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ACCURACY AND CONVERGENCE OF THE SEPARATION OF RIGID BODY DISPLACEMENTS FOR SPACE FRAMES

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In the finite displacement analysis of space frames, a formulation by means of the ordinary direct Lagrangian method makes the governing equations highly nonlinear and complicated largely due to the finite rotations in the space. For this reason, space frames are more often analyzed by the method with the separation of rigid body displacements, compared with plane frames. Nevertheless, its theoretical equivalence to the solutions of the direct Lagrangian method have not been examined so far except that for plane frames.

This paper examines the theoretical convergence and accuracy of the method applied for the analysis of space frames.

1. INTRODUCTION

In the finite displacement analysis of three-dimensional space frames, a precise evaluation of large rotations is of great concern. A formulation by means of the ordinary direct Lagrangian method with the displacement components defined in terms of the coordinates fixed in space makes the governing equations highly nonlinear and complicated largely due to the finite rotations in the space. This sort of governing equations is not only difficult to derive, but also, even if derived, generally requires much complexed and cumbersome procedures to obtain numerical solutions[1)~3)]. For this reason, space frames with large displacements are often analyzed by the method with the separation of rigid body displacements[4)~11)], called here in acronym the SRBD method[12)]. This method divides a structure of concern into an assemblage of finite elements, and then a large portion of finite displacements is removed from respective elements. This portion is regarded as rigid body displacements, and treated as finite rotations without any restriction on the magnitude of displacements. Helped by the geometrical observation, the remaining portion of the displacements due to the deformation of each element is considered small, and thus can be well approximated by the simplified linear[8),9)] or nonlinear[4)~7),10)] governing differential equations defined for respective local coordinates. With those simplified local differential equations, the solution procedures for the SRBD method become easier, compared with those for the direct Lagrangian method.

Considerable works[4),5),7)~9)] have been done for the precise evaluations of rigid body rotations, and reliable numerical solutions are found to be available, to some extent, without proof for the accuracy.

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However, in addition to the evaluations of finite rotations, a way of the simplification of the local differential equations seems to influence the accuracy of solutions. In other words, an appropriate simplification of the local differential equations is none the less important to produce accurate solutions, however adequately rigid body rotations are evaluated. In the present procedures for the SRBD method for space frames, a simplified linear or nonlinear goverinig equation is introduced only from physical considerations without mathematical proof and its validity as well as accuracy has not been examined theoretically so far except that for plane frames[11),12)].

This paper examines theoretically the convergence and accuracy of the SRBD method for the finite displacement analysis of space frames, in comparison with the solutions of the highly nonlinear differential equations in direct Lagrangian expressions.

Firstly, the direct Lagrangian governing equations for space frames without any restriction on the magnitude of displacements are presented, and then typical simplified local differential equations after the separation of rigid body displacements are given to prepare for the use of the SRBD method. Next, by making use of the Taylor expansion with respect to the element length, the discrete forms of the governing equations in terms of nodal forces and displacements of a finite element are derived both for the direct Lagrangian and the SRBD methods under the same basis of coordinates helped by appropriate coordinate transformations. The accuracy and convergence of the SRBD method are examined by comparing the coincidence of the coefficients of the derived power series[11),12)]. For the ease of mathematical manipulations in this paper, space frames are assumed to consist of doubly symmetric solid straight members under the basic beam assumptions of no change of cross sectional shapes and the Bernoulli-Euler hypothesis, where the warping due to torsion is neglected.

2. DIFFERENTIAL EQUATIONS IN DIRECT LAGRANGIAN EXPRESSION

Consider a space member subject to distributed external forces as shown in Fig. 1. Rectangular Cartesian coordinate system (x, y, z) with base vectors $(\boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{g}_z)$ is introduced at the initial configuration of the member. The coordinates (x, y) are chosen as doubly symmetrical axes of the cross section with their origin at the centroid, and the coordinate z is taken along the centroidal axis of the member.

As is well known, if the Lagrangian differential equations are expressed by the displacement components with respect to the coordinate system (x, y, z) fixed in space, the governing equations become much complicated and the corresponding discrete equations in terms of nodal physical quantities can hardly be obtained. For this reason, a similar formulation is used as that for an inextensional rod originally introduced by Love[13)] in order to simplify the governing equations without introducing any approximation. Instead of common displacement components, in this formulation, the deformation of centroidal axis is represented by four unknowns of κ_x, κ_y, τ and $\sqrt{g_0}-1$ which are defined by

$$[D]\equiv\begin{bmatrix} 0 & \tau & -\kappa_y \\ -\tau & 0 & \kappa_x \\ \kappa_y & -\kappa_x & 0 \end{bmatrix} \quad \cdots\cdots (1\cdot a)$$

such that

$$d(\hat{\boldsymbol{i}}_x, \hat{\boldsymbol{i}}_y, \hat{\boldsymbol{i}}_z)^T/dz=[D](\hat{\boldsymbol{i}}_x, \hat{\boldsymbol{i}}_y, \hat{\boldsymbol{i}}_z)^T \quad \cdots\cdots (1\cdot b)$$

and

$$\hat{\boldsymbol{g}}_{z0}=\sqrt{g_0}\,\hat{\boldsymbol{i}}_z \quad \cdots\cdots (1\cdot c)$$

where $(\hat{\boldsymbol{i}}_x, \hat{\boldsymbol{i}}_y, \hat{\boldsymbol{i}}_z)$ are the unit vectors obtained by normalizing the deformed base vectors $(\hat{\boldsymbol{g}}_{x0}, \hat{\boldsymbol{g}}_{y0}, \hat{\boldsymbol{g}}_{z0})$ of the centroidal axis which turn to be orthogonal due to the beam assumptions cited before. Physically, $(\kappa_x/\sqrt{g_0}, \kappa_y/\sqrt{g_0})$, $\tau/\sqrt{g_0}$ and $\sqrt{g_0}-1$ correspond to the

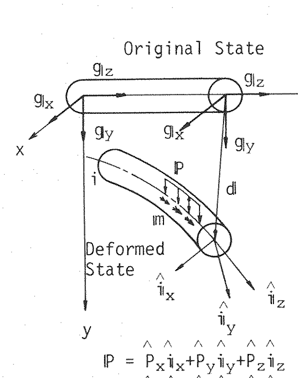

Fig. 1 Coordinate Systems for Direct Lagrangian Expressions.

components of curvature in the directions of the symmetrical axes of cross section, the torsional rate and the extensional rate, respectively, of the deformed centroidal axis. External distributed force $\boldsymbol{p}$ and distributed moment $\boldsymbol{m}$ on the centroidal axis as shown in Fig. 1 are expressed by the components of vectors $(\hat{\boldsymbol{i}}_x, \hat{\boldsymbol{i}}_y, \hat{\boldsymbol{i}}_z)$ as $(\hat{p}_x, \hat{p}_y, \hat{p}_z)$ and $(\hat{m}_x, \hat{m}_y, \hat{m}_z)$. Using the four unknowns and the external force components defined above, the governing differential equations in direct Lagrangian expressions are obtained through a similar way as the Love's formulation. The results are summarized in Table 1, where M_z vs. stress resultant relations as well as the stress resultant vs. $(\kappa_x, \kappa_y, \tau, \sqrt{g_0})$ relations are classified into two levels of nonlinearity.

First is called the theory of ⓐ finite displacements with small strains and has been obtained by the approximation of small strains as

$$|\sqrt{g_0}-1|\ll 1,\quad |\kappa_x y|\ll 1,\quad |\kappa_y x|\ll 1,\quad |\tau y|\ll 1,\quad |\tau x|\ll 1 \quad \cdots\cdots (2\cdot a\sim e)$$

The constitutive equations have been assumed to follow the linear elastic relations as

$$\sigma_{zz}=Ee_{zz},\quad \sigma_{zx}=2\,Ge_{zx},\quad \sigma_{zy}=2\,Ge_{zy} \quad \cdots\cdots (3\cdot a\sim c)$$

in which $(\sigma_{zz}, \sigma_{zx}, \sigma_{zy})$ and (e_{zz}, e_{zx}, e_{zy}) are 2nd Piola-Kirchhoff stress tensor and Green strain tensor defined for the (x, y, z) coordinates. Strain components are found to be expressed under the Bernoulli-Euler hypothesis and the condition of small strains by $(\kappa_x, \kappa_y, \tau, \sqrt{g_0})$ as

Table 1 Direct Lagrangian Expressions.

