

A METHOD TO DETERMINE FRACTILE VALUES FROM STATISTICAL DATA

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A method to determine fractile values from statistical data is proposed: the fractile value for an exceedance probability about 0.5 is first determined from the data and next, the correction factor which is also calculated from the data is multiplied to that fractile, yielding the fractile value of a required exceedance probability. By means of Monte Carlo simulation, its accuracy and those of commonly used methods are compared. The result shows that proposed method is the most stable to various distribution models and coefficients of variation.

1. INTRODUCTION

The design format in structural codes generally takes the form

$$\nu(S_d/R_d) \leq 1 \quad (1)$$

where R_d and S_d represent the design strength and the design load effect respectively, while ν is the so-called safety factor accounting for the importance of the structure, social and economical effect caused by the failure of structure, and so on. It has been put forward by Freudenthal et al.^{1,2)} and Nishino et al.³⁾ that the design strength R_d , the design load effect S_d and the factor ν should be determined such that a specified probability of failure is achieved. Therefore, the design strength R_d and the design load effect S_d are usually taken to be the characteristic values (fractile values) corresponding to certain exceedance probabilities e_R and e_s respectively. On the other hand, in the design codes proposed by ISO⁴⁾ and CEB⁵⁾, the design values are determined by multiplying partial safety factors by the fractile values or the nominal values corresponding to a priori specified exceedance probabilities.

In case eq. (1) is adopted for a design format as mentioned above, it is required to estimate the fractile values used as design values from statistical data. The probability of failure of civil engineering structures is regarded as about $10^{-4} \sim 10^{-6}$. Therefore at least either the values of e_R or e_s corresponding to the fractile values to be estimated is very small, though this depends on the selection of the value of ν . Enough information about each random variable is usually needed to estimate the fractile values corresponding to the very small exceedance or non-exceedance probability e_R or e_s accurately. However, some of the design variables have to be estimated from small size of statistical data at present. It is then important to discuss

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how to estimate the fractile value accurately under the condition that the available data is limited.

In general, the fractile value corresponding to the specified exceedance probability is determined based on the optimum distribution type and the values of its parameters estimated from statistical data. It has been shown in reference (7) that the method of maximum likelihood is the most stable one among the commonly used statistical methods. As for the estimation of fractile value by use of the method of maximum likelihood, however, the information about data size used is not included in its process. It may be considered that the data size influences the probability that the estimated optimum distribution type coincides with the true distribution type of the population. This probability can be approximately expressed by the function of likelihood calculated from data⁷⁾. So it may be desirable to incorporate the information about data size into the determination process based on the method of maximum likelihood in order to increase the accuracy of estimation of fractile values. Few works discuss reliability-based design taking into account that the available data is of finite size. Among them, Ref. (8) has discussed the decision on the optimum number of data required to determine the design value of random variable which is the function of some stochastic variables. But in that work the distribution type of population is assumed to be known. In Ref. (9), decision on the optimum distribution type based on the method of maximum likelihood has been discussed, but the determination of fractile value is not an object of study.

The first purpose of this paper is to formulate several methods to determine a fractile value which includes the information about data size. And the second is to compare the accuracy of these methods with that of commonly used method by means of Monte Carlo simulation.

2. DETERMINATION METHOD OF FRACTILE VALUES AND ITS ACCURACY

(1) Formulation of the determination method

In the method of maximum likelihood, the distribution type whose likelihood calculated from statistical data takes the largest value among distribution models considered as a population distribution is selected as the optimum distribution type. Then the following four methods to determine a fractile value corresponding to a required exceedance or non-exceedance probability e based on the maximum likelihood method can be formulated :

① Method in which the fractile value is determined by use of only the optimum distribution type and the values of its parameters estimated from statistical data. This is the commonly used method and is represented by "ES-1 method" for the convenience of description in this paper.

