【PROC. OF JSCE Structural Eng. ／Earthquake Eng. Vol. 2, No. 1 April 1985】

# A CONCISE AND EXPLICIT FORMULATION OF OUT-OF-PLANE INSTABILITY OF THIN-WALLED MEMBERS

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The purpose of this study is to present a solution scheme for the problem of out-of-plane instability of thin-walled members. Based on the second order kinematic field, the stiffness equation of linearized finite displacements is formulated for thin-walled members, and given in a concise and explicit form. As a particular case, an important and practical application is made for the lateral-torsional buckling of in-plane beams and frames. Numerical examples are given for straight and curved members, and are compared with existing results. The analysis scheme presented is proved accurate, efficient and versatile.

## 1. INTRODUCTION

The out-of-plane instability of thin-walled members has been a research subject of very keen interest during the past decades. Followed by the earlist comprehensive work by Timoshenko and Gere[1], a number of studies have been made in this area by various researchers. Among them are Vlasov[2], Roik·Carl·Linder[3], and Sakai[4], all of whom presented works on thin-walled members subject to axial force, biaxial bending and torsional moments. Both Vlasov and Roik·Carl·Linder linearized the stress and kinematic fields present in members, and hence derived the governing differential equations, making use of some geometrical observations, whereas Sakai adopted the initial stress concept and made the formulations based on the energy theorems, in which the initial stress field as well as the kinematic field were linearized also. Subsequently, Nishino·Kasemset·Lee[5] and Nishino·Kurakata·Hasegawa·Okumura[6),7)] have derived the governing equations for the problem, based strictly on the finite displacement theory of elasticity. Most of those literature have contributed mainly to the closed form presentations of the boundary value problems in a style of differential equations from a very theoritical standpoint. In contrast, the corresponding discrete equations and solution procedures for computer applications have been treated to some extent by Bazant·Nimeiri[8], Ram·Osterrieder[9] and others. However, those available at present for this sort of discrete procedures may have lacked in the theoretical consistency with rigorous treatment of the finite displacement theory of thin-walled members[8),9)] and/or involve complicated and black-box type

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formulations[9] which most readers migth feel some difficulty to apply for their own problems.

For the case of curved members, Nishino · Fukasawa[7),10] formulated the govering equations only of small displacements for thin-walled circular members with a rigorous but somewhat complexed presentation. Yoo[11] investigated recently the flexural-torsional instability of thin-walled curved beams in which the governing differential equations were presented to determine the elastic buckling loads of in-plane as well as out-of-plane buckling modes. However, closed form solutions were obtainable only for circular uniform members with very limited types of support conditions and loadings. When a discrete method is applied for the analysis of curved members, it is common and well-known that a curved member can be well approximated by an assemblage of straight elements, rather than resorting to the direct use of cumbersome curved elements, although it has been proved only for limited cases[12]. For this reason, a discrete equation for out-of-plane instability of thin-walled straight members can serve not only for its own purpose but also for that of curved members.

Under the circumstances described above, this paper presents an efficient analysis scheme for the problem of out-of-plane instability of thin-walled beams and frames. Based on the second order strain-displacement relationship obtained through the approximation of the rigorous kinematic field of finite displacements, the theorem of virtual work for linearized finite displacement theory of continua is used to derive the discrete stiffness equation of interest. This general stiffness equation for thin-walled straight members of linearized finite displacements is given in a concise and explicit expression, which can easily be adopted for practical usage in design offices. An application of practical importance is made to evaluate the lateral-torsional buckling of thin-walled members. Numerical examples are given both for straigth and curved beams, and are compared with existing results. It is found that the stiffness equation presented is accurate, reliable and versatile, helped by its simplicity of expressions.

## 2. KINEMATIC FIELD OF THIN-WALLED MEMBERS

As is well-known, shells of sufficiently longer length compared to the cross-sectional dimensions can be analysed as thin-walled members. Using the following basic assumptions reduces the problem into that of one-dimentional continuum.

(1) The cross-section is assumed to be undeformed in its plane while free to warp out of its plane.

(2) Shear deformation in the middle surface resulting from the equilibrium with normal stresses towards the member axis is neglected and also transverse shear across the thickness is neglected.

