}\right\} \qquad (18.\,a\sim d)$$

$$\left.\begin{array}{l}\hat{F}^c_x=\int_A\sigma^{\alpha}_{sx}\,dA,\quad \hat{F}^c_y=\int_A\sigma^{\alpha}_{sy}\,dA,\quad \hat{F}^c_z=\int_A\sigma^{\alpha}_{sz}\,dA\\ \hat{M}^c_x=\int_A\sigma^{\alpha}_{sz}\,y\,dA,\quad \hat{M}^c_y=-\int_A\sigma^{\alpha}_{sz}\,x\,dA,\quad \hat{M}^c_z=\int_A(\sigma^{\alpha}_{sy}x-\sigma^{\alpha}_{sx}y)\,dA\end{array}\right\} \qquad (19.\,a\sim f)$$

Stress tensor resultants used in eq. (18) are defined by

$$\begin{array}{l}N=\int_A\sigma^{zz}\sqrt{g}\,dA,\quad M_x=\int_A\sigma^{zz}x\sqrt{g}\,dA,\quad M_y=\int_A\sigma^{zz}y\sqrt{g}\,dA\\ M_{xx}=\int_A\sigma^{zz}x^2\sqrt{g}\,dA,\quad M_{yy}=\int_A\sigma^{zz}y^2\sqrt{g}\,dA,\quad M_{xy}=M_{yx}=\int_A\sigma^{zz}xy\sqrt{g}\,dA\\ T_s=\int_A(\sigma^{zy}x-\sigma^{zx}y)\sqrt{g}\,dA,\quad K=\int_A\sigma^{zz}(x^2+y^2)\sqrt{g}\,dA\end{array} \qquad (20.\,a\sim h)$$

It should be noted that $\tilde{N}$, $(-\tilde{M}_x,\ \tilde{M}_y)$ and $\tilde{M}_z$ of eq. (18) are the physical components of stress resultants, corresponding respectively to the axial force toward the deformed member axis, the components of bending mement around the deformed y and x axes and the torsional moment around the deformed z axis. For the help of physical interpretation, the stress resultants of eq. (18) are directly derived by the integration of stress over the cross sectional area in Appendix B. The mechanical quantities of eq. (19) with superscript c denote the stress resultants acting on the ends of the member.

Equilibrium equations and the associated mechanical boundary conditions are obtained form the necessary and sufficient conditions for eq. (17) to hold for any arbitrary virtual displacements. The terms in the bracket under the integral sign yield the equilibrium equations

$$\begin{aligned}
&\{(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)/\sqrt{\hat{g}_0}\}'-(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)\,\hat{\tau}/\sqrt{\hat{g}_0}+\tilde{N}\hat{\kappa}_y+\hat{p}_x=0\\
&\{(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)/\sqrt{\hat{g}_0}\}'+(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)\,\hat{\tau}/\sqrt{\hat{g}_0}-\tilde{N}\hat{\kappa}_x+\hat{p}_y=0\\
&\tilde{N}'+(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)\,\hat{\kappa}_x/\sqrt{\hat{g}_0}-(\tilde{M}'_x-\tilde{M}_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)\,\hat{\kappa}_y/\sqrt{\hat{g}_0}+\hat{p}_z=0\\
&\tilde{M}'_z-\tilde{M}_y\hat{\kappa}_y-\tilde{M}_x\hat{\kappa}_x+\hat{m}_z=0
\end{aligned}\qquad\cdots\cdots\ (21.\mathrm{a}\sim\mathrm{d})$$

and the integrated terms give the associated mechanical boundary conditions

$$\begin{aligned}
&n_z\hat{F}^c_x=(\tilde{M}'_x-\tilde{M}'_y\hat{\tau}+\tilde{M}_z\hat{\kappa}_x-\hat{m}_y)/\sqrt{\hat{g}_0},\quad n_z\hat{F}^c_y=(\tilde{M}'_y+\tilde{M}_x\hat{\tau}+\tilde{M}_z\hat{\kappa}_y+\hat{m}_x)/\sqrt{\hat{g}_0}\\
&n_z\hat{F}^c_z=\tilde{N},\quad n_z\hat{M}^c_x=\tilde{M}_y,\quad n_z\hat{M}^c_y=-\tilde{M}_x,\quad n_z\hat{M}^c_z=\tilde{M}_z
\end{aligned}\qquad\cdots\cdots\ (22.\mathrm{a}\sim\mathrm{f})$$

As clear from the components of the virtual displacements defined in eq. (11), the equilibrium equations and the mechanical boundary conditions are expressed in components in the directions of the vectors ($\hat{\boldsymbol{i}}_{x0}$, $\hat{\boldsymbol{i}}_{y0}$, $\hat{\boldsymbol{i}}_{z0}$), which are physically interpreted as the expression with separation of rigid body rotations[8]. The geometrical boundary conditions corresponding to eq. (22) are given by the components the same as those of the small virtual displacements ($\delta\hat{u}_0$, $\delta\hat{v}_0$, $\delta\hat{w}_0$, $\delta\hat{\alpha}_x$, $\delta\hat{\alpha}_y$, $\delta\hat{\alpha}_z$). However, these components are dependent on the deformed configuration of the member and can hardly be used except approximately in incremental procedures. Then, for the exact analysis, the boundary conditions have to be tramsformed into those expressed by the customary physical components in the directions of the coordinates fixed in space. These transformations are shown later in section 8.

6. STRESS RESULTANT-DEFORMATION RELATIONS

Consider a beam made of linear elastic materials. The stress resultant-deformation relations differ[3] according to whether the linear elastic relations between stress and strain are defined with respect to tensor quantities or physical quantities[6]. Hence, here examined are the two constitutive equations derived from the two definitions stated above.