② According to Ref. (7), the probability that each distribution type coincides with the true distribution type of population can be approximately expressed by

$$l_i = \alpha_i / \sum_j^m \alpha_j \dots\dots\dots (2)$$

where α_i is the likelihood corresponding to the i -th distribution model and m is the number of distribution models considered as a population distribution. l_i is named likelihood ratio in this paper. The fractile value X_2 may then be determined by

$$X_2 = \sum_i^m l_i \cdot y_i \dots\dots\dots (3)$$

where y_i is the fractile value of exceedance or non-exceedance probability e under the condition that the i -th distribution model is assumed to be the true population distribution. This method is represented by "ES-2 method".

③ Method in which the fractile value X_3 is determined so as to satisfy the

$$\begin{aligned} e &= \sum_i^m l_i \cdot e_i \\ &= \sum_i^m l_i \cdot F_i(X_3) \quad \text{or} \quad \sum_i^m l_i \cdot (1.0 - F_i(X_3)) \dots\dots\dots (4) \end{aligned}$$

where $F_i(\cdot)$ is the cumulative distribution function of i -th distribution model. This method is represented by "ES-3 method". The difference between ES-2 method and ES-3 method is as follows ; in the former, fractile values corresponding to each distribution model are weighted by likelihood ratio, in the latter, the values of exceedance or non-exceedance probability corresponding to each distribution model are weighted by likelihood ratio.

④ Method in which the fractile value X'_4 for an exceedance probability e' about 0.5 is first determined from the data by means of ES-2 method and next, the correction factor ϕ which is also calculated from the data is multiplied to that fractile, yielding the fractile value X_4 of a required (non-) exceedance probability e as shown in Fig. 1. In this formulation, ϕ is assumed to include both the uncertainty inherent in statistical data and the uncertainty due to the lack of data. So the value of ϕ is calculated by

$$\phi = 1 - k\sigma \quad [\text{in case of the determination of lower fractile}] \quad (5-1)$$

$$\phi = 1 + k\sigma \quad [\text{in case of the determination of upper fractile}] \quad (5-2)$$

where σ denotes a sample standard deviation and k is the constant calculated from the data. Eqs. (5) is introduced intuitively here and the rationale is insufficient. This method is represented by "ES-4 method".

The reason why ES-4 method is formulated here is as follows :

Fig. 2 shows the relation between data size and root mean square error given by eq. (6) when the upper fractile values corresponding to the exceedance probability e are determined in terms of ES-1 method. This figure indicates that the accuracy regarding the estimation of fractile of small exceedance probability is extremely low in case of small data size. For example, the accuracy corresponding to $e=0.01\%$ is about one-fifth as bad as that corresponding to $e=10\%$ for data size $n=10$. Therefore it may be possible that the accuracy of ES-4 method is higher than that of ES-1 method.

For the ES-4 method to be applied in practice, the determination of the constant k in eqs. (5) has to be investigated. The effectiveness of ES-4 method is discussed in this section by examining the existence of the constant k by use of which the accuracy of ES-4 method becomes better than those of ES-1, ES-2 and ES-3 method. This examination is executed by changing the value of k successively.

(2) Process of numerical experiment

The accuracy of the four methods mentioned above is compared based on Monte Carlo simulation. The simulation procedure is as follows ;

- generate the pseudo random realizations of the specified sample size n ($=10, 20, 50, 100$) whose distribution type and population mean are known in the computer.
- determine the fractile corresponding to the exceedance probability e in terms of ES- i method ($i=1, 2, 3, 4$).
- calculate the root mean square error R. M. S. defined by

$$\text{R. M. S.} = \sqrt{\sum_{j=1}^N (X_{i,j} - X_0)^2 / N} / \mu \quad (6)$$

where $X_{i,j}$ is the fractile value estimated in terms of ES- i method at the j -th simulation, X_0 is true fractile value, N is the number of simulations ($=200$) and μ is the population mean that is known.

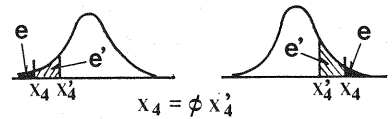


Fig. 1 Method into which a particular factor ϕ is introduced [ES-4 method].