(3) The material is assumed to be homogeneous, isotropic and to display the linearly elastic uniaxial law for normal stresses.

A right hand Cartesian coordinate system $(x, y, z)$ is being used in this study as shown in Fig. 1, with $x$ along the member axis, and $y$ and $z$ being the principal axes with its origin at the centroid of the section. In addition, for explanation purposes, another orthogonal set of coordinates $(x, n, s)$ is introduced, with $s$ being the length coordinate measured along the middle surface of the thin-wall, starting from an arbitrary origin. Variables $u$, $v$, and $w$ are the displacements in the directions $x$, $y$, and $z$ respectively, while $\phi$ is the rotation of the cross section about the shear center $S$ located at point $(y_S, z_S)$ of the section. Variable $\omega$ is the normalized unit warping with respect to the shear center defined as

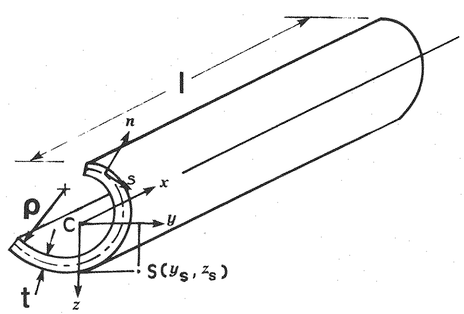


Fig. 1 Coordinate Systems of a Thin-walled Beam.

$$\omega(s)=\Omega(s)-\frac{1}{A}\int_A \Omega(s)dA \cdots\cdots (1)$$

where $A$ is cross-sectional area and

$$\Omega(s)\equiv\begin{cases}\displaystyle\int_0^s hds & \text{(open part)}\\ \displaystyle\int_0^s hds-\frac{\oint hds}{\oint(1/t)ds}\int_0^s(1/t)ds & \text{(closed part)}\end{cases}\quad\cdots\cdots(2)$$

in which $t$ is the thickness of thin-wall, and $h$ is the distance from the shear center to the tangent at the point s with positive value toward the $n$ coordinate.

Subscripts $c$ and $s$ are used to indicate quantities related to the centroid $C$ and shear center $S$, respectively. Also it should be noted that the axial forces are applied at the centroid, while the lateral force components are applied at the shear center.

Assuming that the derivatives of axial displacement are much smaller than those of other displacements for the Green's strain-displacement relations, use of the first and second assumptions for the kinematic field cited above gives the rigorous displacements as

$$u=u_c-v_s'(y\cos\phi-z\sin\phi)-w_s'(z\cos\phi+y\sin\phi)-\omega\phi'\cdots\cdots(3\cdot\mathrm{a})$$

$$v=v_s-(y-y_s)(1-\cos\phi)-(z-z_s)\sin\phi\cdots\cdots(3\cdot\mathrm{b})$$

$$w=w_s+(y-y_s)\sin\phi-(z-z_s)(1-\cos\phi)\cdots\cdots(3\cdot\mathrm{c})$$

where prime $(\ )'$ denotes differentiation with respect to $x$.

Substituting Eq. (3) into the assumed Green's strain displacement relations again leads to

$$\begin{aligned}e_{xx}=&u_c'-v_s''(y\cos\phi-z\sin\phi)-w_s''(z\cos\phi+y\sin\phi)-\omega\phi''+1/2[(v_s')^2+(w_s')^2]\\&+v_s'\phi'(z_s\cos\phi+y_s\sin\phi)+w_s'\phi'(-y_s\cos\phi+z_s\sin\phi)+1/2[(z-z_s)^2\\&+(y-y_s)^2](\phi')^2\cdots\cdots(4\cdot\mathrm{a})\end{aligned}$$

$$e_{sx}=(1/2)\,\Theta\phi',\quad e_{ss}=e_{nn}=e_{ns}=e_{xn}=0\cdots\cdots(4\cdot\mathrm{b,\ c})$$

in which

$$\Theta\equiv\begin{cases}2n & \text{(open part)}\\ 2n+\dfrac{\oint hds}{t\oint\frac{1}{t}ds} & \text{(closed part)}\end{cases}\quad\cdots\cdots(5)$$

including the contribution from the relative difference across thickness.

