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HOW MUCH CONTRIBUTION DOES THE SHEAR DEFORMATION HAVE IN A BEAM THEORY ?

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A beam theory including the effect of shear deformation is formulated in finite deformation. The governing equations are organized in terms of two non-dimensional parameters, one of which is the slenderness ratio. In order to quantify the contribution of shear, the elastic and inelastic buckling of a simple beam is analysed, and results are examined with respect to those two parameters. The significant reduction of critical stresses is observed for shorter columns, and the range of the slenderness ratio is obtained, in which the difference between this theory and the Bernoulli-Euler beam theory becomes prominent. Moreover, judging from the fact that shorter columns buckle inelastically, this theory becomes more important for the inelastic analyses of such deep beams.

1. INTRODUCTION

In small deformation, a beam theory including shear deformation is known as the Timoshenko beam theory[1]. This theory is derived by the relaxation of the Bernoulli-Euler hypothesis which assumes that the cross-section remains planar and normal to the axis of a beam all through the deformation. The significance of this shear component becomes eminent especially for rather deep beams and for the high-frequency or impulsive response of beams[2]. These are natural consequences, because the Bernoulli-Euler assumption is the result from the fact that the shear component turns out negligible in the two-dimensional, elastic analyses of slender bodies. Therefore, the shorter the slender body is, the more contribution the shear deformation has in its mechanical behavior. This suggests the existence of some relationship between the effect of shear and the slenderness ratio. And thus, the beam theory needs to be rearranged in terms of the slenderness ratio and/or other physical parameters.

To this end, a beam theory which corresponds to the Timoshenko beam theory will be formulated in finite deformation. Such a beam theory was formulated, for example, by Reissner[3),4)], Ziegler[5)], Sheinman[6)] and Taweep *et al.*[7)]. However it was not intended to clarify the differences between these theories and the Bernoulli-Euler beam theory. In this report, the main objective is to make these differences clear in terms of non-dimensional parameters, and, furthermore, some discussion on the constitutive relations and their approximations will be given as well. The governing equations are derived by the commonly used method with the aid of the principle of virtual work[8)], and turn out identical to those by Reissner[4)] except the constitutive law. In terms of stress resultants and their corresponding "generalized" strains, some discussion on the constitutive law is also found in connection with the buckling problem in Refs. 4) and 5).

Non-dimensionalization of the governing equations automatically yields two physical parameters, one of which is

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the slenderness ratio, and another is directly associated with the effects of shear. Since the instability of one beam member can be examined analytically, the buckling of a simply supported beam will be solved in order to find this effect, and results will be compared with those presented in Ref. 9).

2. GENERAL FORMULATION

Since the following formulation has been studied in detail elsewhere[8], we here enumerate only the necessary governing equations and other relations.

Consider a spacially fixed, rectangular Cartesian coordinate system with unit base vectors, $\boldsymbol{i}_j$ $(j=1.2.3)$, and write three axes by the x-, y- and z-axis for convenience. Let x-axis coincide with the axis of a straight beam in the reference configuration, and assume that the deformation occurs in the $x-y$-plane. Let X and Y denote the x- and y-component of the position vector in the reference state. As is shown in Fig. 1, if the cross-section remains planar but its normal, $\boldsymbol{n}$, and the axis of a beam make an angle, $\Lambda(X)$, due to shear deformation, and if the cross-section does not change its shape and area, then the displacement field can be given by

$$u_1(X, Y)=u(X)-Y\sin\lambda(X), \quad u_3(X, Y)=0,$$
$$u_2(X, Y)=v(X)+Y\{\cos\lambda(X)-1\}, \qquad (1)$$
$$\tan(\lambda+\Lambda)=v'/(1+u'),$$

where $u(X)$ and $v(X)$ are the displacement components of a particle on the beam axis, in the x- and y- direction, and $\lambda(X)$ is the rotation of the cross-section around $z-$axis ; and prime indicates the differentiation with respect to X.