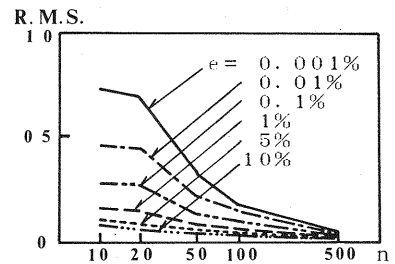


Fig. 2 Relations between data size n and root mean square error R. M. S. of ES-1 method (Population : Log-normal distribution).

The six distribution models described below are adopted in order to generate the sample data because these distribution types are usually adopted as the probabilistic models of the strength and load¹⁰⁾. These models are also used as the supposable population distributions when the estimation of optimum distribution type is made by means of the maximum likelihood method.

- Normal distribution,
- Log-normal distribution,
- Extreme value distribution Type 1 (smallest value),
- Extreme value distribution Type 1 (largest value),
- Extreme value distribution Type 2 (largest value),
- Extreme value distribution Type 3 (smallest value)

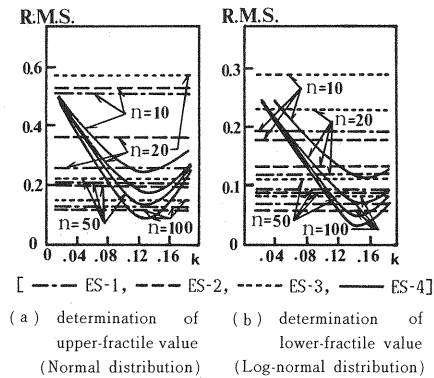


Fig. 3 Relations between factor k and root mean square error R.M.S. of ES-4 method [n : data size].

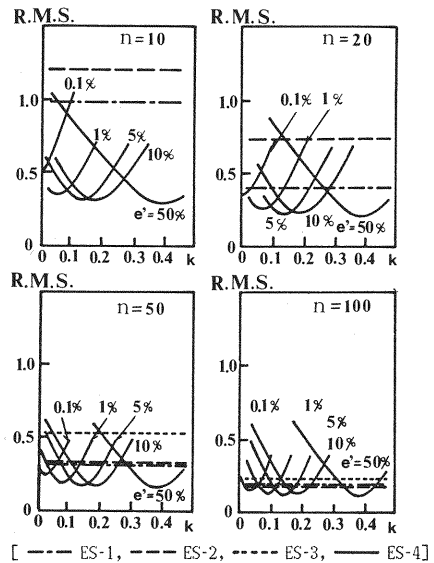


Fig. 4 Relations between value of exceedance probability e' and root mean square error R.M.S. in case that 0.01 % exceedance probability value is determined (population: Normal distribution, C.O.V.=0.3) [n : data size].

(3) Results of numerical experiments and discussion

Typical results of numerical experiments are presented in Fig. 3 (a) and (b), where the abscissa is the factor k included in eqs. (5) and the ordinate is the root mean square error R.M.S. of ES-4 method given by eq. (6). Fig. 3 (a) and (b) correspond to the following cases.

- a) determination of the upper fractile value of 0.1 % exceedance probability from the sample data whose distribution type is Normal and coefficient of variation C.O.V.=0.3
- b) determination of the lower fractile value of 0.1 % non-exceedance probability from the sample data whose distribution type is Log-normal and C.O.V. = 0.3

Though the value of e' used in ES-4 method is assumed to be 10 % here, discussion about the value of e' will be made later on. The values of R.M.S. of ES- i method ($i=1, 2, 3$) are also presented in Fig. 3 (a) and (b).

These two figures indicate that there exists a value of k by use of which the R.M.S. of ES-4 method becomes smaller than those of the other three methods and that this fact is remarkable in the case of small data sizes. The same results are also obtained in cases of other distribution types and other values of C.O.V. ($=0.05\sim0.3$) though they are not shown in figures.