Expanding the trigonometric terms by Taylor series and neglecting third and higher order terms, Eq. (4·a) can be reduced to

$$\begin{aligned}e_{xx}=&u_c'-v_s''(y-z\phi)-w_s''(z+y\phi)-\omega\phi''+1/2[(v_s')^2+(w_s')^2]+(v_s'z_s-w_s'y_s)\phi'\\&+1/2[(z-z_s)^2+(y-y_s)^2](\phi')^2\cdots\cdots(6)\end{aligned}$$

## 3. GENERAL STIFFNESS OF THIN-WALLED MEMBERS

In this chapter, the general stiffness equation of a thin-walled uniform straight beam element in linearized finite displacements is formulated. In the linearized finite displacement theory of continua, the loads and displacements of concern, sometimes referred to as the increments, are measured from an arbitrary reference state of equilibrium where some of the internal forces and stresses, sometimes referred to as the initial stresses, may exist. For the sake of simplicity and practical importance of the discrete procedures, loads are considered to be applied only at the member ends for the state of concern as well as the reference state. All quantities related to the reference state are denoted with superscript o to distinguish them from the quantities of concern. For a thin-walled beam, the normal stress $\sigma_{xx}^0$ at the reference state is expressed in terms of stress resultants which are internal axial force $N^0$, bending moments $M_y^0$ and $M_z^0$, and warping moment $M_\omega^0$, all of which are assumed constant along the element length, by

$$\sigma_{xx}^0=\frac{N^0}{A}+\frac{M_y^0}{I_{yy}}y+\frac{M_z^0}{I_{zz}}z+\frac{M_\omega^0}{I_{\omega\omega}}\omega\cdots\cdots(7)$$

in which $N^0$ is taken positive for tensile forces, $A$ is the cross sectional area, and

$$I_{yy}\equiv\int_A y^2 dA,\quad I_{zz}\equiv\int_A z^2 dA,\quad I_{\omega\omega}\equiv\int_A \omega^2 dA \qquad (8\cdot\text{a, b, c})$$

Considering that the system is in equilibrium after the application of the set of loads, the virtul work equation without body forces for general continua with volume $V$ and surface $S$ can be written in the form as

$$\int_V (\sigma_{ij}^0+\sigma_{ij})\,\delta e_{ij} dV-\int_S (T_i^0+T_i)\,\delta u_i dS=0 \qquad (9)$$

in which $\sigma_{ij}$ and $e_{ij}$ are 2nd Piola-Kirchhoff's stress tensor and Green's strain tensor, respectively, and $T_i$ and $u_i$ refer to the external load components applied at the ends and corresponding displacements, respectively, all are measured from the reference state, while $\sigma_{ij}^0$ and $T_i^0$ are defined for the reference state.

For explanation purposes, $\sigma_{ij}$ and $e_{ij}$ are separated into their first and second order quatities denoted with superscripts (1) and (2) respectively, as

$$\sigma_{ij}=\sigma_{ij}^1+\sigma_{ij}^2,\quad e_{ij}=e_{ij}^1+e_{ij}^2 \qquad (10\cdot\text{a, b})$$

Making use of Eq. (10), Eq. (9) can be rewritten in the form as

$$\int_V (\sigma_{ij}^0+\sigma_{ij}^1+\sigma_{ij}^2)\delta(e_{ij}^1+e_{ij}^2)dV-\int_S (T_i^0+T_i)\delta u_i dS=0 \qquad (11)$$

Taking into consideration that the system is in equilibrium in terms of small displacements for the reference state which is before the application of loads, as expressed by

$$\int_V \sigma_{ij}^0\delta e_{ij}^1 dV-\int_S T_i^0\delta u_i dS=0 \qquad (12)$$

and also neglecting the third and higher order quantities, Eq. (11) can be simplified to

$$\int_V (\sigma_{ij}^0\delta e_{ij}^2+\sigma_{ij}^1\delta e_{ij}^1)dV-\int_S T_i\delta u_i dS=0 \qquad (13)$$