Let $\boldsymbol{g}_j(X, Y)$, $(j=1,2)$, denote the unit convected base vector parallel to $\boldsymbol{i}_j$ in the reference configuration, and $\boldsymbol{G}_j(X, Y)$ the corresponding base vector in the current state. Since Green's strain tensor, $E_{ij}(X, Y)$, can be defined by $E_{ij}=(\boldsymbol{G}_i\cdot\boldsymbol{G}_j-\boldsymbol{g}_i\cdot\boldsymbol{g}_j)/2$, and since $\boldsymbol{G}_i=(\delta_{ji}+u_{j,i})\boldsymbol{g}_j$, the non-zero components of E_{ij} associated with the displacement field, (1), are obtained as

$$2E_{11}(X, Y)=g(X, Y)-1, \quad 2E_{12}(X, Y)=\sqrt{g_0(X)}\sin\Lambda, \qquad (2)$$

where

$$g_0(X)=|\boldsymbol{G}_1(X, 0)|^2=(1+u')^2+(v')^2,$$
$$g(X, Y)=|\boldsymbol{G}_1(X, Y)|^2=\{\sqrt{g_0}\cos\Lambda-Y\kappa(X)\}^2+(\sqrt{g_0}\sin\Lambda)^2, \qquad (3)$$
$$\kappa(X)=\lambda'.$$

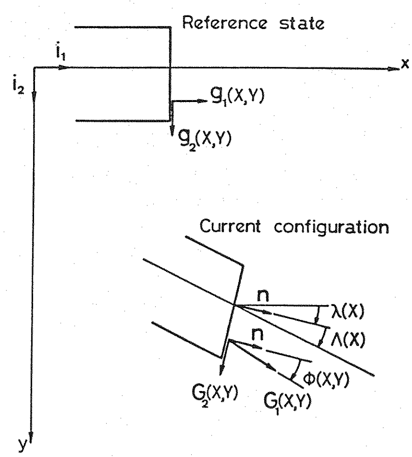

Fig. 1 Definitions of kinematic quantities ; $\boldsymbol{n}$ is a unit normal to the cross-section ; $\lambda(X)$ is a rotation of the cross-section ; $\Lambda(X)$ and $\phi(X,Y)$ are the measures of shear deformation.

In (3), $g(X, Y)$ is square of the stretch of a fiber parallel to x-axis in the reference state.

Since the Green strain is merely a kinematic function, it is necessary to define the corresponding physical quantities in conjunction with the constitutive relation. The most straightforward definitions of such quantities may be the extension of a fiber which is parallel to the x-axis in the reference state, and the change of the angle between two fibers, which are orthogonal to each other in the reference configuration. Let e denote the extension of such a fiber that initially lies parallel to the x-axis. Then from (3)

$$e=|\boldsymbol{G}_1(X, Y)|-1=\sqrt{g}-1=[\{1+\varepsilon(X)-Y\kappa(X)\}^2+\{\gamma(X)\}^2]^{1/2}-1, \qquad (4)$$

where

$$\varepsilon(X)=\sqrt{g_0}\cos\Lambda-1, \quad \gamma(X)=2E_{12}=\sqrt{g_0}\sin\Lambda. \qquad (5)$$

Physically, $\varepsilon(X)$ is the component of the extension of an axis, normal to the cross-section, and $\gamma(X)$ is its in-plane component. It must be noted that neither E_{11} nor e are linear functions of Y because of shear. On the other hand, in the Bernoulli-Euler beam, E_{11} has not, but e has the linear distribution with respect to Y.

Since $\{\pi/2-\phi(X, Y)\}$ is the angle between $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ in the current state (see Fig. 1), ϕ can be a candidate for the physical measure of shear deformation. As $\boldsymbol{G}_2$ remains unit on the cross-section, from (2) and (3), ϕ is given by

$$\cos\phi(X, Y)=(1+\varepsilon-Y\kappa)/\sqrt{g}, \quad \sin\phi(X, Y)=\gamma/\sqrt{g}. \qquad (6)$$

Note that E_{12} in (2) or γ in (5) is a function of X only ; i. e. E_{12} is uniform in the cross-section, but that ϕ in (6) is not.

Using the Kirchhoff stress tensor, S^{ij}, and the Green strain tensor, we write the internal virtual work for a beam of the length l, as

$$\int_0^l\int_A(S^{11}\,\delta E_{11}+2S^{12}\,\delta E_{12})da\,dX, \qquad (7)$$

where $\int_A$ da denotes the integration over the cross-section. Substitution of (2) into (7) yields

$$\int_0^l\int_A(\sigma\delta e+\tau\delta\gamma)da\,dX, \qquad (8)$$

where σ is the normal stress in G_1-direction and τ is the shear stress in G_2-direction on the cross-section, defined by

$$\sigma=\sqrt{g}\,S^{11}, \quad \tau=S^{12}. \qquad (9)$$

Then the principle of virtual work gives the equilibrium equations in $0<X<l$, as