Theories	M_z	Mechanical Boundary Conditions	Stress Resultants vs. $\kappa_x, \kappa_y, \tau, \sqrt{g_0}$
Equilibrium Equations: $\hat{F}_x'-\hat{F}_y\tau+\hat{F}_z\kappa_y+\hat{p}_x=0$, $\hat{F}_y'+\hat{F}_x\tau-\hat{F}_z\kappa_x+\hat{p}_y=0$, $\hat{F}_z'+\hat{F}_y\kappa_x-\hat{F}_x\kappa_y+\hat{p}_z=0$, $M_z'-M_y\kappa_y-M_x\kappa_x+\hat{m}_z=0$ where $\hat{F}_x=M_x'-M_y\tau+M_z\kappa_x-\hat{m}_y$, $\hat{F}_y=M_y'+M_x\tau+M_z\kappa_y+\hat{m}_x$, $\hat{F}_z=N$			
ⓐ Finite Displacements with Small Strains	$M_z=T_s+K\tau$	$\hat{F}_x=\hat{F}_x^c$ $\hat{F}_y=\hat{F}_y^c$ $\hat{F}_z=\hat{F}_z^c$ $M_x=\hat{M}_x^c$ $M_y=\hat{M}_y^c$ $M_z=\hat{M}_z^c$	$N=EA(\sqrt{g_0}-1)+EJ\tau^2/2$ $M_x=-EI_x\kappa_y$, $M_y=EI_y\kappa_x$ $T_s=GJ\tau$, $K=EJ(\sqrt{g_0}-1)+EJ_{rr}\tau^2/2$
ⓑ Love[13]	$M_z=T_s$		$N=EA(\sqrt{g_0}-1)$ $M_x=-EI_x\kappa_y$, $M_y=EI_y\kappa_x$ $T_s=GJ\tau$

Remarks: The following notations are used throughout Tables and Equations

$M_x=\int_A\sigma_{zz}x\,dA$, $M_y=\int_A\sigma_{zz}y\,dA$, $T_s=\int_A(\sigma_{zy}x-\sigma_{zx}y)dA$, $K=\int_A\sigma_{zz}(x^2+y^2)dA$

$A=\int_A dA$, $I_x=\int_A x^2dA$, $I_y=\int_A y^2dA$, $J=\int_A(x^2+y^2)dA$, $J_{rr}=\int_A(x^2+y^2)^2dA$, $(\cdot)'=d(\cdot)/dz$

$$e_{zz}=\sqrt{g_0}-1+y\kappa_x-x\kappa_y+(x^2+y^2)\tau^2/2,\quad e_{zx}=-y\tau,\quad e_{zy}=x\tau \quad \cdots\cdots (4\cdot a\sim c)$$

With the conditions of small strains, it should be noted that the constitutive equation of (3) with Eq. (4) is coincident with that defined between physical components of stress and strain.

Second is the same as the governing equations originally given by Love[13], excepting additional inclusion of the N vs. $(\sqrt{g_0}-1)$ relation, and is called the theory of ⓑ Love. In this formulation, the constitutive equations have been assumed to relate M_x, M_y, M_z, and N proportional to κ_y, κ_x, τ, and $\sqrt{g_0}-1$, respectively. As clear from Table 1, the difference between the two theories is found only in the constitutive equations for N and M_z. The physical interpretation of this difference is that the theory of ⓐ takes into account the contribution of the axial stress σ_{zz} due to the torsional deformation to the torsional moment as well as that of torsional rate τ to the axial strain e_{zz}, while both of them are neglected in the theory of ⓑ. Since the constitutive equation for M_z in the theory of ⓐ is expressed from Table 1 as

$$M_z=T_s+K\tau=GJ\tau\{1+E(\sqrt{g_0}-1)/G+EJ_{rr}\tau^2/2\,GJ\} \quad \cdots\cdots (5)$$

and both of $E(\sqrt{g_0}-1)/G$ and $\sqrt{EJ_{rr}/GJ}\cdot\tau$ are considered the order of the magnitude of strain, the second and the third terms of the right hand side of Eq. (5) can be ignored compared with unity by the reason of Eqs. (2·a, d, e). In a similar way, the constitutive equation for N in the theory of ⓐ is transformed as

$$N=EA(\sqrt{g_0}-1)\{1-\sqrt{J/A}\,\tau\cdot\sqrt{J/A}\,\tau/2(\sqrt{g_0}-1)\} \quad \cdots\cdots (6)$$

where it is noted that $\sqrt{J/A}\,\tau$ and $\sqrt{g_0}-1$ are the same magnitude as shear strain and axial strain, respectively. If the shear strain is assumed the same order of magnitude as the axial strain in the centroidal axis, the term $J\tau^2/A(\sqrt{g_0}-1)$ can also be neglected compared with unity because of the conditions of small strains. Therefore, with the conditions of small strains and the equivalence of magnitude between the axial and shear strains, the constitutive equations for the theory of ⓐ are nearly equal to those for the theory of ⓑ, and thus, it is concluded that little difference is expected between the solutions of the two theories classified in Table 1.

3. SIMPLIFIED LOCAL DIFFERENTIAL EQUATIONS FOR THE SRBD METHOD

As shown in Fig. 2, the local coordinate system $(\bar{x}, \bar{y}, \bar{z})$ is introduced and defined for the finite element i, $i+1$ after eliminating rigid body displacements such that the base vectors coincide with the unit orthogonal vectors $(\hat{\boldsymbol{i}}_x, \hat{\boldsymbol{i}}_y, \hat{\boldsymbol{i}}_z)$ at node i, indicating that the local coordinates move with the rigid body displacement of node i. With the local coordinates defined above, the displacement vector $\bar{\boldsymbol{d}}_0$ of the centroidal axis after eliminating the rigid body displacements can be expressed by the position vector $\hat{\boldsymbol{R}}_0$ of the deformed centroidal axis as

$$\bar{\boldsymbol{d}}_0=\hat{\boldsymbol{R}}_0-\hat{\boldsymbol{R}}_{0i}-\bar{z}\hat{\boldsymbol{i}}_{zi},\quad \bar{z}=z-z_i \qquad (7\cdot\text{a, b})$$

where subscript i indicates the quantities at node i.

The local differential equations of a specific finite element are expressed by the force and displacement components in terms of the local $(\bar{x}, \bar{y}, \bar{z})$ coordinates as

$$\bar{\boldsymbol{d}}_0=(\bar{u}_0, \bar{v}_0, \bar{w}_0)^T,\quad \bar{\boldsymbol{p}}=(\bar{p}_x, \bar{p}_y, \bar{p}_z)^T,\quad \boldsymbol{m}=(\bar{m}_x, \bar{m}_y, \bar{m}_z)^T \qquad (8\cdot\text{a}\sim\text{c})$$

It is noted henceforth that the components of vectors $(\hat{\boldsymbol{i}}_x, \hat{\boldsymbol{i}}_y, \hat{\boldsymbol{i}}_z)$ are distinguished from those of $(\hat{\boldsymbol{i}}_{xi}, \hat{\boldsymbol{i}}_{yi}, \hat{\boldsymbol{i}}_{zi})$ by the notations respectively as $(\hat{\cdot})$ and $(\bar{\cdot})$.

The local simplified differential equations to be examined here are summarized in Table 2, which are called the theories of ⓒ Nishino[14] and Yuki[10], ⓓ Maeda·Hayashi[5], ⓔ beam-column[4),6),7] and ⓕ small displacements[8),9]. All of them have been used so far for the SRBD method to analyze space frames, and can be understood as further simplifications of the theory of ⓐ or ⓑ in direct Lagrangian expression.

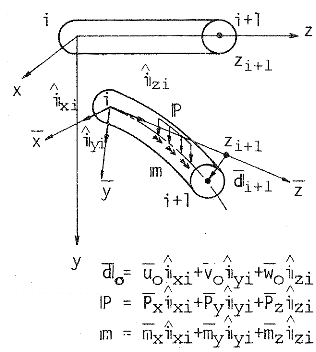

Fig. 2 Coordinate Systems for the Separation of Rigid Body Displacements.

Among those listed in Table 2, the equation of ⓒ is considered most accurate in the sense that it covers wider range of nonlinearity, compared with the other equations, although this equation is too complicated to apply for the SRBD method[10].

The equation of ⓓ corresponds to the local displacement field used in the SRBD method of Ref 5), where the local stiffness equation has been derived directly from the theory of minimum potential energy. Although this equation includes nonlinear terms similar to the equation of ⓒ, it seems less accurate in the consideration on equilibrium.