In order to define the linear elastic relations, the local rectangular Cartesian coordinate system ($\bar{x}$, $\bar{y}$, $\bar{z}$)[1),4),6)] is introduced at the initial configuration of the member with the coordinates ($\bar{x}$, $\bar{y}$) coincident with the coordinates (x, y) and the $\bar{z}$ axis directed toward the tangent of the z axis.

First, the constitutive equations are derived for the case where the linear elastic relations are defined between 2 nd Piola-Kirchhoff stress tensor and Green strain tensor. In the rectangular Cartesian coordinate system ($\bar{x}$, $\bar{y}$, $\bar{z}$), the linear relations for rods are defined by

$$\sigma^{\bar{z}\bar{z}}=Ee_{\bar{z}\bar{z}},\quad \sigma^{\bar{z}\bar{x}}=2Ge_{\bar{z}\bar{x}},\quad \sigma^{\bar{z}\bar{y}}=2Ge_{\bar{z}\bar{y}}\cdots\cdots\ (23.\mathrm{a}\sim\mathrm{c})$$

where E and G are elastic constants. In order to derive the relations between the stress resultants of Eq. (18) and the deformation components, the linear elastic relations of eq. (23) have further to be tansformed into those for the tensors in the coordinates (x, y, z). According to the rule of tensor transformation, the stress and the strain tensors in the coordinates (x, y, z) are related to those in the coordinates ($\bar{x}$, $\bar{y}$, $\bar{z}$) as follows.

$$\sigma^{zz}=\sigma^{\bar{z}\bar{z}}/\rho^2,\quad \sigma^{zx}=\sigma^{\bar{z}\bar{x}}/\rho+\sigma^{\bar{z}\bar{z}}y\tau/\rho^2,\quad \sigma^{zy}=\sigma^{\bar{z}\bar{y}}/\rho-\sigma^{\bar{z}\bar{z}}x\tau/\rho^2\cdots\cdots\ (24.\mathrm{a}\sim\mathrm{c})$$

$$e_{\bar{z}\bar{z}}=(e_{zz}-2e_{zy}x\tau+2e_{zx}y\tau)/\rho^2,\quad e_{\bar{z}\bar{x}}=e_{zx}/\rho,\quad e_{\bar{z}\bar{y}}=e_{zy}/\rho\cdots\cdots\ (25.\mathrm{a}\sim\mathrm{c})$$

where

$$\rho=1-\kappa_y x+\kappa_x y\cdots\cdots\ (26)$$

With the help of eqs. (24) and (25), eq. (23) yields

$$\sigma^{zz}=E(e_{zz}-2e_{zy}x\tau+2e_{zx}y\tau)/\rho^4,\quad \sigma^{zx}=2Ge_{zx}/\rho^2+E(e_{zz}-2e_{zy}x\tau+2e_{zx}y\tau)y\tau/\rho^4$$
$$\sigma^{zy}=2Ge_{zy}/\rho^2-E(e_{zz}-2e_{zy}x\tau+2e_{zx}y\tau)x\tau/\rho^4 \quad \cdots\cdots (27.\text{a}\sim\text{c})$$

The relations between the stress resultants and the deformation components are obtained by substituting eq. (27) into eq. (18), helped by eqs. (8) and (20). These relations are a little simplified if the coordinates are selected such that

$$\int_A x/\rho^4\sqrt{g}\,dA=\int_A y/\rho^4\sqrt{g}\,dA=\int_A xy/\rho^4\sqrt{g}\,dA=0 \quad\cdots\cdots (28)$$

Next, it is examined how the constitutive equations differ when the linear elastic relations are defined between the physical components of stress and strain. These relations in the local orthogonal Cartesian coordinate system $(\bar{x},\ \bar{y},\ \bar{z})$ are defined by

$$\sigma_{\bar{z}}=E\varepsilon_{\bar{z}},\quad \tau_{\bar{z}\bar{x}}=G\varepsilon_{\bar{z}\bar{x}},\quad \tau_{\bar{z}\bar{y}}=G\varepsilon_{\bar{z}\bar{y}} \quad\cdots\cdots (29.\text{a}\sim\text{c})$$

where the physical components of stress $(\sigma_{\bar{z}},\ \tau_{\bar{z}\bar{x}},\ \tau_{\bar{z}\bar{y}})$ and those of strain $(\varepsilon_{\bar{z}},\ \varepsilon_{\bar{z}\bar{x}},\ \varepsilon_{\bar{z}\bar{y}})$ are related to the tensor components of stress and strain in the local orthogonal Cartesian coordinate system, respectively, as

$$\sigma_{\bar{z}}=\sigma^{\bar{z}\bar{z}}|\hat{\boldsymbol{g}}_{\bar{z}}|=\sigma^{\bar{z}\bar{z}}\sqrt{2e_{\bar{z}\bar{z}}+1},\quad \tau_{\bar{z}\bar{x}}=\sigma^{\bar{z}\bar{x}},\quad \tau_{\bar{z}\bar{y}}=\sigma^{\bar{z}\bar{y}} \quad\cdots\cdots (30.\text{a}\sim\text{c})$$

$$\left.\begin{aligned}&\varepsilon_{\bar{z}}=|\hat{\boldsymbol{g}}_{\bar{z}}|-1=\sqrt{2e_{\bar{z}\bar{z}}+1}-1,\quad \sin\varepsilon_{\bar{z}\bar{x}}=\hat{\boldsymbol{g}}_{\bar{z}}\cdot\hat{\boldsymbol{g}}_{\bar{x}}/(|\hat{\boldsymbol{g}}_{\bar{z}}|\ |\hat{\boldsymbol{g}}_{\bar{x}}|)=2e_{\bar{z}\bar{x}}/\sqrt{2e_{\bar{z}\bar{z}}+1},\\ &\sin\varepsilon_{\bar{z}\bar{y}}=\hat{\boldsymbol{g}}_{\bar{z}}\cdot\hat{\boldsymbol{g}}_{\bar{y}}/(|\hat{\boldsymbol{g}}_{\bar{z}}|\ |\hat{\boldsymbol{g}}_{\bar{y}}|)=2e_{\bar{z}\bar{y}}/\sqrt{2e_{\bar{z}\bar{z}}+1}\end{aligned}\right\}\quad\cdots\cdots (31.\text{a}\sim\text{c})$$