Substituting Eqs. (4·b), (6) and (7) into Eq. (13) and making use of linear constitutive equations both for the normal and shear stresses lead to

$$\int_V\Big[\Big(\frac{N^0}{A}+\frac{M_y^0}{I_{yy}}y+\frac{M_z^0}{I_{zz}}z+\frac{M_\omega^0}{I_{\omega\omega}}\omega\Big)\delta\Big[\frac{1}{2}\{(v_s')^2+(w_s')^2\}+(v_s'z_s-w_s'y_s)\phi'+\frac{1}{2}\{(z-z_s)^2$$
$$+(y-y_s)^2\}(\phi')^2+v_s''z\phi-w_s''y\phi\Big]+E(u_c'-v_s''y-w_s''z-\omega\phi'')\delta(u_c'-v_s''y-w_s''z-\omega\phi'')$$
$$+G\Theta\phi'\delta(\Theta\phi')\Big]dV-\int_S T_i\delta u_i dS=0 \qquad (14)$$

in which $E$ and $G$ are Young's modulus and shear modulus of elasticity, respectively.

At this stage, the following set of well-known interpolation functions of Hermite polynomials is introduced as

$$N_1\equiv 1-x/l \qquad (15\cdot\text{a})$$
$$N_2\equiv x/l \qquad (15\cdot\text{b})$$
$$N_3\equiv 1-3x^2/l^2+2x^3/l^3 \qquad (15\cdot\text{c})$$
$$N_4\equiv -x+2x^2/l-x^3/l^2 \qquad (15\cdot\text{d})$$
$$N_5\equiv 3x^2/l^2-2x^3/l^3 \qquad (15\cdot\text{e})$$
$$N_6\equiv x^2/l-x^3/l^2 \qquad (15\cdot\text{f})$$

Using the interpolation functions above, the displacement components at arbitrary cross-section $(0<x<l)$ can be written in terms of the displacements at the ends in the approximate form as

$$u_c=\boldsymbol{A}^T\boldsymbol{U},\quad v_s=\boldsymbol{B}^T\boldsymbol{V},\quad w_s=\boldsymbol{B}^T\boldsymbol{W},\quad \phi=\boldsymbol{B}^T\boldsymbol{\Phi} \qquad (16\cdot\text{a, b, c, d})$$

where, $\boldsymbol{A}\equiv\langle N_1,\ N_2\rangle^T,\ \boldsymbol{B}\equiv\langle N_3,\ N_4,\ N_5,\ N_6\rangle^T$ ⋯ (17·a, b)

and, $\boldsymbol{U}\equiv\langle u_{ci},\ u_{cj}\rangle^T,\ \boldsymbol{V}\equiv\langle v_{si},\ -v_{si}',\ v_{sj},\ -v_{sj}'\rangle^T$ ⋯ (18·a, b)

$\boldsymbol{W}\equiv\langle w_{si},\ -w_{si}',\ w_{sj},\ -w_{sj}'\rangle^T,\ \boldsymbol{\Phi}\equiv\langle\phi_i,\ -\phi_i',\ \phi_j,\ -\phi_j'\rangle^T$ ⋯ (18·c, d)

in which $\langle\ \ \rangle$ indicates row matrix, and the displacement components with subscripts $i$ and $j$ refer to the cross-sections at $x=0$ and $l$, respectively as shown in **Fig. 2**.

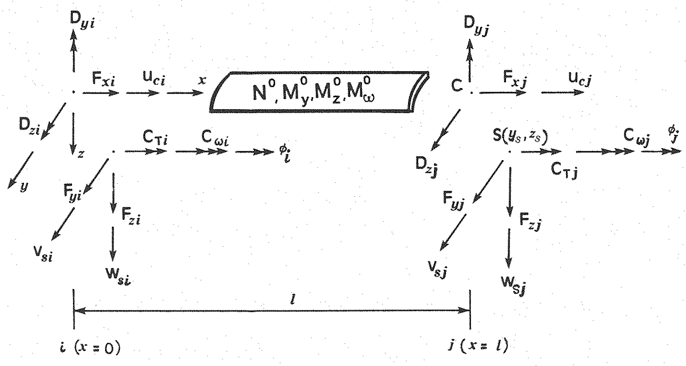



Fig. 2 Generalized Forces and Displacements.