The equation of ⓔ is obtained from the equation of ⓒ or ⓓ by neglecting the nonlinear terms relating to torsional displacement. This equation is relatively simple and accurate to apply for the SRBD method and thus has been used most frequently and conveniently in both for plane and space frames.

The equation of ⓕ is well known as the small displacement theory, and is the simplest only with linear terms. Although rarely applied for the analysis of plane frames, this equation of small displacements is sometimes used for that of space frames for the ease of computation.

All the local equations listed in Table 2 have been derived through the considerable approximations by the condition of relatively small displacements. Because of those approximations, some of the physical

Table 2 Expressions with Separation of Rigid Body Displacements.

Theories	Equilibrium Equation	Boundary Conditions		Stress Resultants vs. Displacements
		Mechanical	Geometrical	
ⓒ Nishino[14] and Yuki[10]	$\{(M_x-M_y\phi)'+N\bar{u}_o'-\bar{m}_y\}'+\bar{p}_x=0$ $\{(M_y+M_x\hat{\phi})'+N\bar{v}_o'+\bar{m}_x\}'+\bar{p}_y=0$ $N'+\bar{p}_z=0$ $M_z'+M_x\bar{v}_o''-M_y\bar{u}_o''+\hat{m}_z=0$	$(M_x-M_y\hat{\phi})'+N\bar{u}_o'-\bar{m}_y=\bar{F}_x^c$ $(M_y+M_x\hat{\phi})'+N\bar{v}_o'+\bar{m}_x+\bar{F}_y^c$ $N=\bar{F}_z^c$ $M_x-M_y\hat{\phi}=\bar{M}_x^c$ $M_y+M_x\hat{\phi}=\bar{M}_y^c$ $M_z=\hat{M}_z^c$	$\bar{u}_o=\bar{u}_o^c$ $\bar{v}_o=\bar{v}_o^c$ $\bar{w}_o=\bar{w}_o^c$ $\bar{u}_o'=\bar{u}_o^{c\prime}$ $\bar{v}_o'=\bar{v}_o^{c\prime}$ $\hat{\phi}=\hat{\phi}^c$	$N=EA\tilde{w}_o'+EJ\hat{\phi}'^2/2$ $M_x=-EI_x(\bar{u}_o''+\bar{v}_o''\hat{\phi})$ $M_y=-EI_y(\bar{v}_o''-\bar{u}_o''\hat{\phi})$ $M_z=T_s+K\hat{\phi}'$
ⓓ Maeda·Hayashi[5]	$(\bar{M}_x'-\bar{M}_y\hat{\phi}'+N\bar{u}_o'-\bar{m}_y)'+\bar{p}_x=0$ $(\bar{M}_y'+\bar{M}_x\hat{\phi}'+N\bar{v}_o'+\bar{m}_x)'+\bar{p}_y=0$ $N'+\bar{p}_z=0$ $(M_z-\bar{M}_y\bar{u}_o'+\bar{M}_x\bar{v}_o')'+\hat{m}_z=0$	$\bar{M}_x'-\bar{M}_y\hat{\phi}'+N\bar{u}_o'-\bar{m}_y=\bar{F}_x^c$ $\bar{M}_y'+\bar{M}_x\hat{\phi}'+N\bar{v}_o'+\bar{m}_x=\bar{F}_y^c$ $N=\bar{F}_z^c$ $\bar{M}_x=\bar{M}_x^c$ $\bar{M}_y=\bar{M}_y^c$ $M_z=\hat{M}_z^c$	$\bar{u}_o=\bar{u}_o^c$ $\bar{v}_o=\bar{v}_o^c$ $\bar{w}_o=\bar{w}_o^c$ $\bar{u}_o'=\bar{u}_o^{c\prime}$ $\bar{v}_o'=\bar{v}_o^{c\prime}$ $\hat{\phi}=\hat{\phi}^c$	$N=EA\tilde{w}'+EJ\hat{\phi}'^2/2$ $\bar{M}_x=-EI_x(\bar{u}_o''-\bar{v}_o'\hat{\phi}')$ $\bar{M}_y=-EI_y(\bar{v}_o''+\bar{u}_o'\hat{\phi}')$ $M_z=T_s+K\hat{\phi}'$
ⓔ Beam -Column	$(\bar{M}_x'+N\bar{u}_o'-\bar{m}_y)'+\bar{p}_x=0$ $(\bar{M}_y'+N\bar{v}_o'+\bar{m}_x)'+\bar{p}_y=0$ $N'+\bar{p}_z=0$ $\bar{M}_z'+\bar{m}_z=0$	$\bar{M}_x'+N\bar{u}_o'-\bar{m}_y=\bar{F}_x^c$ $\bar{M}_y'+N\bar{v}_o'+\bar{m}_x=\bar{F}_y^c$ $N=\bar{F}_z^c$ $\bar{M}_x=\bar{M}_x^c$ $\bar{M}_y=\bar{M}_y^c$ $\bar{M}_z=\bar{M}_z^c$	$\bar{u}_o=\bar{u}_o^c$ $\bar{v}_o=\bar{v}_o^c$ $\bar{w}_o=\bar{w}_o^c$ $\bar{u}_o'=\bar{u}_o^{c\prime}$ $\bar{v}_o'=\bar{v}_o^{c\prime}$ $\bar{\alpha}_z=\bar{\alpha}_z^c$	$N=EA\tilde{w}_o'$ $\bar{M}_x=-EI_x\bar{u}_o''$ $\bar{M}_y=-EI_y\bar{v}_o''$ $\bar{M}_z=GJ\bar{\alpha}_z'$
ⓕ Small Displacements	$(\bar{M}_x'-\bar{m}_y)'+\bar{p}_x=0$ $(\bar{M}_y'+\bar{m}_x)'+\bar{p}_y=0$ $N'+\bar{p}_z=0$ $\bar{M}_z'+\bar{m}_z=0$	$\bar{M}_x'-\bar{m}_y=\bar{F}_x^c$ $\bar{M}_y'+\bar{m}_x=\bar{F}_y^c$ $N=\bar{F}_z^c$ $\bar{M}_x=\bar{M}_x^c$ $\bar{M}_y=\bar{M}_y^c$ $\bar{M}_z=\bar{M}_z^c$	$\bar{u}_o=\bar{u}_o^c$ $\bar{v}_o=\bar{v}_o^c$ $\bar{w}_o=\bar{w}_o^c$ $\bar{u}_o'=\bar{u}_o^{c\prime}$ $\bar{v}_o'=\bar{v}_o^{c\prime}$ $\bar{\alpha}_z=\bar{\alpha}_z^c$	$N=EA\bar{w}_o'$ $\bar{M}_x=-EI_x\bar{u}_o''$ $\bar{M}_y=-EI_y\bar{v}_o''$ $\bar{M}_z=GJ\bar{\alpha}_z'$

Remarks: $\tilde{w}=\bar{w}_o'+(\bar{u}_o'^2+\bar{v}_o'^2)/2$, $T_s=GJ\hat{\phi}'$, $K=EJ\tilde{w}_o'+EJ_{rr}\hat{\phi}'^2/2$

quantities can hardly be identified whether they are defined in terms of the original or the deformed base vectors. Since the difference of the definitions for those physical quantities is expected to influence on the accuracy of the SRBD method, however, the physical quantities are so defined in Table 2 that the accuracy and the convergence of the SRBD method are improved, as discussed later in Section 5. Specifically, the definitions of the physical quantities appeared in Table 2 are that $\hat{\phi}$ in the equations of ⓒ and ⓓ is relatively small rotation around the deformed centroidal axis, while $\bar{\alpha}_z$ in the equations of ⓔ and ⓕ is that around $\bar{z}$ axis, and $(\bar{M}_x, \bar{M}_y, \bar{M}_z)$ are components of internal moment $\boldsymbol{M}$ which are defined for a positive cross section with its normal towards the positive direction of $\bar{z}$ axis as

$$\boldsymbol{M}=\bar{M}_y\hat{\boldsymbol{i}}_{xi}-\bar{M}_x\hat{\boldsymbol{i}}_{yi}+\bar{M}_z\hat{\boldsymbol{i}}_{zi} \quad \cdots\cdots (9)$$

4. DISCRETE EQUATIONS FOR THE NODAL FORCES AND DISPLACEMENTS

(1) Derivation of discrete equations

The discrete equations in terms of the physical quantities at both ends of a finite element i, $i+1$ are obtained from the governing differential equations in Tables 1 and 2 by using the Taylor expansion with respect to the element length $\Delta z=z_{i+1}-z_i$[11),12)]. These discrete equations can be understood to correspond to the solutions of the differential equations. The discrete equations to be derived here are expressed in terms of nodal components for the $(\bar{x}, \bar{y}, \bar{z})$ coordinates after eliminating the rigid body rotation at node i rather than those for the (x, y, z) coordinates fixed in space in order to facilitate simpler mathematical expressions. The nodal values of interest consist of mechanical and geometrical quantities.