$$\begin{aligned}&\{N(X)\cos\lambda-V(X)\sin\lambda\}'+p(X)=0,\\&\{N(X)\sin\lambda+V(X)\cos\lambda\}'+q(X)=0,\\&M'(X)+\sqrt{g_0}\,\{V(X)\cos\Lambda-N(X)\sin\Lambda\}+m(X)=0,\end{aligned} \qquad (10)$$

and the boundary conditions at $X=0$ and l as

$$\begin{aligned}&u=\bar{u} \quad \text{or} \quad (N\cos\lambda-V\sin\lambda)\,n=\bar{N},\\&v=\bar{v} \quad \text{or} \quad (N\sin\lambda+V\cos\lambda)\,n=\bar{V},\\&\lambda=\bar{\lambda} \quad \text{or} \quad Mn=\bar{M},\end{aligned} \qquad (11)$$

where $N(X)$ and $V(X)$ are the stress resultants normal to G_2 and in G_2-direction, and $M(X)$ is the bending moment around z-axis produced by the normal component of σ on the cross-section. They are defined by

$$N(X)=\int_A\sigma\cos\phi\,da, \quad M(X)=\int_A\sigma\cos\phi(-Y)da, \quad V(X)=\int_A(\tau+\sigma\sin\phi)da, \qquad (12)$$

where the second term in the integrand of V is the shear component of the normal stress, which will play an important role in the buckling problem later on. In (10), $p(X)$ and $q(X)$ are the distributed load per unit reference length in the x-and y-direction, and $m(X)$ is the distributed moment per unit initial length around z-axis. In (11), $\bar{u}$, $\bar{v}$ and $\bar{\lambda}$ are the corresponding displacement components specified on the boundary, and $\bar{N}$, $\bar{V}$ and $\bar{M}$ are the axial, and shear forces in x- and y-direction, and the moment around z-axis applied on the boundary, respectively. n denotes the x-component of the outer normal vector of the cross-section on the boundary in the reference configuration, and thus is given by

$$n=1 \text{ at } X=l, \quad -1 \text{ at } X=0. \qquad (13)$$

The remaining portion of the governing equations is the most controversial part ; i. e. constitutive relations. Since the components of the stress and strain tensors are not usually measured in the experiments, it might be stringent to use their physical components discussed above. But the form of the internal virtual work, (8), suggests reasonable pairs for the constitutive law, which are σ and e, and τ and γ. Therefore, from the theoretical viewpoint, it is pertinent that each pair of those is linked by some physically meaningful functionals, even in plasticity.

In the case of elasticity, we employ the simplest form as

$$\sigma=Ee, \quad \tau=G\gamma, \qquad (14)$$

where E and G are the material coefficients which are not necessarily constant but may depend on state variables. As is pointed out, although γ includes the effect of shear deformation, it is merely the in-plane component of the axial extension, but ϕ given in (6) purely represents the shear deformation. Therefore it may not be proper to relate γ to τ directly. But, since there exists ambiguity or arbitrariness to define the shearing part of constitutive relations by τ and ϕ, $(14)_2$ can be accepted as an expedient formula from a purely mathematical point of view.

As for linear elasticity, E and G correspond to Young's modulus and shear modulus, which are constant. But, even in such a case, the stress resultants are not linear functions of ε, $\varkappa$ and γ, because of the nonlinearity due to shear deformation in the definition of e, (4).

3. BERNOULLI-EULER BEAM

Neglect of shear deformation ; i. e. setting $\phi(X, Y)$ and $\Lambda(X)$ zero in the governing equations above, leads to the Bernoulli-Euler beam theory. If we use the constitutive law $(14)_1$ in the *linear elasticity*, with a constant E, then we have

$$N=EA\varepsilon+EB\varkappa,\quad M=EB\varepsilon+EI\varkappa, \qquad (15)$$

where A is a cross-sectional area and

$$B=\int_A(-Y)da,\quad I=\int_A Y^2\,da. \qquad (16)$$

In this case, the total potential associated with the virtual work can be determined as

$$\phi_0=1/2\int_0^l(EA\varepsilon^2+2EB\varepsilon\varkappa+EI\varkappa^2)dX-\int_0^l(pu+qv+m\lambda)dX-(\overline{N}u+\overline{V}v+\overline{M}\lambda)_{X=0,l}. \qquad (17)$$

The inextensible version of the Bernoulli-Euler beam is known as an elastica[9], which assumes the inextensibility of the beam axis. This condition can be expressed in the present theory as $\sqrt{g_0}-1=0$. Therefore, the governing equations with this condition can be derived from the stationary condition of a functional

$$\Psi_0=\phi_0+\int_0^l T(\sqrt{g_0}-1)dX, \qquad (18)$$

where T is introduced as a Lagrange's multiplier but can be physically interpreted as an axial reaction force due to the constraint produced by inextensibility.