Noting that $u_i$ in the last term of the left hand side of Eq. (14) is the displacement at the member ends defined by Eq. (18) and $T_i$ is the corresponding nodal force integrated over the cross section, substitution of Eq. (16) into Eq. (14) leads to the general stiffness equation of a thin-walled beam element, as expressed by

$$\boldsymbol{F}=\boldsymbol{Kd} \text{ i.e. } \begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{F}_y \\ \boldsymbol{F}_z \\ \boldsymbol{T} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{K}_{11} & & & \\ 0 & \boldsymbol{K}_{22} & & \text{sym} \\ 0 & 0 & \mathbf{K}_{33} & \\ 0 & \boldsymbol{K}_{42} & \boldsymbol{K}_{43} & \boldsymbol{K}_{44} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \\ \boldsymbol{\Phi} \end{Bmatrix} \quad \cdots\cdots (19)$$

where the left hand side of Eq. (19) is the nodal force vector, shown in Fig. 2, as given by

$$\boldsymbol{P}\equiv\langle F_{xi},\ F_{xj}\rangle^T,\quad \boldsymbol{F}_y\equiv\langle F_{yi},\ D_{yi},\ F_{yj},\ D_{yj}\rangle^T \quad \cdots\cdots (20\cdot\text{a, b})$$

$$\boldsymbol{F}_z\equiv\langle F_{zi},\ D_{zi},\ F_{zj},\ D_{zj}\rangle^T,\quad \boldsymbol{T}\equiv\langle C_{Ti},\ C_{\omega i},\ C_{Tj},\ C_{\omega j}\rangle^T \quad \cdots\cdots (20\cdot\text{c, d})$$

and, the block stiffnesses appeared in Eq. (19) are given by

$$\boldsymbol{K}_{11}\equiv\int_0^l EA(\boldsymbol{A}')(\boldsymbol{A}')^T dx \quad \cdots\cdots (21\cdot\text{a})$$

$$\boldsymbol{K}_{22}\equiv\int_0^l [EI_{yy}(\boldsymbol{B}'')(\boldsymbol{B}'')^T+N^0(\boldsymbol{B}')(\boldsymbol{B}')^T]dx \quad \cdots\cdots (21\cdot\text{b})$$

$$\boldsymbol{K}_{33}\equiv\int_0^l [EI_{zz}(\boldsymbol{B}'')(\boldsymbol{B}'')^T+N^0(\boldsymbol{B}')(\boldsymbol{B}')^T]dx \quad \cdots\cdots (21\cdot\text{c})$$

$$\boldsymbol{K}_{44}\equiv\int_0^l [EI_{\omega\omega}(\boldsymbol{B}'')(\boldsymbol{B}'')^T+GJ(\boldsymbol{B}')(\boldsymbol{B}')^T+(N^0 r_s^2+M_y^0\beta_y+M_z^0\beta_z+M_\omega^0\beta_\omega)(\boldsymbol{B}')(\boldsymbol{B}')^T]dx \quad \cdots\cdots (21\cdot\text{d})$$

$$\boldsymbol{K}_{42}\equiv\int_0^l [N^0 z_s(\boldsymbol{B}')(\boldsymbol{B}')^T+M_z^0(\boldsymbol{B})(\boldsymbol{B}'')^T]dx \quad \cdots\cdots (21\cdot\text{e})$$

$$\boldsymbol{K}_{43}\equiv\int_0^l [-N^0 y_s(\boldsymbol{B}')(\boldsymbol{B}')^T-M_y^0(\boldsymbol{B})(\boldsymbol{B}'')^T]dx \quad \cdots\cdots (21\cdot\text{f})$$

in which

$$J\equiv\int_A \Theta^2 dA,\quad r_s^2\equiv[I_{yy}+I_{zz}+A(y_s^2+z_s^2)]/A,\quad \beta_y\equiv-2\,y_s+\frac{1}{I_{yy}}\int_A(y^2+z^2)y\,dA \quad \cdots\cdots (22\cdot\text{a, b})$$

$$\beta_z\equiv-2\,z_s+\frac{1}{I_{zz}}\int_A(y^2+z^2)z\,dA,\quad \beta_\omega\equiv\frac{1}{I_{\omega\omega}}\int_A(y^2+z^2)\omega\,dA \quad \cdots\cdots (22\cdot\text{c, d})$$