As for mechanical quantities, the components of nodal force $\boldsymbol{F}_k$ and nodal moment $\boldsymbol{M}_k$ are defined at node i and $i+1$ of the finite element as

$$\begin{array}{ll} \boldsymbol{F}_i = -\overline{F}_{xi}\hat{\boldsymbol{i}}_{xi} - \overline{F}_{yi}\hat{\boldsymbol{i}}_{yi} - \overline{F}_{zi}\hat{\boldsymbol{i}}_{zi} & \boldsymbol{M}_i = -\overline{M}_{yi}\hat{\boldsymbol{i}}_{xi} + \overline{M}_{xi}\hat{\boldsymbol{i}}_{yi} - \overline{M}_{zi}\hat{\boldsymbol{i}}_{zi} \\ \boldsymbol{F}_{i+1} = \overline{F}_{xi+1}\hat{\boldsymbol{i}}_{xi} + \overline{F}_{yi+1}\hat{\boldsymbol{i}}_{yi} + \overline{F}_{zi+1}\hat{\boldsymbol{i}}_{zi}, & \boldsymbol{M}_{i+1} = \overline{M}_{yi+1}\hat{\boldsymbol{i}}_{xi} - \overline{M}_{xi+1}\hat{\boldsymbol{i}}_{yi} + \overline{M}_{zi+1}\hat{\boldsymbol{i}}_{zi} \end{array} \quad \cdots\cdots (10\cdot\mathrm{a}\sim\mathrm{d})$$

Regarding geometrical quantities, the translational and rotational displacements at node i become zero, after eliminating the rigid body displacement of node i, and hence, only those at node $i+1$ are considered. At node $i+1$, the translation is defined as $(\overline{u}_{0i+1}, \overline{v}_{0i+1}, \overline{w}_{0i+1})$ according to Eq. (8・a), and the rotation is evaluated by the direction cosines $[l_{ab}]$ between the orthogonal unit vectors $(\hat{\boldsymbol{i}}_{xi+1}, \hat{\boldsymbol{i}}_{yi+1}, \hat{\boldsymbol{i}}_{zi+1})$ and $(\hat{\boldsymbol{i}}_{xi}, \hat{\boldsymbol{i}}_{yi}, \hat{\boldsymbol{i}}_{zi})$ which are identical to the base vectors of the $(\overline{x}, \overline{y}, \overline{z})$ coordinates, which are mathematically expressed by

$$\begin{Bmatrix} \hat{\boldsymbol{i}}_{xi+1} \\ \hat{\boldsymbol{i}}_{yi+1} \\ \hat{\boldsymbol{i}}_{zi+1} \end{Bmatrix} = [l_{ab}]_{i+1} \begin{Bmatrix} \hat{\boldsymbol{i}}_{xi} \\ \hat{\boldsymbol{i}}_{yi} \\ \hat{\boldsymbol{i}}_{zi} \end{Bmatrix}, \quad [l_{ab}] = \begin{bmatrix} l_{\hat{x}\overline{x}} & l_{\hat{x}\overline{y}} & l_{\hat{x}\overline{z}} \\ l_{\hat{y}\overline{x}} & l_{\hat{y}\overline{y}} & l_{\hat{y}\overline{z}} \\ l_{\hat{z}\overline{x}} & l_{\hat{z}\overline{y}} & l_{\hat{z}\overline{z}} \end{bmatrix} \cdots\cdots (11\cdot\mathrm{a,\ b})$$

For the simplicity of writing, the vector $\{Q_j\}$ is introduced as

$$\{Q_j\}^t = (\overline{F}_x, \overline{F}_y, \overline{F}_z, \overline{M}_x, \overline{M}_y, \overline{M}_z, \overline{u}_0, \overline{v}_0, \overline{w}_0) \quad (j=1\sim 9) \cdots\cdots (12)$$

The discrete equations for the nodal physical quantities are to be expressed by the power series with the element length Δz and can be derived by the method of the Taylor expansion after tansforming the governing equations of Tables 1 and 2 into the first order differential equations in terms of the physical quanities of concern[11),12)]. Those discrete equations take the form of transferring the physical quanities from node i to $i+1$ as

$$Q_j|_{i+1} = Q_j|_i + \sum_{n=1}^{\infty} \overset{(n)}{Q_j}|_i \Delta z^n/n!, \quad l_{ab}|_{i+1} = \delta_{ab} + \sum_{n=1}^{\infty} \overset{(n)}{l_{ab}}|_i \Delta z^n/n!, \quad \Delta z = z_{i+1} - z_i \cdots\cdots (13\cdot\mathrm{a}\sim\mathrm{c})$$

where δ_{ab} is a Kronecker delta and $(\overset{(n)}{Q_j}|_i, \overset{(n)}{l_{ab}}|_i)$ are the nth order derivatives of (Q_j, l_{ab}) at node i. Those drivatives can be expressed by the physical quantities $\{Q_j\}$ and $[l_{ab}]$ at node i with the successive differentiation and substitution for the derived first order differential equations as well as the introduction of the boundary conditions at node i as

$$(\overline{u}_0, \overline{v}_0, \overline{w}_0)^T = \{0\}, \quad [l_{ab}] = [E] \cdots\cdots (14\cdot\mathrm{a,\ b})$$

where $[E]$ is a unit matrix.

The accuracy and convergence of the SRBD method are examined by the coincidence of the derivatives with those of the direct Lagrangian method[11),12)].

The discrete equations for the nodal force $\boldsymbol{F}_k$ are derived simply from the considration of force equilibrium, irrelevant of geometrical nonlinearity[11),12)]. Because of this, if the rigid body rotation is exactly evaluated at node i, the local equilibrium equations for the SRBD method naturally coincide with those for the direct Lagrangian equations and thus the coincidence of the discrete equations is always assured for the nodal force components $(\overline{F}_x, \overline{F}_y, \overline{F}_z)$. Therefore, the physical quantities whose accuracy have to be examined by the coincidence of the coefficients of Eq. (13) are

$$(\overline{M}_x, \overline{M}_y, \overline{M}_z, \overline{u}_0, \overline{v}_0, \overline{w}_0), \quad [l_{ab}] \cdots\cdots (15\cdot\mathrm{a,\ b})$$

(2) The discrete equations for the direct Lagrangian differential equations

The first order differential equations for the physical quantities are obtained as given in Table 3. Those for the mechanical quantities derived from the differential equations of Table 1 are expressed using the components of $(\hat{F}_x, \hat{F}_y, \hat{F}_z, M_x, M_y, M_z)$. In order to be compared with the SRBD method discussed later, however, the derivatives determined from the first order differential equations of Table 3 have to be transformed into those of Eq. (12) by means of the following relations as

$$\begin{array}{l} (\overline{M}_y, -\overline{M}_x, \overline{M}_z)^T = [l_{ab}](M_y, -M_x, M_z)^T, \quad (\overline{F}_x, \overline{F}_y, \overline{F}_z)^T = [l_{ab}](\hat{F}_x, \hat{F}_y, \hat{F}_z)^T \\ (\overline{m}_x, \overline{m}_y, \overline{m}_z)^T = [l_{ab}](\hat{m}_x, \hat{m}_y, \hat{m}_z)^T, \quad (\overline{p}_x, \overline{p}_y, \overline{p}_z)^T = [l_{ab}](\hat{p}_x, \hat{p}_y, \hat{p}_z)^T \end{array} \cdots\cdots (16\cdot\mathrm{a}\sim\mathrm{d})$$

Table 3 First Order Differential Equations in Lagrangian Expressions.