Next, the relations between the value of exceedance probability e' and the root mean square error R.M.S. in the case that the fractile of 0.01 % exceedance probability is determined from sample data are shown in Fig. 4, where the distribution type of the sample data is Normal and its value of C.O.V. equals 0.3. Fig. 4 shows the following three facts :

- (i) If the value of e' satisfies the condition $e < e' < 50\%$, the R.M.S. of ES-4 method is smaller than those of the other three methods regardless of the data size.

(ii) For values of e' between 5 % and 50 %, significant difference among the minimum values of R. M. S. corresponding to each value of e' cannot be recognized in all data sizes. However the minimum values of R. M. S. increase gradually as the value of e' becomes smaller when the data size n is small.

(iii) When the data size n equals 100, there is no significant difference among the minimum values of R. M. S. corresponding to each value of e' regardless of the value of e' .

The same results are obtained in the cases of other distribution types, values of C. O. V. and of the determination of lower fractile values.

The results discussed in this section are summarized as follows : there exists a value of k by use of which the accuracy of ES-4 method becomes better than those of ES- i method ($i=1, 2, 3$) ; it is desirable that e' used in ES-4 method should take the value of about 50 % ; there is no significant difference among the four methods for data size more than 100.

3. DETERMINATION OF FRACTILE VALUE BY MEANS OF CORRECTION FACTOR

It has been revealed in the previous section that ES-4 method is appropriate for the determination of fractile value corresponding to a small (non-) exceedance probability when the size of available data is small. The population distribution of statistical data is however unknown, and it is impossible to find the value of factor k in the same way as executed in section 2. The determination of the factor k is discussed in this section. Calculation formula of the value of k will be proposed first and then the accuracy of ES-4 method using this value will be compared with those of the other three methods based on Monte Carlo simulations.

(1) Calculation formula of correction factor

Fig. 5 presents both the relations between the values of e' and the minimum values of the root mean square error R. M. S. min., and the relations between e' and the factor k with regard to ES-4 method. These two relations can be extracted from Fig. 4, and the values obtained analytically through the following eq. are drawn in broken line in Fig. 5.

$$k_0 = (X_e/X'_e) - 1/\sigma \quad (7)$$

X_e : fractile value corresponding to the exceedance probability e

X'_e : fractile value corresponding to the exceedance probability e'

σ : population standard deviation

It can be recognized from Fig. 5 that the value of k , by means of which the R. M. S. of ES-4 method takes the minimum value, is nearly equal to the value of k_0 for the case where the upper fractile value is determined from the sample data whose distribution type is Normal and C. O. V. = 0.3. Other cases have shown the same results. In other words, it can be concluded that the value of k used in ES-4 method must take approximately the same value as k_0 and that correction factor ϕ is given by

$$\phi = X_e/X'_e \quad (8)$$

where X_e and X'_e are the fractile values of the exceedance probability e and e' respectively, calculated analytically. The following eq. is adopted here as the calculation formula of ϕ .

$$\phi = \sum_{i=1}^m l_i \cdot \phi_i \quad (9)$$

where ϕ_i and l_i ($i=1 \sim m$) are the correction factor given by (8) and likelihood ratio corresponding to the i -th distribution model respectively. In eq. (9) ϕ_i is calculated analytically from sample mean, sample variance and so on. Eq. (9) is formulated taking into

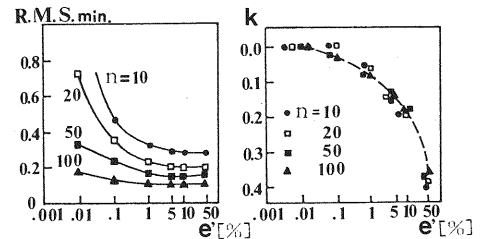


Fig. 5 Relations between value of exceedance probability e' and minimum root mean square error R. M. S. min., and relations between e' and factor k of ES-4 method [n : data size].

account the characteristics of the likelihood ratio mentioned previously.