in which $(\hat{\boldsymbol{g}}_{\bar{x}},\ \hat{\boldsymbol{g}}_{\bar{y}},\ \hat{\boldsymbol{g}}_{\bar{z}})$ are the deformed base vectors of the coordinates $(\bar{x},\ \bar{y},\ \bar{z})$. The linear elastic relations defined by eq. (29) are transformed into those expressed by the tensor components in the coordinates $(x,\ y.\ z)$ so as to be compared with eq. (27). Substituting eqs. (30) and (31) into eq. (29) with the help of eqs. (24) and (25), eq. (29) yields

$$\sigma^{zz}=E(\sqrt{\hat{g}_{\bar{z}}}-1)/(\sqrt{\hat{g}_{\bar{z}}}\rho^2),\quad \sigma^{zx}=G\sin^{-1}\{2e_{zx}/(\sqrt{\hat{g}_{\bar{z}}}\rho)\}/\rho+Ey\tau(\sqrt{\hat{g}_{\bar{z}}}-1)/(\sqrt{\hat{g}_{\bar{z}}}\rho^2)$$
$$\sigma^{zy}=G\sin^{-1}\{2e_{zy}/(\sqrt{\hat{g}_{\bar{z}}}\rho)\}/\rho-Ex\tau(\sqrt{\hat{g}_{\bar{z}}}-1)/(\sqrt{\hat{g}_{\bar{z}}}\rho^2) \quad\cdots\cdots (32.\text{a}\sim\text{c})$$

where

$$\sqrt{\hat{g}_{\bar{z}}}=|\hat{\boldsymbol{g}}_{\bar{z}}|=\sqrt{\rho^2+2e_{zz}-4x\tau e_{zy}+4y\tau e_{zx}}/\rho \quad\cdots\cdots (33)$$

Eq. (32) becomes highly nonlinear, if expanded into power series with respect to strain tensor. Therefore, it is clear that the stress resultant-deformation relations derived from eq. (32) are much more complicated than those from eq. (27). However, in the specific case when the torsional rate is zero at both of the initial and the deformed states[3], the constitutive equations derived from eq. (32) are considerably simplified as

$$\tilde{N}=E\tilde{A}(\sqrt{\hat{g}_0}-1),\quad \tilde{M}_x=-E\tilde{I}_{xx}(\hat{\kappa}_y-\kappa_y),\quad \tilde{M}_y=E\tilde{I}_{yy}(\hat{\kappa}_x-\kappa_x) \quad\cdots\cdots (34.\text{a}\sim\text{c})$$

where

$$\tilde{A}=\int_A 1/\rho\,dA,\quad \tilde{I}_{xx}=\int_A x^2/\rho\,dA,\quad \tilde{I}_{yy}=\int_A y^2/\rho\,dA \quad\cdots\cdots (35.\text{a}\sim\text{c})$$

with the coordinates $(x,\ y)$ defined such that

$$\int_A x/\rho\,dA=\int_A y/\rho\,dA=\int_A xy/\rho\,dA=0 \quad\cdots\cdots (36)$$

7. DISPLACEMENT-DEFORMATION RELATIONS

The deformation components of $(\hat{\kappa}_x,\ \hat{\kappa}_y,\ \hat{\tau},\ \sqrt{\hat{g}_0})$ have to be related to displacement components in order to analyze the geometry of the deformed member.

The translational displacements on the member axis are expressed by the components $(u_0,\ v_0,\ w_0)$ in the directions of the vectors $(\boldsymbol{g}_{x0},\ \boldsymbol{g}_{y0},\ \boldsymbol{g}_{z0})$. With regard to rotational displacements, the customary expression by the rotational angles considerably complicates the formulation[4)~6)], since the angles of finite rotation can not be treated as a vector quantity. Furthermore, the rotational angles are often transformed into direction cosines in the formulation. For these reasons, rotational displacements are expressed here by the direction cosines between the orthogonal unit vectors $(\hat{\boldsymbol{i}}_{x0},\ \hat{\boldsymbol{i}}_{y0},\ \hat{\boldsymbol{i}}_{z0})$ and the base vectors $(\boldsymbol{g}_{x0},\ \boldsymbol{g}_{y0},$

$\boldsymbol{g}_{z0}$) before deformation, which are defined by

$$\begin{Bmatrix} \hat{\boldsymbol{i}}_{x0} \\ \hat{\boldsymbol{i}}_{y0} \\ \hat{\boldsymbol{i}}_{z0} \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} \boldsymbol{g}_{x0} \\ \boldsymbol{g}_{y0} \\ \boldsymbol{g}_{z0} \end{Bmatrix}, \quad [l_{\hat{a}b}] = \begin{bmatrix} l_{\hat{x}x}, & l_{\hat{x}y}, & l_{\hat{x}z} \\ l_{\hat{y}x}, & l_{\hat{y}y}, & l_{\hat{y}z} \\ l_{\hat{z}x}, & l_{\hat{z}y}, & l_{\hat{z}z} \end{bmatrix} \quad \cdots\cdots (37.\text{a, b})$$

Differentiating eq. (37. a) with respect to z and helped by eqs. (1), (5) and (37. a), the direction cosine-deformation relations are given by

$$d[l_{\hat{a}b}]/dz = [\hat{D}][l_{\hat{a}b}] - [l_{\hat{a}b}][D] \quad \cdots\cdots (38)$$

Next derived are the translational displacement-deformation relations. The deformed position vector $\hat{\boldsymbol{R}}_0$ on the member axis is expressed by the components of translational displacements (u_0, v_0, w_0) as

$$\hat{\boldsymbol{R}}_0 = \boldsymbol{R}_0 + u_0 \boldsymbol{g}_{x0} + v_0 \boldsymbol{g}_{y0} + w_0 \boldsymbol{g}_{z0} \quad \cdots\cdots (39)$$