First Order Differential Equations	
$F_x'=\hat{F}_y\tau-\hat{F}_z\kappa_y-\hat{p}_x$ $F_y'=-\hat{F}_x\tau+\hat{F}_z\kappa_x-\hat{p}_y$ $F_z'=\hat{F}_x\kappa_y-\hat{F}_y\kappa_x-\hat{p}_z$ $M_x'=\hat{F}_x+M_y\tau-M_z\kappa_x+\hat{m}_y$ $M_y'=\hat{F}_y-M_x\tau-M_z\kappa_y-\hat{m}_x$ $M_z'=M_y\kappa_y+M_x\kappa_x-\hat{m}_z$	$\overline{u}_o'=\sqrt{g_o}\,l_{\hat{z}\overline{x}}$ $\overline{v}_o'=\sqrt{g_o}\,l_{\hat{z}\overline{y}}$ $\overline{w}_o'=\sqrt{g_o}\,l_{\hat{z}\overline{z}}-1$ $d[l_{ab}]/dZ=[D][l_{ab}]$ $[D]=\begin{bmatrix} 0, & \tau, & -\kappa_y \\ -\tau, & 0, & \kappa_x \\ \kappa_y, & -\kappa_x, & 0 \end{bmatrix}$
ⓐ Finite Displacements with Small Strains	$\kappa_x=M_y/EI_y$, $\kappa_y=-M_x/EI_x$ $E(J_{rr}-J^2/A)\tau^3/2+(G+\hat{F}_z/A)J\tau-M_z=0$ $\sqrt{g_o}=\hat{F}_z/EA+1-J\tau^2/2A$
ⓑ Love	$\kappa_x=M_y/EI_y$, $\kappa_y=-M_x/EI_x$ $\tau=M_z/GJ$, $\sqrt{g_o}=\hat{F}_z/EA+1$

Table 4 First Order Differential Equations with Separation of Rigid Body Displacements.

Theories	First Order Differential Equations
ⓒ Nishino and Yuki	$\overline{F}_x'=-\overline{p}_x$, $\overline{M}_x'=\overline{F}_x-\overline{F}_z\overline{\alpha}_y+\overline{m}_y$ $\overline{F}_y'=-\overline{p}_y$, $\overline{M}_y'=\overline{F}_y+\overline{F}_z\overline{\alpha}_x-\overline{m}_x$ $\overline{F}_z'=-\overline{p}_z$, $M_z'=\{(\overline{M}_x+\overline{M}_y\hat{\phi})\overline{\alpha}_x'+(\overline{M}_y-\overline{M}_x\hat{\phi})\overline{\alpha}_y'\}/(1+\hat{\phi}^2)-\hat{m}_z$ $\overline{u}_o'=\overline{\alpha}_y$, $\overline{\alpha}_x'=\{(\overline{M}_x+\overline{M}_y\hat{\phi})\hat{\phi}/I_x+(\overline{M}_y-\overline{M}_x\hat{\phi})/I_y\}/E(1+\hat{\phi}^2)^2$ $\overline{v}_o'=-\overline{\alpha}_x$, $\overline{\alpha}_y'=-\{(\overline{M}_x+\overline{M}_y\hat{\phi})/I_x-(\overline{M}_y-\overline{M}_x\hat{\phi})\hat{\phi}/I_y\}/E(1+\hat{\phi}^2)^2$ $\overline{w}_o'=\overline{F}_z/EA-(\overline{\alpha}_x^2+\overline{\alpha}_y^2)/2-J\hat{\phi}'^2/2A$ $\hat{\phi}'$ is the solution of $E(J_{rr}-J^2/A)\hat{\phi}'^3/2+(G+\overline{F}_z/A)J\hat{\phi}'-M_z=0$
ⓓ Maeda • Hayashi	$\overline{F}_x'=-\overline{p}_x$, $\overline{M}_x'=\overline{F}_x-\overline{F}_z\overline{\alpha}_y+\overline{M}_y\hat{\phi}'+\overline{m}_y$ $\overline{F}_y'=-\overline{p}_y$, $\overline{M}_y'=\overline{F}_y+\overline{F}_z\overline{\alpha}_x-\overline{M}_x\hat{\phi}'-\overline{m}_x$ $\overline{F}_z'=-\overline{p}_z$, $M_z'=(\overline{M}_x\overline{\alpha}_x+\overline{M}_y\overline{\alpha}_y)'-\hat{m}_z$ $\overline{u}_o'=\overline{\alpha}_y$, $\overline{\alpha}_x'=\overline{M}_y/EI_y+\overline{\alpha}_y\hat{\phi}'$ $\overline{v}_o'=-\overline{\alpha}_x$, $\overline{\alpha}_y'=-\overline{M}_x/EI_x+\overline{\alpha}_x\hat{\phi}'$ $\overline{w}_o'=\overline{F}_z/EA-(\overline{\alpha}_x^2+\overline{\alpha}_y^2)/2-J\hat{\phi}'^2/2A$ $\hat{\phi}'$ is the solution of $E(J_{rr}-J^2/A)\hat{\phi}'^3/2+(G+\overline{F}_z/A)J\hat{\phi}'-M_z=0$
ⓔ Beam-Column	$\overline{F}_x'=-\overline{p}_x$, $\overline{M}_x'=\overline{F}_x-\overline{F}_z\overline{\alpha}_y+\overline{m}_y$ $\overline{F}_y'=-\overline{p}_y$, $\overline{M}_y'=\overline{F}_y+\overline{F}_z\overline{\alpha}_x-\overline{m}_x$ $\overline{F}_z'=-\overline{p}_z$, $\overline{M}_z'=-\overline{m}_z$ $\overline{u}_o'=\overline{\alpha}_y$, $\overline{\alpha}_x'=\overline{M}_y/EI_y$ $\overline{v}_o'=-\overline{\alpha}_x$, $\overline{\alpha}_y'=-\overline{M}_x/EI_x$ $\overline{w}_o'=\overline{F}_z/EA-(\overline{\alpha}_x^2+\overline{\alpha}_y^2)/2$, $\overline{\alpha}_z'=\overline{M}_z/GJ$
ⓕ Small Displacements	$\overline{F}_x'=-\overline{p}_x$, $\overline{M}_x'=\overline{F}_x+\overline{m}_y$ $\overline{F}_y'=-\overline{p}_y$, $\overline{M}_y'=\overline{F}_y-\overline{m}_x$ $\overline{F}_z'=-\overline{p}_z$, $\overline{M}_z'=-\overline{m}_z$ $\overline{u}_o'=\overline{\alpha}_y$, $\overline{\alpha}_x'=\overline{M}_y/EI_y$ $\overline{v}_o'=-\overline{\alpha}_x$, $\overline{\alpha}_y'=-\overline{M}_x/EI_x$ $\overline{w}_o'=\overline{F}_z/EA$, $\overline{\alpha}_z'=\overline{M}_z/GJ$

As for the geometrical quantities, the first order differential equations for displacements are obtained by differentiating Eq. (7·a) with respect to z helped by Eqs. (1) and (11), and those for the direction cosines are derived by substituting Eq. (11) into Eq. (1·b).

Combined use of Table 3 and Eq. (13) helped by the boundary conditions of Eq. (14) leads to the derivatives for the physical quantities at node i. The derivatives obtained are summarized in Table 5 to be compared with those derived from the SRBD method. In addition, since the elongation of the member axis seems very small for common structures of practical importance, the derivatives with the condition of inextensional member axis are given in Table 6, which corresponds to a particular case of the derivatives in Table 5 with the cross sectional area A tending to infinity. The order of the derivatives listed in Tables 5 and 6 is determined such that the coincidence of the respective methods can completely be examined. Since the derivatives both for $(\overline{M}_x, \overline{M}_y, \overline{M}_z)$ and $[l_{ab}]$ do not depend on whether the member axis is extensional or inextensional, the expressions for these derivatives are given only in Table 5. It should be noted that the physical quantities in Tables 5 and 6 are those at node i, although subscript i is omitted for simplicity.

(3) The discrete equations for the SRBD method

In a similar way as for the direct Lagrangian method, the first order differential equations for the physical quantities are obtainable for the SRBD method from the local governing equations of Table 2 and the results are summarized in Table 4, where angles $(\overline{\alpha}_x, \overline{\alpha}_y, \overline{\alpha}_z)$ represent components of the member rotation around the $(\overline{x}, \overline{y}, \overline{z})$ coordinate axes after eliminating the rigid body rotation. Those angles are regarded as small as angles $(\overline{\alpha}_x, \overline{\alpha}_y)$ can well be approximated using displacements $(\overline{u}_0, \overline{v}_0)$ as

$$\overline{\alpha}_x=\overline{u}_0', \quad \overline{\alpha}_y=-\overline{v}_0' \qquad (17\cdot a, b)$$

and the direction cosines $[l_{ab}]$ are also approximated[4),7)] as

$$l_{\hat{x}\overline{x}}=1, \quad l_{\hat{x}\overline{y}}=\overline{\alpha}_z, \quad l_{\hat{x}\overline{z}}=-\overline{\alpha}_y, \quad l_{\hat{y}\overline{x}}=-\overline{\alpha}_z, \quad l_{\hat{y}\overline{y}}=1$$
$$l_{\hat{y}\overline{z}}=\overline{\alpha}_x, \quad l_{\hat{z}\overline{x}}=\overline{\alpha}_y, \quad l_{\hat{z}\overline{y}}=-\overline{\alpha}_x, \quad l_{\hat{z}z}=1 \qquad (18\cdot a\sim i)$$

Use of nonlinear equations as in Ref. 5) instead of the above linear equations (17) and (18) may not improve the accuracy of solutions because the local differential equations of Table 2 have already been approximated considerably by the conditions of relatively small displacements[11)].