Performing integrations, Eq. (21) can be expressed in a concise and explicit form as

$$\boldsymbol{K}_{11}\equiv\frac{EA}{l}\boldsymbol{K}_0,\quad \boldsymbol{K}_{22}\equiv\frac{EI_{yy}}{l^3}\boldsymbol{K}_1+\frac{N^0}{l}\boldsymbol{K}_2,\quad \boldsymbol{K}_{33}\equiv\frac{EI_{zz}}{l^3}\boldsymbol{K}_1+\frac{N^0}{l}\boldsymbol{K}_2 \quad \cdots\cdots (23\cdot\text{a, b, c})$$

$$\boldsymbol{K}_{44}\equiv\frac{EI_{\omega\omega}}{l^3}\boldsymbol{K}_1+\frac{GJ}{l}\boldsymbol{K}_2+\left(\frac{N^0}{l}\,r_s^2+\frac{M_y^0}{l}\beta_y+\frac{M_z^0}{l}\beta_z+\frac{M_\omega^0}{l}\beta_\omega\right)\boldsymbol{K}_2 \quad \cdots\cdots (23\cdot\text{d})$$

$$\boldsymbol{K}_{42}\equiv\frac{N^0 z_s}{l}\boldsymbol{K}_2+\frac{M_z^0}{l}\boldsymbol{K}_3,\quad \boldsymbol{K}_{43}\equiv-\frac{N^0 y_s}{l}\boldsymbol{K}_2-\frac{M_y^0}{l}\boldsymbol{K}_3 \quad \cdots\cdots (23\cdot\text{e, f})$$

in which

$$K_0=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad K_1=\begin{bmatrix} 12 & & & \\ -6l & 4l^2 & \text{sym} & \\ -12 & 6l & 12 & \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix} \quad \cdots\cdots (24\cdot\text{a, b})$$

$$K_2=\begin{bmatrix} 6/5 & & & \\ -l/10 & 2l^2/15 & \text{sym} & \\ -6/5 & l/10 & 6/5 & \\ -l/10 & -l^2/30 & l/10 & 2l^2/15 \end{bmatrix} \quad \cdots\cdots (24\cdot\text{c})$$

$$K_3=\begin{bmatrix} -6/5 & 11l/10 & 6/5 & l/10 \\ l/10 & -2l^2/15 & -l/10 & l^2/30 \\ 6/5 & -l/10 & -6/5 & -11l/10 \\ l/10 & l^2/30 & -l/10 & -2l^2/15 \end{bmatrix} \quad \cdots\cdots (24\cdot\text{d})$$

Eq. (19) constitutes (14×14) matrix equation. It is noted for Eq. (24) that $K_0$ and $K_1$ are found to be stiffness matrices of small displacements for axial and bending problems of beams, respectively, and $K_2$ turns to be the well-known geometric stiffness matrix of the in-plane beam-column. Stiffness $K_3$ appeared only in the non-diagonal blocks $K_{42}$ and $K_{43}$ of Eq. (19) proves identical to $(-K_2)$ with exceptions of the elements of (1, 2) and (3, 4).

As is clear from the expressions of Eq. (23), stiffness matrix $K$ in Eq. (19) can be divided into two parts which are stiffness of small displacements $K_E$ with terms multiplied by $A$, $I_{yy}$, $I_{zz}$, $I_{\omega\omega}$, or $J$ and geometric stiffness $K_G$ involving coefficients of $N^0$, $M_y^0$, $M_z^0$, or $M_\omega^0$ resulting from the stress at the reference state as given by Eq. (7). It should be noted that the stiffness matrix obtained in Eq. (19) is identical to the tangential stiffness matrix in the overall evaluation of the nonlinear behavior of thin-walled members, since the force and displacements of Eq. (19) can be regarded as increments from the reference state.

## 4. LATERAL-TORSIONAL BUCKLING OF THIN-WALLED MEMBERS

Of most practical importance for the out-of-plane instability of thin-walled members is the lateral torsional buckling of the beam-column. Consider a thin-walled beam element with the shear center lying on the $z$ axis ($y_S=0$) subject to axial compressive force $P^0$ and the in-plane bending moment $M^0$, as shown in Fig. 3. The stress resultants for the reference state appeared in the stiffness of Eq. (23) are evaluated by the small displacement theory of beams as

$$N^0=-P^0,\ M_y^0=0,\ M_z^0=M^0,\ M_\omega^0=0 \quad \cdots\cdots (25)$$

in which $P^0$ is defined positive for compression.