Differentiating eq. (39) with respect to z and helped by eq. (1), the deformed base vector $\hat{\boldsymbol{g}}_{z0}$ on the member axis is obtained by

$$\partial \hat{\boldsymbol{R}}_0 / \partial z = \hat{\boldsymbol{g}}_{z0} = (u_0' - \tau v_0 + \kappa_y w_0)\boldsymbol{g}_{x0} + (v_0' + \tau u_0 - \kappa_x w_0)\boldsymbol{g}_{y0} + (1 + w_0' - u_0 \kappa_y + v_0 \kappa_x)\boldsymbol{g}_{z0} \quad \cdots\cdots (40)$$

Substituting eqs. (5. b) and (37) into eq. (40), the derivatives of the translational displacement components are expressed by the deformation components and the direction cosines as

$$u_0' = \sqrt{\hat{g}_0}\, l_{\hat{z}x} + \tau v_0 - \kappa_y w_0, \quad v_0' = \sqrt{\hat{g}_0}\, l_{\hat{z}y} - \tau u_0 + \kappa_x w_0, \quad w_0' = \sqrt{\hat{g}_0}\, l_{\hat{z}z} - 1 + \kappa_y u_0 - \kappa_x v_0 \quad \cdots\cdots (41.\text{a}\sim\text{c})$$

8. EXPRESSIONS OF GOVERNING EQUATIONS

As stated in Section 5, it is difficult to use the geometrical boundary conditions corresponding to the mechanical ones of eq. (22). Hence, the mechanical boundary conditions is transformed into those corresponding to the geometrical ones expressed by the physical components in the directions of the coordinates fixed in space. In addition to the boundary conditions, the equilibrium equations of eq. (21) are expressed here by the customary components in the direction of the coordinates (x, y, z) fixed in space in order to compare the present theory with the customary ones.

For the derivation of the governing equations in the customary expression, the components of virtual displacements, forces and moments in the directions of the base vectors ($\boldsymbol{g}_{x0}$, $\boldsymbol{g}_{y0}$, $\boldsymbol{g}_{z0}$) are defined as

$$\begin{aligned} &\delta \boldsymbol{d}_0 = \delta u_0 \boldsymbol{g}_{x0} + \delta v_0 \boldsymbol{g}_{y0} + \delta w_0 \boldsymbol{g}_{z0}, \quad \delta \boldsymbol{\alpha} = \delta \alpha_x \boldsymbol{g}_{x0} + \delta \alpha_y \boldsymbol{g}_{y0} + \delta \alpha_z \boldsymbol{g}_{z0} \\ &\boldsymbol{p} = p_x \boldsymbol{g}_{x0} + p_y \boldsymbol{g}_{y0} + p_z \boldsymbol{g}_{z0}, \quad \boldsymbol{m} = m_x \boldsymbol{g}_{x0} + m_y \boldsymbol{g}_{y0} + m_z \boldsymbol{g}_{z0} \\ &\boldsymbol{F}_S = \overline{F}{}^c_x \boldsymbol{g}_{x0} + \overline{F}{}^c_y \boldsymbol{g}_{y0} + \overline{F}{}^c_z \boldsymbol{g}_{z0}, \quad \boldsymbol{M}_S = \overline{M}{}^c_x \boldsymbol{g}_{x0} + \overline{M}{}^c_y \boldsymbol{g}_{y0} + \overline{M}{}^c_z \boldsymbol{g}_{z0} \end{aligned} \quad \cdots\cdots (42.\text{a}\sim\text{f})$$

in which $\boldsymbol{F}_S$ and $\boldsymbol{M}_S$ are the force and the moment vectors of stress resultants due to surface force σ^α_S acting on the end cross sections of the member, while $\delta \boldsymbol{d}_0$, $\delta \boldsymbol{\alpha}$, $\boldsymbol{p}$ and $\boldsymbol{m}$ are already defined in Section 5. With the help of eq. (37), the components of eq. (42) are related to those in the directions of ($\hat{\boldsymbol{i}}_{x0}$, $\hat{\boldsymbol{i}}_{y0}$, $\hat{\boldsymbol{i}}_{z0}$) as

$$\begin{aligned} &\begin{Bmatrix} \delta\hat{u}_0 \\ \delta\hat{v}_0 \\ \delta\hat{w}_0 \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{Bmatrix}, \quad \begin{Bmatrix} \delta\hat{\alpha}_x \\ \delta\hat{\alpha}_y \\ \delta\hat{\alpha}_z \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} \delta\alpha_x \\ \delta\alpha_y \\ \delta\alpha_z \end{Bmatrix}, \quad \begin{Bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}, \\ &\begin{Bmatrix} \hat{m}_x \\ \hat{m}_y \\ \hat{m}_z \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}, \quad \begin{Bmatrix} \hat{F}{}^c_x \\ \hat{F}{}^c_y \\ \hat{F}{}^c_z \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} \overline{F}{}^c_x \\ \overline{F}{}^c_y \\ \overline{F}{}^c_z \end{Bmatrix}, \quad \begin{Bmatrix} \hat{M}{}^c_x \\ \hat{M}{}^c_y \\ \hat{M}{}^c_z \end{Bmatrix} = [l_{\hat{a}b}] \begin{Bmatrix} \overline{M}{}^c_x \\ \overline{M}{}^c_y \\ \overline{M}{}^c_z \end{Bmatrix} \end{aligned} \quad \cdots\cdots (43.\text{a}\sim\text{f})$$

In the first place, the force-equilibrium equations of eqs. (21. a~c) are transformed into those expressed by the customary components. For simplicity, eqs. (21. a~c) are expressed here by

$$\hat{f}_x + \hat{p}_x = 0, \quad \hat{f}_y + \hat{p}_y = 0, \quad \hat{f}_z + \hat{p}_z = 0 \quad \cdots\cdots (44.\text{a}\sim\text{c})$$