(2) Results of numerical experiments and discussion

The comparison of the accuracy of ES-4 method, in which correction factor ϕ is given by eq. (9), with those of the other three methods are shown in Fig. 6 and Fig. 7, where the abscissa is data size n and the ordinate is root mean square error R. M. S. given by eq. (6). In Fig. 6 the case that upper fractile of the exceedance probability 0.1 % is determined is presented, and in Fig. 7 the case of estimation of the fractile corresponding to the non-exceedance probability 0.1 % is presented. Distribution type and the value of C. O. V. symbolized by V of the data are shown in each figure. The value of e' used in ES-4 method is equal to 10 % here. From these figures it is found that the accuracy of each method does not depend on the value of C. O. V. of the data and that there is no significant difference among the accuracy of four methods for data size $n=100$. The same results are found in other cases not shown in figures. Consequently, discussion about the accuracy of the four methods will be narrowed down to the cases that the data sizes are small i. e. $n=10\sim50$. The results of qualitative comparison of accuracy among four methods are presented in Table 1, where three symbols \bigcirc , \triangle and \times indicate that accuracy is 'relatively good', 'medium', and 'relatively bad', respectively. This comparison is made for each distribution type of data. The following facts are obtained from Table 1 ;

(i) ES-4 method is the most stable for various distribution types and coefficients of variation, and

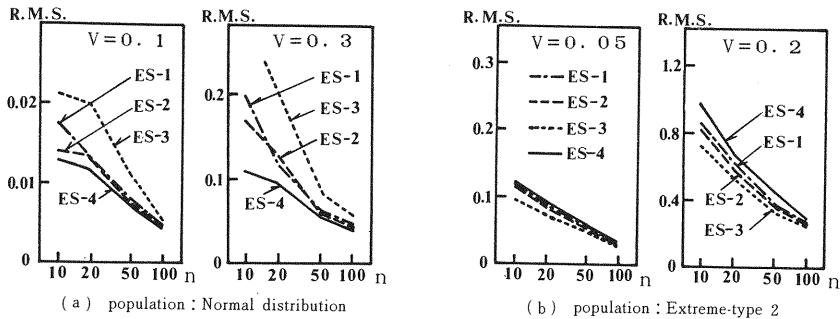


Fig.6 Comparison of accuracy among four methods in case that value of 0.1 % exceedance probability is determined.

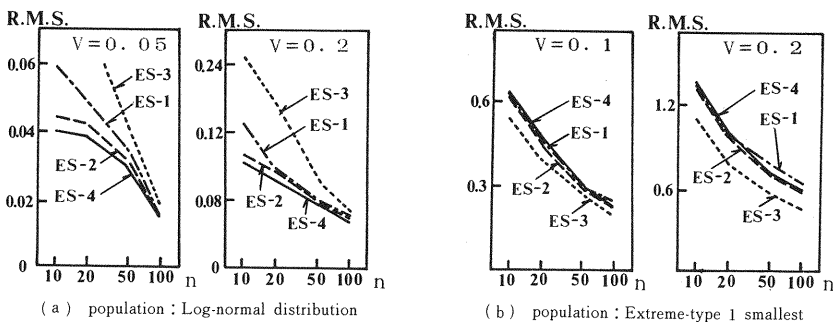


Fig.7 Comparison of accuracy among four methods in case that value of 0.1 % non-exceedance probability is determined.

Table 1 Relative comparison of accuracy among four methods.

(a) determination of upper-fractile value

Method	Population Distribution					
	N	LN	EXT1S	EXT1L	EXT2	EXT3
ES-1	\triangle	\times	\bigcirc	\triangle	\triangle	\triangle
ES-2	\triangle	\triangle	\bigcirc	\triangle	\triangle	\triangle
ES-3	\times	\times	\times	\bigcirc	\bigcirc	\times
ES-4	\bigcirc	\bigcirc	\bigcirc	\bigcirc	\times	\bigcirc