For this particular case, the governing stiffness equation of concern can be extracted from the general expression of Eq. (19), and expressed as

$$\begin{Bmatrix} F_y \\ T \end{Bmatrix}=\begin{bmatrix} K_{22} & \text{sym} \\ K_{42} & K_{44} \end{bmatrix}\begin{Bmatrix} V \\ \Phi \end{Bmatrix} \quad \cdots\cdots (26)$$

which constitutes the (8×8) matrix equation. Rewriting Eq. (26) as

$$F=[K_E+K_G]d \quad \cdots\cdots (27)$$

stiffness matrix of small displacements $K_E$ and geometric stiffness $K_G$ are given respectively by

$$K_E=\begin{bmatrix} \dfrac{EI_{yy}}{l^3}K_1 & 0 \\ 0 & \left(\dfrac{EI_{\omega\omega}}{l^3}K_1+\dfrac{GJ}{l}K_2\right) \end{bmatrix} \quad \cdots\cdots (28\cdot\text{a})$$

$$K_G=-\frac{P^0}{l}\begin{bmatrix} K_2 & z_S K_2 \\ z_S K_2 & r_S^2 K_2 \end{bmatrix}+\frac{M^0}{l}\begin{bmatrix} 0 & K_3^T \\ K_3 & \beta_z K_2 \end{bmatrix} \quad \cdots\cdots (28\cdot\text{b})$$

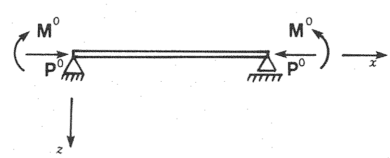


Fig. 3 Beam-column.

in which $K_1 \sim K_3$ are given by Eq. (24).

Assembling the element stiffness equation given by Eq. (26) over the whole structure of interest, helped by the coordinate transformation, if necessary, and introducing the kinematic boundary conditions, the critical buckling loads can be obtained from the singularity of the martix. Even for this particular case of Eq. (26) for lateral torsional buckling, the method presented in this paper is much more versatile and comprehensive than the existing ways of analysis, helped by its concise and explicit stiffness expressions, and can be applicable not only for straigh members but also for curved members.

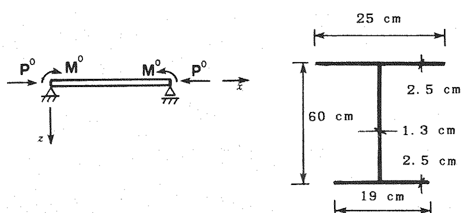
## 5. NUMERICAL EXAMPLES

First, out-of-plane instabilty of a straight singly symmetric simply supported I beam (i. e. $z_s \neq 0$, $v_i = v_j = \phi_i = \phi_j = D_{yi} = D_{yj} = C_{\omega i} = C_{\omega j} = 0$) under uniform in-plane moment $M^0$, as shown in Table 1, is examined. The case with constant axial force is also considered. In both cases, critical moment $M^0_{cr}$ is computed. The computed results are found to be in good agreement with the existing analytical results (See, for example, Ref. 13), as shown in Table 1.

The out-of-plane instability of a circular arch with doubly symmetric I section simply supported at the sections of both ends under horizontal end-force, as shown in Table 2, is considered next. The curved member is assumed to be formed by a set of straight members, connecting the nodal points introduced in the original configuration and appropriate transformation matrices are employed. The critical compressive as well as tensile forces are obtained and compared with corresponding results from Ref. 14, as shown in Table 2. The computed results are seen to deviate much from those given in Ref. 14. Remembering that the value of zero should result in the case of the semicircular arch ($\theta = 180^\circ$) due to its instability of rotating about the diameter connecting the two ends, irrespective of the load applied, the results in the Ref. 14 are dubious, and the present analysis seems to be right. It is also noted that the critical forces (compressive as well as tensile) of the simply supported circular arch decrease rapidly with increasing subtended angles.

Finally, the out-of-plane buckling of a simply supported circular arch subject to uniform in-plane

Table 1 Critical Moment of Straight Member, in kN-m.