4. APPROXIMATION OR ALTERNATIVE THEORY

A rigorous nonlinear theory of a beam with shear has been expressed completely by the governing equations from (1) through (14) in elasticity. They form a nonlinear boundary-value problem, but the constitutive relation is so highly nonlinear that it will be a formidable task to solve for general boundary conditions. In this section, an attempt will be made to approximate this theory by some reasonable assumption on the constitutive law. Only the *uniform cross-section* will be considered.

(1) Approximation

Since Λ appears because of shear, Λ or γ is in the order of strain. And from the form of (15), ε and $\varkappa$ are also in the same order as strain. Therefore if *small strain* is assumed, then from (4) and (6), the first-order approximation can be expressed as

$$e\simeq\varepsilon-Y\varkappa,\quad \cos\phi\simeq1,\quad \sin\phi\simeq\gamma. \qquad (19)$$

Substituting (14) and (19) into (12) and taking only linear terms for ε, $\varkappa$ and γ, we arrive at the approximated constitutive law for the stress resultants as

$$N\simeq EA\varepsilon+EB\varkappa,\quad V\simeq GA\gamma,\quad M\simeq EB\varepsilon+EI\varkappa, \qquad (20)$$

in *linear elasticity*, where A, B and I are defined in the previous section.

In order to non-dimensionalize the governing equations, the following notation is introduced ;

$$\zeta=X/l,\quad (\dot{\ })=d/d\zeta. \qquad (21)$$

Choose x-axis such that $B=O$. Introduction of the "thickness parameter"[10]; i. e. the inverse of the slenderness ratio ; β, and of a new parameter, α, defined by

$$\beta=r_0/l,\quad r_0^2=I/A,\quad \alpha=E/G, \qquad (22)$$

results in the following non-dimensional field equations in $0<\zeta<1$;

$$\dot{z}_1=-q_1,\quad \dot{z}_2=-q_2,$$

$$\dot{z}_3=-\{1+\beta^2(1-\alpha)y_1\}y_2-q_3,$$

$$\dot{z}_4=(1+\beta^2 y_1)\cos z_6-\alpha\beta^2 y_2\sin z_6-1, \qquad (23)$$

$\dot{z}_5=(1+\beta^2 y_1)\sin z_6+\alpha\beta^2 y_2 \cos z_6,$

$\dot{z}_6=z_3,$

where

$$z_1=l^2(N\cos\lambda-V\sin\lambda)/(EI),\quad z_2=l^2(N\sin\lambda+V\cos\lambda)/(EI),$$
$$z_3=Ml/(EI),\quad z_4=u/l,\quad z_5=v/l,\quad z_6=\lambda, \qquad (24)$$
$$y_1=z_1\cos z_6+z_2\sin z_6,\quad y_2=-z_1\sin z_6+z_2\cos z_6,$$
$$q_1=pl^3/(EI),\quad q_2=ql^3/(EI),\quad q_3=ml^2/(EI),$$

and the boundary conditions are, at $\zeta=0$ and 1, as

$$z_4=\overline{z}_4 \ \text{ or } \ z_1 n_\zeta=\overline{z}_1,$$
$$z_5=\overline{z}_5 \ \text{ or } \ z_2 n_\zeta=\overline{z}_2, \qquad (25)$$
$$z_6=\overline{z}_6 \ \text{ or } \ z_3 n_\zeta=\overline{z}_3,$$

where

$$\overline{z}_1=\overline{N}l^2/(EI),\quad \overline{z}_2=\overline{V}l^2/(EI),\quad \overline{z}_3=\overline{M}l/(EI),$$
$$\overline{z}_4=\overline{u}/l,\quad \overline{z}_5=\overline{v}/l,\quad \overline{z}_6=\overline{\lambda}, \qquad (26)$$
$$n_\zeta=1 \ \text{ at } \ \zeta=1,\ -1 \ \text{ at } \ \zeta=0.$$

The Bernoulli-Euler theory can be obtained as a limiting case as $\alpha\to 0$; i. e. rigidity in shear. Therefore the parameter, α, represents the factor of the contribution of shear, together with the thickness parameter, β. And the theory of an elastica is formally resumed as the further limiting case as $\beta\to 0$. The assumption of small strain is adopted only in the constitutive relations here.