(b) determination of lower-fractile value

Method	Population Distribution					
	N	LN	EXT1S	EXT1L	EXT2	EXT3
ES-1	\times	\times	\triangle	\triangle	\bigcirc	\times
ES-2	\triangle	\triangle	\triangle	\triangle	\triangle	\triangle
ES-3	\times	\times	\bigcirc	\times	\times	\bigcirc
ES-4	\bigcirc	\bigcirc	\triangle	\bigcirc	\triangle	\bigcirc

ES-2 method is more stable than both ES-1 and ES-2 methods ;

- (ii) the accuracy of ES-1 method is relatively bad in comparison with the other three methods despite the fact that it is the commonly used method ;
 - (iii) the accuracy of ES-3 method is relatively good in cases of determination of upper fractile from the data whose distribution type is Extreme Type 2 and determination of lower fractile from the data of Extreme Type 1 smallest value distribution. However, in other cases, ES-3 method is unstable.
- As for ES-4 method, fractile value determined from statistical data X_4 is expressed by

$$\begin{aligned} X_4 &= (\sum l_i \cdot \phi_i) \times (\sum l_i \cdot y'_i) \\ &= \sum l_i^2 \cdot (\phi_i y'_i) + \sum \sum l_i \cdot l_j \cdot \phi_i \cdot y'_j \\ &= \sum l_i^2 \cdot y_i + \sum \sum l_i \cdot l_j \cdot \phi_i \cdot y'_j \dots \dots \dots (10) \end{aligned}$$

where y'_i is the characteristic value of the exceedance probability e' corresponding to the i -th distribution model. Comparing eq. (10) with eq. (3), the former is approximately equivalent to the case that l_i^2 is substituted into the latter in place of l_i , the weight factor for y_i . Taking account of this, it may be considered that the weight factor for y_i included in ES-2 method should be replaced by w_i defined as

$$w_i = \alpha_i^z / \sum_j \alpha_j^z \dots \dots \dots (11)$$

in place of l_i and that the optimum value of z should be searched. The accuracy of ES-2 method in which eq. (11) replaces l_i and that of ES-4 method are compared by changing the value of z . Though its result is not shown in figure or table here, the former cannot become better than the latter for $z=1/2, 2, 3$, and 4.

As mentioned in section 2, it is necessary to select the value of e' which is the (non-) exceedance probability of characteristic value estimated in the first step of ES-4 method. Its value should be selected in order to minimize the root mean square error given by eq. (6). Fig. 8 (a) and (b) show the relations between the value of e' and root mean square error R. M. S. for four different values of data size n . In those cases the (non-) exceedance probability of fractile value is 0.01 %. The distribution type and C. O. V. are shown in each figure. From Fig. 8 the same results as obtained from Fig. 4 can be concluded. That is to say, the optimum value of e' used in ES-4 method may be about 50 %.

(3) In the case that distribution type of data is known a priori

If the deduction or formulation of the distribution model that represents the properties of random phenomenon is easy, the appropriate distribution type may be obtained theoretically¹¹⁾. In this case the following method to determine a fractile can be formulated in addition to the four methods proposed in section 2.

⑤ Method in which theoretically derived distribution type and the values of distribution parameters calculated from statistical data are used in order to determine a fractile value. This method is represented

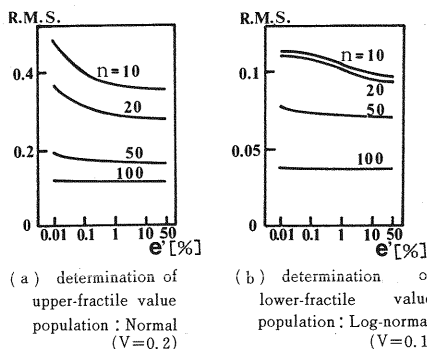


Fig. 8 Relations among value of exceedance probability e' , root mean square error R. M. S. and data size n in regard to ES-4 method.