Since the first order differential equations of Table 4 are expressed by the components of Eq. (12), excepting M_z, $\hat{m}$ and $\hat{\phi}$ in the theories of ⓒ and ⓓ which are defined with respect to the vectors $(\hat{i}_x, \hat{i}_y, \hat{i}_z)$, the derivatives for the physical quantities are mostly derived directly from Table 4 without transformation procedures. The physical components M_z, $\hat{m}$ and $\hat{\phi}$ are transformed to those of the $(\overline{x}, \overline{y}, \overline{z})$ coordinates using the direction cosines of Eq. (18) as

$$\overline{M}_z=M_z-M_y\overline{\alpha}_y-M_x\overline{\alpha}_x,$$
$$\overline{m}_z=\hat{m}_z-\hat{m}_x\overline{\alpha}_y+\hat{m}_y\overline{\alpha}_x,$$
$$\overline{\alpha}_z=\hat{\phi}-\hat{\alpha}_x\overline{\alpha}_y+\hat{\alpha}_y\overline{\alpha}_x \qquad (19\cdot a\sim c)$$

where $(\hat{\alpha}_x, \hat{\alpha}_y)$ are the components of small angles representing the member rotation around the vectors $(\hat{i}_x, \hat{i}_y)$.

With Table 4 helped by Eqs. (17) and (19), the results for the derivatives of the physical quantities for the SRBD method are summarized in Table 5 and those for the inextensional deformation of the member axis are given in Table 6, in a similar way as for the direct Lagrangian method.

Table 5 Derivatives of Physical Quantities : (a) ~ (c).

Theories		$\overline{u}_0'$, $\overline{v}_0'$, $\overline{w}_0'$	$\overline{u}_0''$, $\overline{v}_0''$, $\overline{w}_0''$
Lagrangians	ⓐ	$\overline{u}_0'=\overline{v}_0'=0$ $\overline{w}_0'=\tilde{\lambda}-1$	$\overline{u}_0''=\tilde{\lambda}\overline{\kappa}_y$, $\overline{v}_0''=-\tilde{\lambda}\overline{\kappa}_x$ $\overline{w}_0''=(\overline{F}_x\overline{\kappa}_y-\overline{F}_y\overline{\kappa}_x-\overline{p}_z-EJ\rho\rho')/a$
	ⓑ	$\overline{u}_0'=\overline{v}_0'=0$ $\overline{w}_0'=\lambda-1$	$\overline{u}_0''=\lambda\overline{\kappa}_y$, $\overline{v}_0''=-\lambda\overline{\kappa}_x$ $\overline{w}_0''=(\overline{F}_x\overline{\kappa}_y-\overline{F}_y\overline{\kappa}_x-\overline{p}_z)/a$
With Separation of Rigid Body Displacements	ⓒ ⓓ	$\overline{u}_0'=\overline{v}_0'=0$ $\overline{w}_0'=\tilde{\lambda}-1$	$\overline{u}_0''=\overline{\kappa}_y$, $\overline{v}_0''=-\overline{\kappa}_x$ $\overline{w}_0''=(-\overline{p}_z-EJ\rho\rho')/a$
	ⓔ ⓕ	$\overline{u}_0'=\overline{v}_0'=0$ $\overline{w}_0'=\lambda-1$	$\overline{u}_0''=\overline{\kappa}_y$, $\overline{v}_0''=-\overline{\kappa}_x$ $\overline{w}_0''=-\overline{p}_z/a$

Remarks: The following notations are used throughout Tables.

$EA=a$, $EI_x=b_x$, $EI_y=b_y$, $\lambda=\overline{F}_z/a+1$, $\tilde{\lambda}=\lambda-J\rho^2/2A$, $\overline{\kappa}_x=\overline{M}_y/b_y$, $\overline{\kappa}_y=-\overline{M}_x/b_x$
$\rho_0=\overline{M}_z/GJ$, ρ is the solution of $E(J_{rr}-J^2/A)\rho^3/2+(G+\overline{F}_z/A)J\rho-\overline{M}_z=0$,

$$\rho'=\frac{[\overline{M}_y\overline{\kappa}_y+\overline{M}_x\overline{\kappa}_x-\overline{m}_z-\{G+(\overline{F}_x\overline{\kappa}_y-\overline{F}_y\overline{\kappa}_x-\overline{p}_z)/A\}J\rho]}{\{3E(J_{rr}-J^2/A)\rho^2/2+(G+\overline{F}_z/A)J\}}, \quad \rho_0'=\frac{\overline{M}_y\overline{\kappa}_y+\overline{M}_x\overline{\kappa}_x-\overline{m}_z}{GJ}$$

Theories		$[l_{ab}]'$	$[l_{ab}]''$
Lagrangians	ⓐ ⓑ	$l_{\hat{x}\overline{x}}'=0$, $l_{\hat{x}\overline{y}}'=\tilde{\rho}$, $l_{\hat{x}\overline{z}}'=-\overline{\kappa}_y$ $l_{\hat{y}\overline{x}}'=-\tilde{\rho}$, $l_{\hat{y}\overline{y}}'=0$, $l_{\hat{y}\overline{z}}'=\overline{\kappa}_x$ $l_{\hat{z}\overline{x}}'=\overline{\kappa}_y$, $l_{\hat{z}\overline{y}}'=-\overline{\kappa}_x$, $l_{\hat{z}\overline{z}}'=0$	$l_{\hat{x}\overline{x}}''=\tilde{\rho}^2+\overline{\kappa}_y^2$, $l_{\hat{x}\overline{y}}''=\tilde{\rho}'+\overline{\kappa}_x\overline{\kappa}_y$, $l_{\hat{x}\overline{z}}''=-\overline{\kappa}_y'+\overline{\kappa}_x\tilde{\rho}$ $l_{\hat{y}\overline{x}}''=-\tilde{\rho}'+\overline{\kappa}_x\overline{\kappa}_y$, $l_{\hat{y}\overline{y}}''=-(\tilde{\rho}^2+\overline{\kappa}_x^2)$, $l_{\hat{y}\overline{z}}''=\overline{\kappa}_x'+\overline{\kappa}_y\tilde{\rho}$ $l_{\hat{z}\overline{x}}''=\overline{\kappa}_y'+\overline{\kappa}_x\tilde{\rho}$, $l_{\hat{z}\overline{y}}''=-\overline{\kappa}_x'+\overline{\kappa}_y\tilde{\rho}$, $l_{\hat{z}\overline{z}}''=-(\overline{\kappa}_x^2+\overline{\kappa}_y^2)$
With Separation of Rigid Body Displacements	ⓒ ⓓ		$l_{\hat{x}\overline{x}}''=0$, $l_{\hat{x}\overline{y}}''=\rho'$, $l_{\hat{x}\overline{z}}''=(\overline{F}_x+\overline{M}_y\rho+\overline{m}_y)/b_x-\overline{\kappa}_x\rho^*$ $l_{\hat{y}\overline{x}}''=-\rho'$, $l_{\hat{y}\overline{y}}''=0$, $l_{\hat{y}\overline{z}}''=(\overline{F}_y-\overline{M}_x\rho-\overline{m}_x)/b_y-\overline{\kappa}_y\rho^*$ $l_{\hat{z}\overline{x}}''=-l_{\hat{x}\overline{z}}''$ $l_{\hat{z}\overline{y}}''=-l_{\hat{y}\overline{z}}''$ $l_{\hat{z}\overline{z}}''=0$
	ⓔ ⓕ		$l_{\hat{x}\overline{x}}''=0$, $l_{\hat{x}\overline{y}}''=\rho_0'$, $l_{\hat{x}\overline{z}}''=(\overline{F}_x+\overline{m}_y)/b_x$ $l_{\hat{y}\overline{x}}''=-\rho_0'$, $l_{\hat{y}\overline{y}}''=0$, $l_{\hat{y}\overline{z}}''=(\overline{F}_y-\overline{m}_x)/b_y$ $l_{\hat{z}\overline{x}}''=-l_{\hat{x}\overline{z}}''$ $l_{\hat{z}\overline{y}}''=-l_{\hat{y}\overline{z}}''$, $l_{\hat{z}\overline{z}}''=0$

$\overline{\kappa}_x'=(\overline{F}_y-\overline{M}_x\tilde{\rho}-\overline{M}_z\kappa_y-\overline{m}_x)/EI_y$, $\overline{\kappa}_y'=-(\overline{F}_x+\overline{M}_y\tilde{\rho}-\overline{M}_z\kappa_x+\overline{m}_y)/EI_x$, $\tilde{\rho}$ and ρ^* differ according to the Theories as $\tilde{\rho}=\rho$ for a) and $\tilde{\rho}=\rho_0$ for b) and $\rho^*=\rho$ for c) and $\rho^*=-\rho$ for d).