(2) Alternative Theory

Thus far the field equations, (23), have been introduced as an approximation of the complete theory formulated in the second section. But these can be interpreted as an alternative theory based on another constitutive relation different from (14). Substituting (4) into (8) and considering (6), we can express the internal virtual work as

$$\int_0^l\int_A\{\sigma\cos\phi\,\delta(\varepsilon-Y\kappa)+(\tau+\sigma\sin\phi)\delta\gamma\}da\,dX. \qquad (27)$$

If we specify the constitutive law *in elasticity*, from (27), as

$$\sigma\cos\phi=E(\varepsilon-Y\kappa),\quad \tau+\sigma\sin\phi=G\gamma, \qquad (28)$$

then (23) is no longer an approximation, but a rigorous field equation associated with the constitutive relations (28), with constant E and G, which is employed by Taweep *et al.*[7]. Physically $(28)_1$ relates the normal component of the stress vector on the cross-section to the normal component of the extension. And $(28)_2$ relates the G_2-components of these two quantities.

5. ELASTIC BUCKLING OF A SIMPLE BEAM

The significance of shear is evaluated by the analysis of the elastic buckling of a simply supported beam subjected to the axial load. A fundamental solution before buckling is obtained as

$$u=-P_0X/(EA),\quad v=0,\quad \lambda=0, \qquad (29)$$

for both theories discussed in the second and fourth section.

(1) Alternative Theory

First we examine the alternative theory derived in the previous section. From (23), a linear boundary-value problem for the increments of v and λ, Δv and $\Delta\lambda$, from the fundamental solution, is obtained as

$$l^2\Delta\lambda'''+z_0\{1+\beta^2 z_0(\alpha-1)\}\Delta\lambda'=0,$$
$$l^2\alpha\beta^2\Delta\lambda''-(1-\beta^2 z_0)\{1+\beta^2 z_0(\alpha-1)\}\Delta\lambda+\{1+\beta^2 z_0(\alpha-1)\}\Delta v'=0, \qquad (30)$$

with the boundary conditions at $X=0$ and l, as $\Delta v=0$ and $\Delta\lambda'=0$, where the equations for Δu are independent of those for Δv and $\Delta\lambda$, and thus are omitted here ; $z_0=\pi^2P_0/P_0^*$, and $P_0^*=\pi^2EI/l^2$ which is the Euler load. (30) with the boundary conditions forms an eigenvalue problem, and the minimum eigenvalue gives the lowest critical load, $(z_0)_{cr}$, as

$$\frac{(P_0)_{cr}}{P_0^*}=\frac{1}{2\pi^2\beta^2(1-\alpha)}\{1-\sqrt{1-4\pi^2\beta^2(1-\alpha)}\}, \quad (31)$$

which is identical to Eq. (2. 67) at p. 143 in Ref. 9), that includes the effects of both shear and extension* before buckling. From (31), it is clear that the buckling does not occur for columns with the slenderness ratio, β^{-1}, smaller than 2π, if shear deformation is neglected.

The corresponding load without the effect of extension before buckling can be obtained by setting β zero with keeping $\alpha\beta^2$ in (31). It then follows that

$$(P_0)_{cr}/P_0^*=(\sqrt{1+4\pi^2\alpha\beta^2}-1)/(2\pi^2\alpha\beta^2), \quad (32)$$

which is again the identical result to that in Ref. 9) (Eq. (2. 59) at p. 135), and is considered by the authors to "be more accurate" than Eq. (2. 57) at p. 133, that is (36) below.

(2) Original Theory

The original theory derived in the second section is now examined. Here the increments of the stress resultants are obtained directly from their integral forms as

$$\begin{aligned} \Delta N&=\int_A(\Delta\sigma)\cos\phi\,da-\int_A\sigma(\Delta\phi)\sin\phi\,da,\\ \Delta M&=\int_A(\Delta\sigma)\cos\phi(-Y)da-\int_A\sigma(\Delta\phi)\sin\phi(-Y)da, \quad (33)\\ \Delta V&=\int_A\{\Delta\tau+(\Delta\sigma)\sin\phi+(\Delta\phi)\,\sigma\cos\phi\}da, \end{aligned}$$