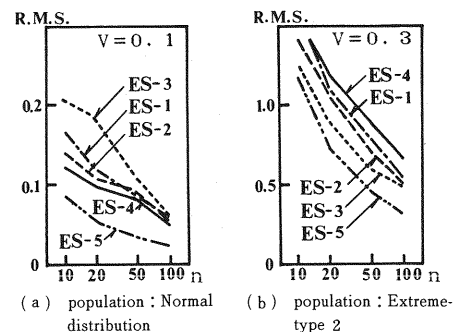


Fig. 9 Comparison of accuracy among 5 methods in case that value of 0.1 % exceedance probability is determined.

by "ES-5 method".

In general ES-5 method may be suitable for the estimation of fractiles in case that the population distribution type is a priori known. But it is necessary to give any confirmation to this thought.

Under the condition that the distribution type of statistical data is a priori known and fractile of 0.1 % exceedance probability has to be determined, the comparisons of accuracy among five methods are presented in Fig. 9 (a) and (b), where the same representation as Fig. 6 and Fig. 7 is adopted. Fig. 9 (a) and (b) correspond to the following cases.

- (a) distribution type of statistical data=Normal, C. O. V. =0.1
- (b) distribution type of statistical data=Extreme value Type 2, C. O. V. =0.3

The value of e' used in ES-4 method is 10% here, too. As expected, it can be seen that ES-5 method is most suitable for the determination of fractile in case that distribution type of statistical data is a priori known.

4. DETERMINATION OF FRACTILE VALUE TAKING ACCOUNT OF THE TOTAL COST OF STRUCTURE

In the previous section the accuracy of four methods is discussed from the point of view of the root mean square error. When the design values included in structural design codes are determined from each statistical data, however, it should be considered that the total cost of structure would depend considerably on whether the design values shift toward safer region or not. In Ref. (12), as a result of investigating the influence of the gap between selected safety level and optimum safety level on the total cost of structure, it has been pointed out that the safety level of structure should be desirable to be shifted more or less to safe region from the view point of minimum total cost. So the accuracy of ES-1, ES-2, ES-3, and ES-4 method is discussed taking this situation into account in this section. Concretely speaking, the accuracy of these four methods is compared by means of the judgement measure called "weighted root mean square error".

(1) Formulation of weighted root mean square error

The function defined by the following equation is assumed as a weight factor.

$$Y = X - 1 + \exp(-X) \dots\dots\dots (12)$$

where X represents the gap between estimated value and true value, and Y represents the weight corresponding to X , as shown in Fig. 10. The variable X ought to be evaluated taking account of the relations between design values and total cost of structure. It is very difficult to do so at present, however, the following assumption is introduced for simplicity ; X takes the value ± 1 when a determined fractile value deviates from the true value by $\pm 1 \sigma$ (σ : population standard deviation). Therefore it is possible only to compare the accuracy among four methods relatively.

According to the assumption mentioned above, the weighted root mean square error is expressed by

$$R. M. S. = \left[\frac{1}{N} \sum_j^N \{ (X_{i,j} - X_0) (t - 1 + e^{-t}) \}^2 \right]^{\frac{1}{2}} / \mu \dots\dots\dots (13)$$

where $t = (X_{i,j} - X_0) / \sigma$, $X_{i,j}$ = fractile value determined by means of ES- i method at the j -th simulation, X_0 = true fractile value, N = number of simulations, σ = population standard deviation, and μ = population mean.

(2) Results of numerical experiments and discussion

The numerical experiments are excuted under the same conditions as shown in Fig. 6 and Fig. 7, and the results are presented in Fig. 11 and Fig. 12, where the abscissa is data size n and the ordinate is the weighted root mean square error calculated through eq. (13) instead of eq. (6). Analogous to the case that eq. (6) is applied, it is found that the accuracy of each method does not

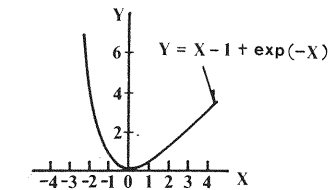


Fig. 10 Weight function in consideration of the difference from optimum level.