Theories		$\overline{M}_x'$, $\overline{M}_y'$, $\overline{M}_z'$	$\overline{M}_x''$, $\overline{M}_y''$, $\overline{M}_z''$
Lagrangians	ⓐ ⓑ	$\overline{M}_x'=\overline{F}_x+\overline{m}_y$ $\overline{M}_y'=\overline{F}_y-\overline{m}_x$ $\overline{M}_z'=-\overline{m}_z$	$\overline{M}_x''=-\overline{p}_x+\overline{m}_y'-\overline{F}_z\overline{\kappa}_y$ $\overline{M}_y''=-\overline{p}_y-\overline{m}_x'+\overline{F}_z\overline{\kappa}_x$ $\overline{M}_z''=-\overline{m}_z'-\overline{F}_y\overline{\kappa}_y-\overline{F}_x\overline{\kappa}_x$
With Separation of Rigid Body Displacements	ⓒ		$\overline{M}_x''=-\overline{p}_x+\overline{m}_y'-\overline{F}_z\overline{\kappa}_y$ $\overline{M}_y''=-\overline{p}_y-\overline{m}_x'+\overline{F}_z\overline{\kappa}_x$ $\overline{M}_z''=-\overline{m}_z'-\overline{F}_y\overline{\kappa}_y-\overline{F}_x\overline{\kappa}_x-(\overline{M}_x\overline{\kappa}_y-\overline{M}_y\overline{\kappa}_x)\rho$
	ⓔ		$\overline{M}_x''=-\overline{p}_x+\overline{m}_y'-\overline{F}_z\overline{\kappa}_y$ $\overline{M}_y''=-\overline{p}_y-\overline{m}_x'+\overline{F}_z\overline{\kappa}_x$ $\overline{M}_z''=-\overline{m}_z'$
	ⓕ		$\overline{M}_x''=-\overline{p}_x+\overline{m}_y'$ $\overline{M}_y''=-\overline{p}_y-\overline{m}_x'$ $\overline{M}_z''=-\overline{m}_z'$
	ⓓ	$\overline{M}_x'=\overline{F}_x+\overline{M}_y\rho+\overline{m}_y$ $\overline{M}_y'=\overline{F}_y-\overline{M}_x\rho-\overline{m}_x$ $\overline{M}_z'=-\overline{m}_z$	$\overline{M}_x''=-\overline{p}_x+\overline{m}_y'-\overline{F}_z\overline{\kappa}_y+(\overline{F}_y-\overline{M}_x\rho-\overline{m}_x)\rho+\overline{M}_y\rho'$ $\overline{M}_y''=-\overline{P}_y-\overline{m}_x'+\overline{F}_z\overline{\kappa}_x-(\overline{F}_x+\overline{M}_y\rho+\overline{m}_y)\rho-\overline{M}_x\rho'$ $\overline{M}_z''=-\overline{m}_z'-\overline{m}_x\overline{\kappa}_y+\overline{m}_y\overline{\kappa}_x$

5. DISCUSSIONS

(1) Converged solutions for the SRBD method

If converged solutions be produced for the SRBD Method, the solution must satisfy the simultaneous first order differential equations for the physical quantities obtained by reducing the element length of the discrete equation (13) infinitesimally

Table 6 Derivatives of $\overline{u}_0$, $\overline{v}_0$, $\overline{w}_0$ with Inextensional Deformations.

<table>
<tr><th colspan="2">Theories</th><th>$\overline{u}_0'$, $\overline{v}_0'$, $\overline{w}_0'$</th><th>$\overline{u}_0''$, $\overline{v}_0''$, $\overline{w}_0''$</th><th>$\overline{u}_0'''$, $\overline{v}_0'''$, $\overline{w}_0'''$</th></tr>
<tr><td rowspan="2">Lagrangians</td><td>ⓐ</td><td rowspan="6">$\overline{u}_0'=0$
$\overline{v}_0'=0$
$\overline{w}_0'=0$</td><td rowspan="6">$\overline{u}_0''=\overline{\kappa}_y$
$\overline{v}_0''=-\overline{\kappa}_x$
$\overline{w}_0''=0$</td><td rowspan="2">$\overline{u}_0'''=-(\overline{F}_x+\overline{M}_y\tilde{\rho}-\overline{M}_z\overline{\kappa}_x+\overline{m}_y)/b_x+\tilde{\rho}\overline{\kappa}_x$
$\overline{v}_0'''=-(\overline{F}_y-\overline{M}_x\tilde{\rho}-\overline{M}_z\overline{\kappa}_y-\overline{m}_x)/b_y+\tilde{\rho}\overline{\kappa}_y$
$\overline{w}_0'''=-(\overline{\kappa}_x^2+\overline{\kappa}_y^2)$</td></tr>
<tr><td>ⓑ</td></tr>
<tr><td rowspan="4">With Separation of Rigid Body Displacements</td><td>ⓒ</td><td rowspan="2">$\overline{u}_0'''=-(\overline{F}_x+\overline{M}_y\rho+\overline{m}_y)/b_x+\rho\overset{*}{\overline{\kappa}}_x$
$\overline{v}_0'''=-(\overline{F}_y-\overline{M}_x\rho-\overline{m}_x)/b_y+\rho\overset{*}{\overline{\kappa}}_y$
$\overline{w}_0'''=-(\overline{\kappa}_x^2+\overline{\kappa}_y^2)$</td></tr>
<tr><td>ⓓ</td></tr>
<tr><td>ⓔ</td><td>$\overline{u}_0'''=-(\overline{F}_x+\overline{m}_y)/b_x$, $\overline{v}_0'''=-(\overline{F}_y-\overline{m}_x)/b_y$
$\overline{w}_0'''=-(\overline{\kappa}_x^2+\overline{\kappa}_y^2)$</td></tr>
<tr><td>ⓕ</td><td>$\overline{u}_0'''=-(\overline{F}_x+\overline{m}_y)/b_x$, $\overline{v}_0'''=-(\overline{F}_y-\overline{m}_x)/b_y$
$\overline{w}'''=0$</td></tr>
</table>

Remarks: ρ^* differs according to the theories as $\rho^*=\rho$ for c) and $\rho^*=-\rho$ for d)

close to zero. The forms of the differential equations can completely be determined from the first order coefficients of the power series of Eq. (13) with respect to Δz[11),12)]. The results of Tables 5 and 6 indicate that the first order coefficients of the SRBD method coincide completely with those derived from the Lagrangian equation of ⓐ for the local differential equation of ⓒ, and those from ⓑ both for ⓔ and ⓕ irrespective of the extensional or inextensional deformation of axis. Hence, it is concluded that the converged solution from the SRBD method with the local differential equation of ⓒ is identical to the analytical solution of the Lagrangian differential equation for the theory of ⓐ finite displacements with small strains, while that with the local differential equation of ⓔ or ⓕ is identical to the analytical solution of the differential equation for the theory of ⓑ Love. The difference of the two Lagrangian differential equations is only that the equation for the theory of ⓐ includes nonlinear terms representing the contributions of the axial stress to the torsional moment M_z as well as that of the torsional rate to the axial strain. Since those nonlinear terms resulting from the torsional deformation of member axis cannot be eliminated by the separation of rigid body displacements, the solution for the differential equation of ⓐ is identical only to the converged solution for the SRBD method with the local equation of ⓒ which reflects the corresponding nonlinear terms. It is noted, however, as mentioned before in Section 2., that there appears little difference in the solutions between the theory of ⓐ and that of ⓑ as far as the conditions of small strains and the equivalence of magnitude between the axial and the shear strains hold. On the other hand, while most of the first order coefficients for the physical quantities in the SRBD method with the local differential equation of ⓓ coincide with those derived from the Lagrangian differential equation of ⓐ, the coefficients of $\overline{M}_x$ and $\overline{M}_y$ differ from those for the theory of ⓐ. It is said for the reason that the solution of the SRBD method with the equation of ⓓ converges neither to the solution for the equation of ⓐ nor to that for the equation of ⓑ. It is reminded that the solution for the equation of ⓓ will converge to that for the equation of ⓐ, if the following relation as

$$\overline{M}_x=M_x-M_y\hat{\phi},\quad \overline{M}_y=M_y+M_x\hat{\phi} \qquad (20\cdot a,\ b)$$

is used in the same manner as in the equation of ⓒ after replacing $\overline{M}_x$ and $\overline{M}_y$ in the equations of ⓓ by M_x and M_y. However, there is little reason for the relation of Eq. (20) to hold for the equation of ⓓ as evident from the expression of the mechanical boundary conditions of $\overline{M}_x=\overline{M}_x^c$ and $\overline{M}_y=\overline{M}_y^c$ given in Table 2.