where the condition that the cross-sectional area does not change is used. Substituting the incremental form of (14) with constant E and G, into the incremental field equations, we obtain the eigenvalue problem for Δv and $\Delta\lambda$, as

$$\left.\begin{aligned} &l^2\Delta\lambda'''+z_0(1-\beta^2z_0)^2\{1-\beta^2z_0(1+\alpha)\}^{-1}\Delta\lambda'=0,\\ &l^2\alpha\beta^2\Delta\lambda''-(1-\beta^2z_0)^2\Delta\lambda+(1-\beta^2z_0)\Delta v'=0, \end{aligned}\right\}\cdots(34)$$

with the same boundary condition as before. From (34), the characteristic equation for $(P_0)_{cr}$ is

$$1-\pi^2\beta^2(1+\alpha)(P_0)_{cr}/P_0^*=\{1-\pi^2\beta^2(P_0)_{cr}/P_0^*\}^2(P_0)_{cr}/P_0^*. \quad (35)$$

Similarly to (32), the critical load without extension effect is given from (35) by

$$(P_0)_{cr}/P_0^*=(1+\pi^2\alpha\beta^2)^{-1}, \quad (36)$$

which is also derived in Ref. 9) (Eq. (2. 57) at p. 133) and is "more on the safe side" than (32). Fig. 2 shows the results from (31), (32), (35) and (36) for $\alpha=0$ and 3 (values for α will be discussed later on). The differences between (31) and (35), or (32) and (36) stem from the third term in the integrand of $(33)_3$ which is neglected as a small term in the approximated theory, and which cannot be directly taken into account in the alternative theory. The *second-order approximation* can be developed in order to include the contribution from this term (see Appendix).

The effect of extension of a beam axis enhances the critical load that causes the dangerous estimate. On the other hand,

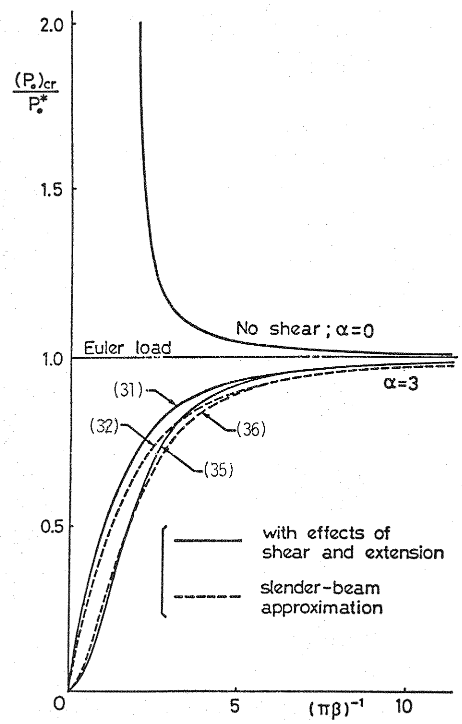

Fig. 2 Elastic critical loads of simply supported beam by general theories, (35) and (36), and by approximated theories, (31) and (32) ; the "slender-beam approximation" neglects the extension before buckling.

* The terminology, "extension" is employed throughout this paper, although the beam axis is shortened in this particular problem. This is simply because this effect stems from the difference between "extensible" and "inextensible" beam theories.

the shear effect has the tendency to reduce the buckling load.

(3) Critical Strain (Stress)

Although the differences from the Euler load in Fig. 2 seem considerable for beams with smaller slenderness ratio, it is necessary to know the corresponding critical strain (or stress) to check whether the fundamental state is in the elastic state or not. The critical strain, ε_{cr}, can be defined by $(P_0)_{cr}/(EA)$ and therefore

$$\varepsilon_{cr}=\pi^2\beta^2(P_0)_{cr}/P_0^*. \qquad (37)$$

Fig. 3 shows the critical strains from (31), (32), (35) and (36).

The difference between those equations is not perceptible when the slenderness ratio is larger than 5π, while the tremendous reduction of the buckling strain is observed for shorter columns. At any rate, the critical strain level is so high that the inelastic analyses become necessary for the ordinary structural materials.