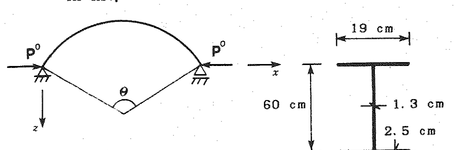



Member Length = 5 m
E = 2. 060 × $10^8$ kPa
G = 7. 946 × $10^7$ kPa

| Axial Force | Critical Positive Moment | Critical Negative Moment |
|---|---|---|
| 0 | 1873. 7 (1873. 7*) | 1032. 0 (1032. 0) |
| 762. 0 kN ( = 0. 2 $\frac{\pi^2 E I_{yy}}{l^2}$ ) | 1646. 1 (1645. 1) | 830. 4 (830. 2) |

*Values in brackets are the analytical results from Ref. 13

Table 2 Critical Horizontal End-force of Circular Arch, in kN.

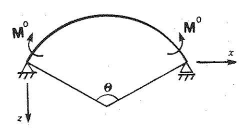



Arch Length = 5 m
E = 2. 060 × $10^8$ kPa
G = 7. 946 × $10^7$ kPa

| Subtended Angle (θ) | Critical Compressive Force | Critical Tensile Force |
|---|---|---|
| 90° | 431. 1 (2402. 5*) | 9387. 9 (913. 8) |
| 135° | 108. 1 (2543. 7) | 4927. 5 (452. 1) |
| 180° | 0. 0 (2788. 0) | 0. 0 (245. 0) |

*Values in brackets are from Ref. 14

Table 3 Critical Moment of Circular Arch, in kN-m.




Arch Length = 10. 244 m
E = 1. 998 × $10^8$ kPa
G = 7. 717 × $10^7$ kPa

| Subtended Angle (θ) | Critical Positive Moment | Critical Negative Moment |
|---|---|---|
| 10° | 343. 6 (344. 0*) | 345. 8 (345. 9) |
| 30° | 333. 0 (333. 6) | 339. 4 (339. 3) |
| 50° | 315. 5 (314. 9) | 324. 8 (323. 8) |
| 90° | 244. 1 (253. 2) | 261. 1 (266. 1) |
| 180° | 0. 0 (0. 0) | 0. 0 (0. 0) |

*Values in brackets are from Ref. 11

moment $M^0$ is investigated. For the sake of comparison with Ref. 11, critical moments of a doubly symmetric member with $A=92.88\ \text{cm}^2$, $I_{yy}=11\,363\ \text{cm}^4$, $I_{zz}=3\,871\ \text{cm}^4$, $I_{\omega\omega}=555\,869\ \text{cm}^6$, $J=58.9\ \text{cm}^4$, and $l=1\,024.4$ cm are computed for a number of subtended angles. The results are presented in Table 3 and compared with those obtained by Yoo[11], and found to be in good agreement. This confirms Yoo's opinion that the results obtained by Vlasov and Timoshenko have been wrong. It should be noted that all the numercal results presented in this paper have been obtained using 16 elements which have been proved sufficient by the convergence study.

## 6. SUMMARY AND CONCLUSIONS

Based on the second order kinematic field which is consistently approximated from the rigorous assumptions for beams, the general stiffness equation of linearized finite displacements for a thin-walled member has been formulated, using the theorem of virtual work. The stiffness matrix obtained is concise and explicit, suitable for versatile computations. The matrix is found to be identical to the tangential stiffness which facilitates the overall evaluation of the nonlinear behavior of thin-walled members.

An important and practical application is made for the evaluation of the lateral-torsional buckling load of in-plane beams and frames. Helped by its simpler stiffness expression which is a particular case of the above, the method presented in this paper can be easily used by individuals and design offices, which may render the existing cumbersome and complicated design charts of concern unnecessary. The stiffness equation obtained can be applicable not only for straight members but for curved members also, utilizing the idea of the assemblage of straight elements.

Numerical examples are given for three cases which are the out-of-plane instabilities of the straight and circular members. Compared with existing literature which has been proved unsatisfactory, the results indicate that the present analysis scheme is accurate, efficient and versatile for wide applications.

## 7. ACKNOWLEDGEMENT

This study is supported in part by the Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture.

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(Received August 15 1984)