For the SRBD method examined above, the rotational angle $\hat{\phi}$, the internal moment M_z, and the external distributed moment $\hat{m}_z$ in the local equations of ⓒ and ⓓ have been defined with respect to the deformed centroidal axis instead of the $\overline{z}$ coordinate axis[10),14)]. The reason is that the first order derivative of $\overline{M}_z$ at node i in Table 5 becomes

$$\overline{M}_z'|_i=(-\overline{m}_z+\overline{M}_x\kappa_x-\overline{M}_y\kappa_y)|_i \qquad (21)$$

under the definition with respect to $\overline{z}$ axis which would fail to coincide with that for the theory of ⓐ.

(2) Accuracy of the SRBD method

In case that the element length is finite for the SRBD method as is the case of practical computations, its accuracy needs to be examined from the view of computational efficiency. It is possible by comparing the coefficients of higher order terms in the Taylor expansions of Eq. (13) between the SRBD method and the direct Lagrangian method[11),12)]. From the results of Tables 5 and 6, the coincidence of the maximum order of derivatives is summarized in Table 7 which is classified into the theory of ⓐ finite displacments with small strains, and that of ⓑ Love. Since the SRBD method with the local differential equation of ⓓ has failed to converge to either solution of the direct Lagrangian differential equations, it is omitted in Table 7.

For a general case with the extensional deformation of axis, the maximum order of coincidence is only one for most of the derivatives of physical quantities, excepting the second order coincidence of $\overline{M}_x$ and $\overline{M}_y$ derived from the local differential equations of ⓒ or ⓔ. Therefore, all the SRBD methods examined here can only be said a method of the first order approximation to the direct Lagrangian method, resulting in the same conclusion for plane frames[11),12)]. However, the order of coincidence for the derivatives of the direction cosine $[l_{ab}]$ representing the rotation of the member has decreased considerably for space frames, compared with those of the rotational angle for plane frames. It is noted that, if the freedom of space is restricted two dimensional, the results of Tables 5 and 6 coincide with those of plane frames given in Ref. 11).

As for a particular case of the inextensional deformation of axis, as indicated in the parentheses of Table 7, only the coincidence of the derivatives for $(\overline{u}_0, \overline{v}_0, \overline{w}_0)$ differs from that for the extensional deformation of axis discussed above such that the approximation is improved up to the second order at least, and the third order at most. However, the coincidence for M_z and $[l_{ab}]$ still remains only the first order. Hence, even for the inextensional deformation, the SRBD method for space frames cannot be the method of the second order approximation different from the case for plane frames[11),12)].

Table 7 Coincidence of the Order of Derivatives.

a. Finite Displacements with Small Strains

	ⓒ
$\overline{u}_o$	1(2)
$\overline{v}_o$	1(2)
$\overline{w}_o$	1(3)
$\overline{M}_x$	2
$\overline{M}_y$	2
$\overline{M}_z$	1
$[l_{ab}]$	1

b. Love

	ⓔ	ⓕ
$\overline{u}_o$	1(2)	1(2)
$\overline{v}_o$	1(2)	1(2)
$\overline{w}_o$	1(3)	1(2)
$\overline{M}_x$	2	1
$\overline{M}_y$	2	1
$\overline{M}_z$	1	1
$[l_{ab}]$	1	1

Remarks: (·) indicates orders for inextensional deformation of member axis only when it differs from the case of extensional deformation.

6. CONCLUDING REMARKS

The convergence and accuracy of the SRBD method have been examined for the finite displacement analysis of space frames. The converged solutions of the SRBD method for infinitesimally small length of element are classified into two levels either for the extensional or inextensional deformation of member axis. One is the converged solution from the local differential equation presented by Nishino[14)] and Yuki[10)] which is identical to the analytical solution for the theory of finite displacements with small strains in the direct Lagrangian expression. The other is the converged solution from the local differential equation of beam-column or small displacements which is identical to the solution for the theory of Love[13)].

Regarding the accuracy in the case of the element length being finite, it can only be said for members with extensional deformation of axis that the SRBD method is of the first order approximation. When the elongation of member axis is negligibly small, the order of coincidence is improved for the derivatives of displacements. Since the order of coincidence for the other derivatives such as the torsional moment and rotations still remain only the first order, however, the SRBD method for space frames with the inextensional deformation cannot be the method of the second order approximation different from the case for plane frames.

REFERENCES

1) Schroeder, F. H. : Allgemeine Stabtheorie des dünnwantigen räumlich vorgekrümmten und vorgewundenen Trägers mit großen Verformungen, Ingenieur-Archiv, 39-Band, s. 87-103, 1970.
2) Yoda, Y., Hirashima, M. and Iura, M. : Finite Displacement Theory of Curved and Twisted Thin-Walled Members, Theoretical and Applied Mechanics, Vol. 28, pp. 105~117, 1980.
3) Kurakata, Y. and Nishino, F. : Finite Displacement Field and Governing Equations of Solid Curved Beams, Proc. of JSCE, No. 317, pp. 15~30, Jan., 1982 (in Japanese).
4) Oran, C. : Tangent Stiffness in Space frames, Journal of Structural Division, ASCE, Vol. 99, No. ST 6, pp. 987~1001, Jun.,

1973.
5) Maeda, Y. and Hayashi, M. : Finite Displacement Analysis of Space Framed Structures, Proc. of JSCE, No. 253, pp. 13~27, Sept., 1976 (in Japanese).
6) Goto, S., Hane, G. and Tanaka, T. : Tangent Stiffness Method for Large Deformation Analysis of Frame Structure, Proc. of JSCE, No. 238, pp. 31~42, Jun., 1975 (in Japanese).
7) Bathe, K. and Bolouchi, S. : Large Displacement Analysis of Three-Dimensional Beam Structures, International Journal for Numerical Methods in Engineering, Vol. 14, pp. 961~986, 1979.
8) Yoshida, Y., Masuda, N., Morimoto, T. and Hirosawa, N. : An Incremantal Formulation for Computer Analysis of Space Framed Structures, Proc. of JSCE, No. 300, pp. 21~31, Aug., 1980 (in Japanese).
9) Goto, S. : A Formulation of Tangent Geometric Stiffness Matrix for Space Structure, Proc. of JSCE, No. 335, pp. 1~11, Jul., 1983 (in Japanese).
10) Yuki, T. and Maeda, Y. : 3-Dimensional Behavior Analysis of Thin-Walled Strucures, Proc. of JSCE, No. 224, pp. 67~78, Apr., 1974 (in Japanese).
11) Goto, Y., Hasegawa, A. and Nishino, F. : Accuracy of Finite Displacement Analysis of Plane Frames, Proc. of JSCE, No. 331, pp. 33~44, Mar., 1983 (in Japanese).
12) Goto, Y., Hasegawa, A. and Nishino, F. : Accuracy and Convergence of the Separation of Rigid Body Displacements for Plane Curved Frames, Proc. of JSCE, Div. of Structural Eng./Earthquake Eng., Vol. 1, No. 1, pp. 29 s~39 s, Apr., 1984.
13) Love, A. E. H. : A Treatise on the Mathematical Theory of Elasticity, Dover, pp. 381~388, 1944.
14) Nishino, F., Kasemest, C. and Lee, S. L. : Variational Formulation of Stability Problems for Thin-Walled Members, Ingenieur-Archiv, 43-Band, s. 58~68, 1973.

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