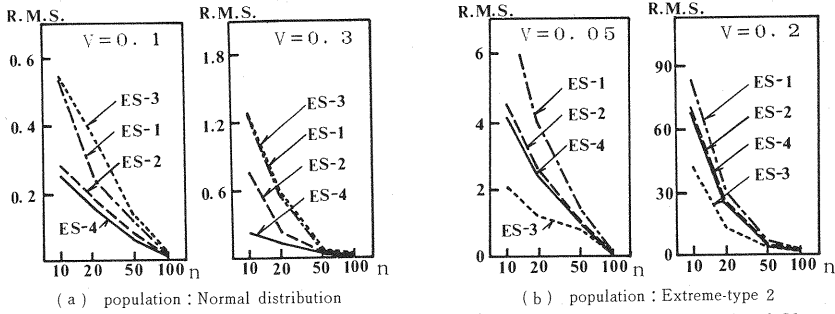


Fig. 11 Comparison of accuracy among four methods in case that weight function is considered [determination of value of 0.1 % exceedance probability].

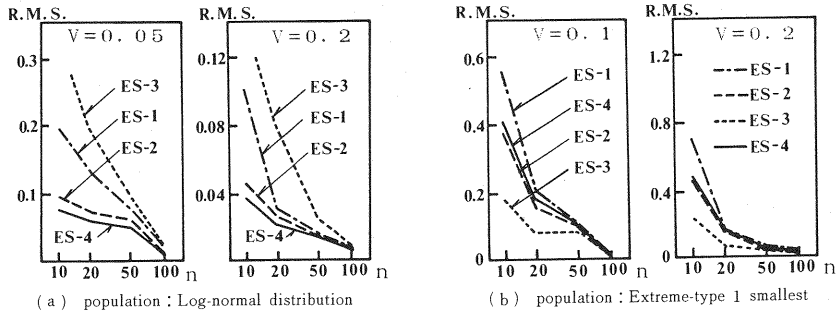


Fig. 12 Comparison of accuracy among four methods in case that weight function is considered [determination of value of 0.1 % non-exceedance probability].

Table 2 Relative comparison of accuracy among four methods in case that weight function is considered.

(a) determination of upper-fractile value

Method	Population Distribution					
	N	LN	EXT1S	EXT1L	EXT2	EXT3
ES-1	×	×	×	×	×	×
ES-2	△	○	△	△	△	△
ES-3	×	△	×	×	○	×
ES-4	○	○	○	△	△	○

(b) determination of lower-fractile value

Method	Population Distribution					
	N	LN	EXT1S	EXT1L	EXT2	EXT3
ES-1	×	×	×	△	△	×
ES-2	△	△	△	○	○	△
ES-3	×	×	×	×	×	×
ES-4	○	○	△	○	○	△

depend on the value of C. O. V. of statistical data and that there is no significant difference among the accuracy of four methods for data size $n=100$. Then in the same manner as Table 1, Table 2 is made in order to compare relatively the accuracy of four methods when the weighted root mean square error are used. From Table 2 the notable differences cannot be recognized in comparison with the case that weight factor is not considered. That is to say, ES-4 method is the most stable for various distribution types and coefficients of variations when the influence of the gap between determined fractile value and true fractile value on the total cost of structure is taken into account. And the accuracy of the commonly used ES-1 method is not good in this situation, too.

5. CONCLUDING REMARKS

Some methods to determine fractile values from statistical data of small size are formulated, and the accuracy of each method is compared with that of commonly used method based on Monte Carlo simulations. As a result, it has been revealed that the following method is the most stable for various distribution types and coefficients of variations ; the fractile value for an exceedance probability of about 0.5 is first determined from the data and next, the correction factor which is also calculated from the data is multiplied to that fractile, yielding the fractile value of a required exceedance probability. On the other hand, the accuracy of the commonly used method is not as good as that of the proposed method. Furthermore the way

to determine fractile value in the case that the distribution type of the data is a priori known has been discussed.

It should be noted that this paper discusses how to determine fractile values from statistical data and not how to select the optimum exceedance probability corresponding to the design values included in structural design codes.

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