Substituting eq. (43. a) into the terms of eq. (17) in the bracket under the integral sign, the components of the force-equilibrium equations in the directions of ($\boldsymbol{g}_{x0}$, $\boldsymbol{g}_{y0}$, $\boldsymbol{g}_{z0}$) are obtained as follows from the conditions for eq. (17) to hold for any arbitrary virtual displacements (δu_0, δv_0, δw_0).

$$[l_{\hat{a}b}]^T \begin{Bmatrix} \hat{f}_x \\ \hat{f}_y \\ \hat{f}_z \end{Bmatrix} + \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = \{0\} \quad (45)$$

With regard to the moment equilibrium equation, the three components in the directions of $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ are necessary in order to carry out the transformation similar to the force equilibrium equations. In eq. (17), howrver, only one component of eq. (21. d) in the direction of $\hat{\boldsymbol{i}}_{z0}$ is found in the term of the work due to the virtual rotation $\delta\hat{\alpha}_z$. Hence, It is necessary to derive the equation of virtual work in the different form which includes the three components, that is to say, the three terms of the work due to the virtual rotations $(\delta\hat{\alpha}_x, \delta\hat{\alpha}_y, \delta\hat{\alpha}_z)$. Helped by eqs. (14) and (16), eq. (12) is transformed into the above mentioned form through the first integration by part which have to be executed in the course of the derivation of eq. (17). Substituting eqs. (43. b, d) into the transformed equation of virtual work and supposing $\delta\alpha_x$, $\delta\alpha_y$, and $\delta\alpha_z$ arbitrary, the components of the moment equilibrium equations in the directions of the vectors $(\boldsymbol{g}_{x0}, \boldsymbol{g}_{y0}, \boldsymbol{g}_{z0})$ are obtained as

$$[l_{\hat{a}b}]^T \begin{Bmatrix} \tilde{M}'_y + \tilde{M}_x\hat{\tau} + \tilde{M}_z\hat{\kappa}_y \\ -(\tilde{M}'_x - \tilde{M}_y\hat{\tau} + \tilde{M}_z\hat{\kappa}_x) \\ \tilde{M}'_z - \tilde{M}_y\hat{\kappa}_y - \tilde{M}_x\hat{\kappa}_x \end{Bmatrix} + \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix} = \{0\} \quad (46)$$

Though eq. (46) is composed of three components, only one of them is independent. It is common that the third component of eq. (52) corresponding to the moment-equilibirum equation around the vector $\boldsymbol{g}_{z0}$ is adopted as an independent equation. However, this component fails to be independent under the specific large rotations which make the direction cosine $l_{\hat{z}z}$ zero. In such a case, an independent equation have to be selected from the remaining components of eq. (46).

The mechanical boundary conditions of eq. (22) are similarly transformed into the components in the directions of the vectors $(\boldsymbol{g}_{x0}, \boldsymbol{g}_{y0}, \boldsymbol{g}_{z0})$. Substituting eqs. (43. a, b) into the integrated terms of eq. (17) and considering δu_0, δv_0, δw_0, $\delta\alpha_x$, $\delta\alpha_y$ and $\delta\alpha_z$ arbitrary, the mechanical boundary conditions in the directions fixed in space are obtained as

$$n_z \begin{Bmatrix} \bar{F}^c_x \\ \bar{F}^c_y \\ \bar{F}^c_z \end{Bmatrix} = [l_{\hat{a}b}]^T \begin{Bmatrix} \{\tilde{M}'_x - \tilde{M}_y\hat{\tau} + \tilde{M}_z\hat{\kappa}_x - (l_{\hat{y}x}m_x + l_{\hat{y}y}m_y + l_{\hat{y}z}m_z)\}/\sqrt{\hat{g}_0} \\ \{\tilde{M}'_y + \tilde{M}_x\hat{\tau} + \tilde{M}_z\hat{\kappa}_y - (l_{\hat{x}x}m_x + l_{\hat{x}y}m_y + l_{\hat{x}z}m_z)\}/\sqrt{\hat{g}_0} \\ \tilde{N} \end{Bmatrix} \quad (47)$$

$$n_z \begin{Bmatrix} \bar{M}^c_x \\ \bar{M}^c_y \\ \bar{M}^c_z \end{Bmatrix} = [l_{\hat{a}b}]^T \begin{Bmatrix} \tilde{M}_y \\ -\tilde{M}_x \\ \tilde{M}_z \end{Bmatrix} \quad (48)$$

The associated geometrical boundary conditions corresponding to eqs. (47) and (48) are given by the displacement components the same as those of $(\delta u_0, \delta v_0, \delta w_0, \delta\alpha_x, \delta\alpha_y, \delta\alpha_z)$.

Eqs. (45) ~ (48) show that the exact governing equations in components in the directions fixed in space are easily derived from eqs. (21) and (22) as a result of orthogonal transformation.

However, even the rotational components $(\delta\alpha_x, \delta\alpha_y, \delta\alpha_z)$ given as geometrical boundary conditions cannot be easily expressed by physical quantities in case of finite rotation, unless more than two components are fully restrained[3]. Therefore, it is necessary further to transform the mechanical boundary conditions into those corresponding to the geometrical boundary conditions expressed by the physical rotational quantities.