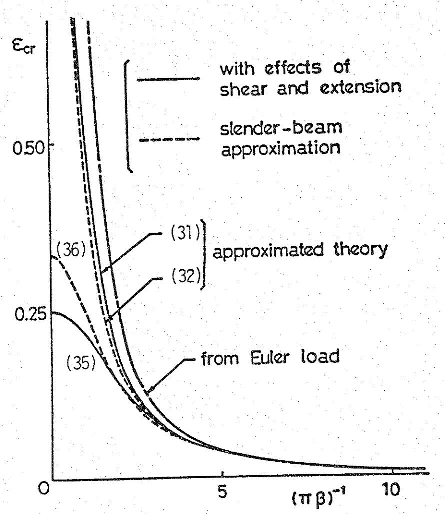

Fig. 3 Corresponding critical axial strains of a simply supported beam by general theories, (35) and (36), and by approximated theories, (31) and (32).

6. INELASTIC BUCKLING

Only the original theory is considered, because the safer estimate of the critical strain is expected. The governing equations in the second section hold except the constitutive equation (14). Without the constitutive law, the fundamental solution for this stability problem of a simply supported beam can be expressed as

$$\left.\begin{array}{l} u \text{ is indeterminate, } v=0,\ \lambda=0, \\ N=-P_0,\ V=0,\ M=0, \end{array}\right\} \qquad (38)$$

but at least we can expect the linearity of u with respect to X, only for this kind of simple problem ; in other words, $\varepsilon=u'$ is constant. No matter what relation is employed for the inelastic part of the constitutive law, we can calculate $E_s=P_0/(A\varepsilon)$, once we know P_0 and ε. This implies a formal relation between stress and strain as

$$\sigma=E_s\,\varepsilon, \qquad (39)$$

where E_s is not constant but can be interpreted as a secant modulus. It must be, however, emphasized that (39) is not generally a constitutive equation but merely a relation to describe the state. Acceptance of (39) allows the expression as

$$\varepsilon=u'=-P_0/(E_sA),\quad Q_0^{-1}=\sqrt{g_0}=1-P_0/(E_sA). \qquad (40)$$

Similarly to (39), suppose that the incremental relations between stresses and strains are given by

$$\Delta\sigma=E_t\,\Delta e,\quad \Delta\tau=G_t\,\Delta\gamma, \qquad (41)$$

where E_t and G_t are the current tangent parameters. (41) may be a special form of the commonly used incremental constitutive equations.

Then defining

$$\overline{EA}=\int_A E_t\,da,\quad \overline{EB}=\int_A E_t(-Y)da,\quad \overline{EI}=\int_A E_t\,Y^2\,da,\quad \overline{GA}=\int_A G_t\,da, \qquad (42)$$

we obtain the incremental equations near the fundamental solution, (38) and (40) as

$$\begin{array}{l} (\overline{EA}\,\Delta u'+\overline{EB}\,\Delta\lambda')'=0, \\ \{P_0\,\Delta\lambda+Q_0(1-P_0\,Q_0/\overline{GA})(\overline{EB}\,\Delta u'+\overline{EI}\,\Delta\lambda')'\}'=0, \\ \Delta\lambda=Q_0\{\Delta v'+Q_0(\overline{EB}\,\Delta u'+\overline{EI}\,\Delta\lambda')'/\overline{GA}\}, \end{array} \qquad (43)$$

with the same boundary condition as before.

If we accept the Shanley theory, then the minimum buckling load can be obtained by the tangent modulus theory in which no unloading zone in the cross-section is assumed[11]. Since the stress state in the beam is uniform all over the body, the integration in (42) can be carried out to obtain

$$\overline{EA}=E_t A,\ \overline{EB}=0,\ \overline{EI}=E_t I,\ \overline{GA}=G_t A, \quad (44)$$

when the x-axis is defined such that $B=O$. Similarly to the elastic case, (43) forms an eigenvalue problem and, therefore, the buckling load is obtained from the following characteristic equation ;

$$1-\pi^2\beta^2(\mu+\eta\alpha)(P_0)_{cr}/P_0^*=\xi\{1-\pi^2\beta^2\mu(P_0)_{cr}/P_0^*\}^2(P_0)_{cr}/P_0^*, \quad (45)$$

where

$$\xi=E/E_t,\ \eta=G/G_t,\ \mu=E/E_s. \quad (46)$$

Once some explicit forms for ξ, η and μ are known, (45) gives the critical load, although ξ, η, and μ are functions of stress or P_0 itself, in general. Therefore (45) is an implicit equation for the critical load. The corresponding equation for elastic buckling, (35), can be retrieved by setting ξ, η and μ unity in (45).