In section 7, direction cosines are introduced as physical quantities to express the finite rotations. and hence, it is convenient that the geometrical boundary conditions are given with respect to the direction cosines instead of the rotational angles. In such a case, the mechanical boundary conditions of eqs. (22. d∼f) or eq. (48) have to be transformed into those corresponding to the direction cosines. By the variation of eq. (37) helped by eq. (A-1) in Appendix A, the virtual increments of the rotational angles ($\delta\hat{\alpha}_x$, $\delta\hat{\alpha}_y$, $\delta\hat{\alpha}_z$) are related to those of direction cosines $[\delta l_{\hat{a}b}]$ as

$$\begin{bmatrix} 0 & \delta\hat{\alpha}_z & -\delta\hat{\alpha}_y \\ -\delta\hat{\alpha}_z & 0 & \delta\hat{\alpha}_x \\ \delta\hat{\alpha}_y & -\delta\hat{\alpha}_x & 0 \end{bmatrix} = [\delta l_{\hat{a}b}][l_{\hat{a}b}]^T \qquad (49)$$

Though the matrix $[\delta l_{\hat{a}b}]$ includes nine components, independent components are reduced to three from eq. (49) and are related to ($\delta\hat{\alpha}_x$, $\delta\hat{\alpha}_y$, $\delta\hat{\alpha}_z$) symbolically as

$$\begin{Bmatrix} \delta\hat{\alpha}_x \\ \delta\hat{\alpha}_y \\ \delta\hat{\alpha}_z \end{Bmatrix} = [f(l_{\hat{a}b})] \begin{Bmatrix} \delta l_{\hat{i}j} \\ \delta l_{\hat{k}l} \\ \delta l_{\hat{m}n} \end{Bmatrix} \qquad (50)$$

where ($\delta l_{\hat{i}j}$, $\delta l_{\hat{k}l}$, $\delta l_{\hat{m}n}$) are the three independent components of $[\delta l_{\hat{a}b}]$ and $[f(l_{\hat{a}b})]$ is the matrix function of $l_{\hat{a}b}$ derived easily from eq. (49). Substituting eq. (50) into the integrated terms of eq. (17) and considering $\delta l_{\hat{i}j}$, $\delta l_{\hat{k}l}$, and $\delta l_{\hat{m}n}$ arbitrary, the mechanical boundary conditions corresponding to the geometrical boundary conditions expressed by independent components of direction cosines ($l_{\hat{i}j}$, $l_{\hat{k}l}$, $l_{\hat{m}n}$) are obtained in the manner similar to the derivations of eq. (48).

Comparing eqs. (21) and (22) with eqs. (45) ∼ (47) and the mechanical boundary conditions derived from eq. (50), the former governing equations are easier to be interpreted physically. Nevertheless, the geometrical boundary conditions associated with eq. (21) and (22) cannot be exactly expressed by physical quantities. Therefore, the equilibrium equations of (21) have to be used with the mechanical boundary conditions of eq. (47) and those derived from eq. (50).

9. CONCLUDING REMARKS

A new Lagrangian formulation through the principle of virtual work is presented for the finite displacement theory of curved and twisted rods with solid cross section. In this formulation, the deformation components as well as the virtual displacement components with separation of rigid body rotations are introduced instead of the customary displacement components defined in terms of the coordinates fixed in space. With these new components, the derivations of the governing equations are considerably simplified and the exact governing equations can be easily obtained for the theory of finite displacements with finite strains.

Appendix A Derivations of Eqs. (14. a∼d) and (15. a, b)

In the first place, eqs. (14. d) and (15. a, b) are derived. The vectors ($\hat{\boldsymbol{i}}_{x0}+\delta\hat{\boldsymbol{i}}_{x0}$, $\hat{\boldsymbol{i}}_{y0}+\delta\hat{\boldsymbol{i}}_{y0}$, $\hat{\boldsymbol{i}}_{z0}+\delta\hat{\boldsymbol{i}}_{z0}$) after small virtual displacements are resolved in the directions of the vectors ($\hat{\boldsymbol{i}}_{x0}$, $\hat{\boldsymbol{i}}_{y0}$, $\hat{\boldsymbol{i}}_{z0}$) by using the virtual rotational angles of $\delta\hat{\alpha}_x$, $\delta\hat{\alpha}_y$ and $\delta\hat{\alpha}_z$ and ignoring nonlinear terms with respect to these rotational angles as

$$\begin{Bmatrix} \hat{\boldsymbol{i}}_{x0}+\delta\hat{\boldsymbol{i}}_{x0} \\ \hat{\boldsymbol{i}}_{y0}+\delta\hat{\boldsymbol{i}}_{y0} \\ \hat{\boldsymbol{i}}_{z0}+\delta\hat{\boldsymbol{i}}_{z0} \end{Bmatrix} = \begin{Bmatrix} \hat{\boldsymbol{i}}_{x0}+\delta\hat{\alpha}_z\hat{\boldsymbol{i}}_{y0}-\delta\hat{\alpha}_y\hat{\boldsymbol{i}}_{z0} \\ \hat{\boldsymbol{i}}_{y0}+\delta\hat{\alpha}_x\hat{\boldsymbol{i}}_{z0}-\delta\hat{\alpha}_z\hat{\boldsymbol{i}}_{x0} \\ \hat{\boldsymbol{i}}_{z0}+\delta\hat{\alpha}_y\hat{\boldsymbol{i}}_{x0}-\delta\hat{\alpha}_x\hat{\boldsymbol{i}}_{y0} \end{Bmatrix} \qquad (A\text{-}1)$$

The third component of eq. (A-1) yields

$$\delta\hat{\alpha}_y\hat{\boldsymbol{i}}_{x0}-\delta\hat{\alpha}_x\hat{\boldsymbol{i}}_{y0}=\delta\hat{\boldsymbol{i}}_{z0} \qquad (A\text{-}2)$$

Eq. (A-2) indicates that the expressions of $\delta\hat{\alpha}_x$ and $\delta\hat{\alpha}_y$ with the independent virtual displacements can be obtained by comparison of the coefficients between both sides of eq. (A-2), if $\delta\hat{\boldsymbol{i}}_{z0}$ is resolved into the directions of ($\hat{\boldsymbol{i}}_{x0}$, $\hat{\boldsymbol{i}}_{y0}$).

The variation of eq. (5. b) leads to

$$\delta\hat{\boldsymbol{i}}_{z0}=\delta\hat{\boldsymbol{g}}_{z0}/\sqrt{\hat{g}_0}-\delta\hat{g}_0/2\hat{g}_0\cdot\hat{\boldsymbol{i}}_{z0} \qquad \text{(A-3)}$$

$\delta\hat{\boldsymbol{g}}_{z0}$ in eq. (A-3) is resolved in the directions of $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ as

$$\delta\hat{\boldsymbol{g}}_{z0}=(\delta\hat{u}_0'+\hat{\kappa}_y\delta\hat{w}_0-\hat{\tau}\delta\hat{v}_0)\hat{\boldsymbol{i}}_{x0}+(\delta\hat{v}_0'+\hat{\tau}\delta\hat{u}_0-\hat{\kappa}_x\delta\hat{w}_0)\hat{\boldsymbol{i}}_{y0}+(\delta\hat{w}_0'+\hat{\kappa}_x\delta\hat{v}_0-\hat{\kappa}_y\delta\hat{u}_0)\hat{\boldsymbol{i}}_{z0} \qquad \text{(A-4)}$$

by substituting eq. (5. a) into $\delta\hat{\boldsymbol{g}}_{z0}$ defined by

$$\delta\hat{\boldsymbol{g}}_{z0}=\partial(\delta\hat{\boldsymbol{R}}_0)/\partial z, \quad \delta\hat{\boldsymbol{R}}_0=\delta\hat{u}_0\hat{\boldsymbol{i}}_{x0}+\delta\hat{v}_0\hat{\boldsymbol{i}}_{y0}+\delta\hat{w}_0\hat{\boldsymbol{i}}_{z0} \qquad \text{(A-5. a, b)}$$

$\delta\hat{g}_0$ in eq. (A-3) is further expressed by the independent virtual displacements as

$$\delta\hat{g}_0=2\sqrt{\hat{g}_0}\,(\delta\hat{w}_0'+\hat{\kappa}_x\delta\hat{v}_0-\hat{\kappa}_y\delta\hat{u}_0) \qquad \text{(A-6)}$$

by substituting eq. (A-4) into the right side of the equation

$$\hat{g}_0+\delta\hat{g}_0=|\hat{\boldsymbol{g}}_{z0}+\delta\hat{\boldsymbol{g}}_{z0}|^2 \qquad \text{(A-7)}$$

and taking the linear terms of virual displacements.

Comparison of the coefficients of the vectors $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0})$ between the right and the left side of eq. (A-2) with the help of eqs. (A-3), (A-4) and (A-6) leads to eqs. (15. a, b), while eq. (14. d) is derived through the substitution of eq. (A-6) into the relation

$$\delta\sqrt{\hat{g}_0}=\delta\hat{g}_0/2\sqrt{\hat{g}_0} \qquad \text{(A-8)}$$

Next derived are eqs. (14. a~c). Differentiation of eq. (A-1) helped by eq. (5. a) leads to

$$\frac{d}{dz}\begin{Bmatrix}\delta\hat{\boldsymbol{i}}_{x0}\\ \delta\hat{\boldsymbol{i}}_{y0}\\ \delta\hat{\boldsymbol{i}}_{z0}\end{Bmatrix}=\begin{bmatrix}-\hat{\tau}\delta\hat{\alpha}_z-\hat{\kappa}_y\delta\hat{\alpha}_y & \delta\hat{\alpha}_z'+\hat{\kappa}_x\delta\hat{\alpha}_y & -\delta\hat{\alpha}_y'+\hat{\kappa}_x\delta\hat{\alpha}_z\\ -\delta\hat{\alpha}_z'+\hat{\kappa}_y\delta\hat{\alpha}_x & -\hat{\tau}\delta\hat{\alpha}_z-\hat{\kappa}_x\delta\hat{\alpha}_x & \delta\hat{\alpha}_x'+\hat{\kappa}_y\delta\hat{\alpha}_z\\ \delta\hat{\alpha}_y'+\hat{\tau}\delta\hat{\alpha}_x & -\delta\hat{\alpha}_x'+\hat{\tau}\delta\hat{\alpha}_y & -\hat{\kappa}_y\delta\hat{\alpha}_y-\hat{\kappa}_x\delta\hat{\alpha}_x\end{bmatrix}\begin{Bmatrix}\hat{\boldsymbol{i}}_{x0}\\ \hat{\boldsymbol{i}}_{y0}\\ \hat{\boldsymbol{i}}_{z0}\end{Bmatrix} \qquad \text{(A-9)}$$

In a similar way, variation of eq. (5. a) helped by eq. (A-1) leads to

$$\frac{d}{dz}\begin{Bmatrix}\delta\hat{\boldsymbol{i}}_{x0}\\ \delta\hat{\boldsymbol{i}}_{y0}\\ \delta\hat{\boldsymbol{i}}_{z0}\end{Bmatrix}=\left[\begin{bmatrix}0 & \delta\hat{\tau} & -\delta\hat{\kappa}_y\\ -\delta\hat{\tau} & 0 & \delta\hat{\kappa}_x\\ \delta\hat{\kappa}_y & -\delta\hat{\kappa}_x & 0\end{bmatrix}+\begin{bmatrix}0 & \hat{\tau} & -\hat{\kappa}_y\\ -\hat{\tau} & 0 & \hat{\kappa}_x\\ \hat{\kappa}_y & -\hat{\kappa}_x & 0\end{bmatrix}\begin{bmatrix}0 & \delta\hat{\alpha}_z & -\delta\hat{\alpha}_y\\ -\delta\hat{\alpha}_z & 0 & \delta\hat{\alpha}_x\\ \delta\hat{\alpha}_y & -\delta\alpha_x & 0\end{bmatrix}\right]\begin{Bmatrix}\hat{\boldsymbol{i}}_{x0}\\ \hat{\boldsymbol{i}}_{y0}\\ \hat{\boldsymbol{i}}_{z0}\end{Bmatrix} \qquad \text{(A-10)}$$

Comparison of the components of matrices between eqs. (A-9) and (A-10) gives eqs. (14. a~c).

Apendix B Derivations of Eqs. (18. a~d) by the Integration of Stress

The internal force $d\boldsymbol{F}$ acting on the small cross sectional area dA $(=dxdy)$ is expressed by stress tensor in the form[1)]

$$d\boldsymbol{F}=(\sigma^{zx}\hat{\boldsymbol{g}}_x+\sigma^{zy}\hat{\boldsymbol{g}}_y+\sigma^{zz}\hat{\boldsymbol{g}}_z)\sqrt{g}\,dA \qquad \text{(B-1)}$$

With the help of eq. (7), $d\boldsymbol{F}$ is resolved in the directions of the vectors $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$ as

$$d\boldsymbol{F}=(\sigma^{zx}-\sigma^{zz}y\hat{\tau})\sqrt{g}\,dA\,\hat{\boldsymbol{i}}_{x0}+(\sigma^{zy}+\sigma^{zz}x\hat{\tau})\sqrt{g}\,dA\,\hat{\boldsymbol{i}}_{y0}+\sigma^{zz}(\sqrt{\hat{g}_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)\sqrt{g}\,dA\,\hat{\boldsymbol{i}}_{z0} \qquad \text{(B-2)}$$

Expression of eq. (B-2) with the physical components of stress $(\sigma_z, \tau_{zx}, \tau_{zy})$ defined in the (x, y, z) coordinates is given by

$$d\boldsymbol{F}=(\tau_{zx}-\sigma_z y\hat{\tau}/\sqrt{\hat{g}})\,dA\,\hat{\boldsymbol{i}}_{x0}+(\tau_{zy}+\sigma_z x\hat{\tau}/\sqrt{\hat{g}})\,dA\,\hat{\boldsymbol{i}}_{y0}+\sigma_z(\sqrt{\hat{g}_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)/\sqrt{\hat{g}}\,dA\,\hat{\boldsymbol{i}}_{z0} \qquad \text{(B-3)}$$

taking into account the relation between the physical and the tensor components of strees[6)] such as

$$\sigma^{zz}=\sigma_z/\sqrt{\hat{g}}\sqrt{g}, \quad \sigma^{zx}=\tau_{zx}/\sqrt{g}, \quad \sigma^{zy}=\tau_{zy}/\sqrt{g} \qquad \text{(B-4. a~c)}$$

where

$$\sqrt{\hat{g}}=|\hat{\boldsymbol{g}}_z|=\sqrt{(\sqrt{\hat{g}_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)^2+(x^2+y^2)\hat{\tau}^2} \qquad \text{(B-5)}$$

The physical meaning of eq. (B-2) is easily understood from eq. (B-3) considering that $(-y\hat{\tau}/\sqrt{\hat{g}}$, $x\hat{\tau}/\sqrt{\hat{g}}$, $(\sqrt{\hat{g}_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)/\sqrt{\hat{g}})$ are the direction cosines of $\hat{\boldsymbol{i}}_z$ referred to the vectors $(\hat{\boldsymbol{i}}_{x0}, \hat{\boldsymbol{i}}_{y0}, \hat{\boldsymbol{i}}_{z0})$. With torsional deformation, it should be noted that $\hat{\boldsymbol{i}}_z$ is not orthogonal to the transverse plane except on the member axis.

Integration of eq. (B-2) over the cross sectional area gives the sectional force $\boldsymbol{F}$

$$\boldsymbol{F}=\int_A(\sigma^{zx}-\sigma^{zz}y\hat{\tau})\sqrt{g}\,dA\,\hat{\boldsymbol{i}}_{x0}+\int_A(\sigma^{zy}+\sigma^{zz}x\hat{\tau})\sqrt{g}\,dA\,\hat{\boldsymbol{i}}_{y0}+\int_A\sigma^{zz}(\sqrt{\hat{g}_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)\sqrt{g}\,dA\,\hat{\boldsymbol{i}}_{z0} \qquad \text{(B-6)}$$

From eq. (B-6), it is clear that the coefficient of $\hat{\boldsymbol{i}}_{z0}$, which is the axial force $\tilde{N}$ toward the deformed member axis, coincides with the right side of eq. (18. a).

The sectional moment $\boldsymbol{M}$ is given by

$$\boldsymbol{M}=\int_A \boldsymbol{r}\times d\boldsymbol{F},\quad \boldsymbol{r}=x\hat{\boldsymbol{i}}_{x0}+y\hat{\boldsymbol{i}}_{y0} \quad \cdots\cdots \quad \text{(B-7. a, b)}$$

Substituting eq. (B-2) into eq. (B-7), eq. (B-7) yields

$$\boldsymbol{M}=\int_A \sigma^{zz}(\sqrt{g_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)\, y\sqrt{g}\; dA\hat{\boldsymbol{i}}_{x0}-\int_A \sigma^{zz}(\sqrt{g_0}-x\hat{\kappa}_y+y\hat{\kappa}_x)\, x\sqrt{g}\; dA\hat{\boldsymbol{i}}_{y0}$$
$$+\left\{\int_A(\sigma^{zy}x-\sigma^{zx}y)\sqrt{g}\; dA+\int_A \sigma^{zz}(x^2+y^2)\,\hat{\tau}\sqrt{g}\; dA\right\}\hat{\boldsymbol{i}}_{z0} \quad \cdots\cdots \quad \text{(B-8)}$$

The coefficients of the vectors $(\hat{\boldsymbol{i}}_{x0},\ \hat{\boldsymbol{i}}_{y0},\ \hat{\boldsymbol{i}}_{z0})$ in eq. (B-8), which are the components of the sectional moments $(\tilde{M}_y,\ -\tilde{M}_x,\ \tilde{M}_z)$, exactly coincide with the right sides of eqs. (18. b～d).

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