For beams with the slenderness ratio larger than 2π, the axis of a beam can be almost inextensible. Therefore, neglecting μ in (45), we obtain the approximate equation for the critical load in lieu of (45) as

$$(P_0)_{cr}/P_0^*=(\xi+\pi^2\eta\alpha\beta^2)^{-1}. \quad (47)$$

The corresponding critical compressive stress, $\sigma_{cr}=(P_0)_{cr}/A$, is then expressed by

$$\sigma_{cr}/\sigma_Y=(P_0)_{cr}/(P_0^*\bar{\lambda}^2), \quad (48)$$

where $\sigma_Y=E\varepsilon_Y$ and ε_Y are the tensile yield stress and strain, and

$$\bar{\lambda}=\sqrt{\varepsilon_Y}/(\pi\beta). \quad (49)$$

Many phenomenological models are proposed to describe the material properties, but most of them are based on the experimental observations, and the micromechanical behavior is not taken into account. In three dimensions, however, Hutchinson has carried out the global estimate of the mechanical behavior of the polycrystalline metals on the basis of the micromechanical approach[12]. Recently similar calculation has been done in two dimensions[13]. And it has been observed in both reports that the change of G_t is more gradual than that of E_t in the numerical tensile test. According to this numerical observation, one phenomenological model with only two parameters has been proposed[13] for rather ductile materials as

$$\xi=\exp\{\eta_1(\sigma/\sigma_Y-1)^{1/2}\},\ \eta=\eta_2(\sigma/\sigma_Y-1)^2+1, \quad (50)$$

for $\sigma/\sigma_Y\geq 1$, and $\xi=\eta=1$ for $\sigma/\sigma_Y<1$. In (50), no Bauschinger effect is considered. To be rigorous, σ in (50) must be replaced by its absolute value. But, since only compression appears in this special buckling problem, the compressive stress is taken positive in this section for convenience. With the aid of the schematic definitions of material parameters in Fig. 4, from (46) and (50), we can obtain

$$\mu=\sigma_Y[1+2\{(s-1/\eta_1)\exp(\eta_1 s)+1/\eta_1\}/\eta_1]/\sigma, \quad (51)$$

for $s\geq 0$, where $s=\sqrt{\sigma/\sigma_Y-1}$.

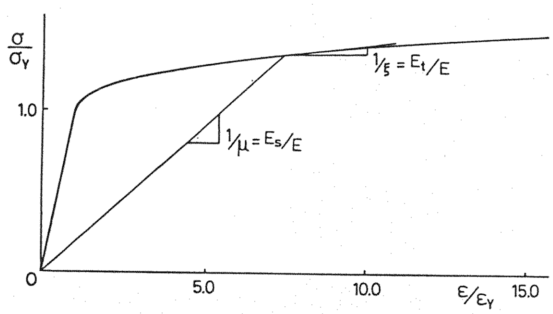

Fig. 4 Schematic concept of material parameters for the inelastic constitutive relation.

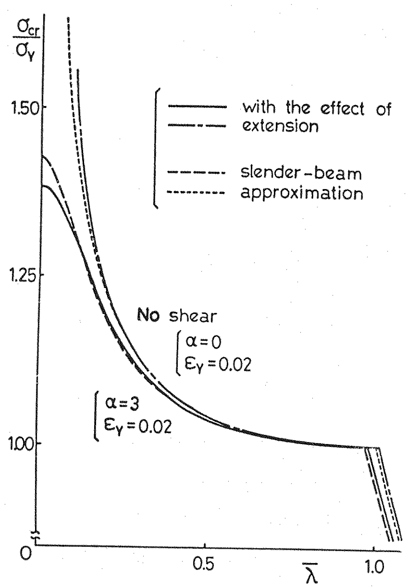

Fig. 5 Inelastic critical stresses for a specific material model, given by (50), with the yield strain, $\varepsilon_Y=2\%$, with and without shear ; the "slender-beam approximation" without shear effect coincides with the ordinary inelastic buckling stress.

In Ref. 13), a specific numerical calculation based on a special micromechanical model has determined that $\eta_1=6.5$ and that $\eta_2=60$. The form of (50) is originally defined as a general deformation-theory-type formula in Ref. 13), but (50) itself is valid only for this kind of simple problems where the stress state is uniform and uniaxial. Otherwise, σ/σ_Y in the right-hand side of (50) must be replaced by some function that reflects the yield condition under the general stress state ; e. g. J_2.

As an illustration, using those parameters and relations, (50) and (51), we can calculate σ_{cr}/σ_Y from its implicit equations (45) and (48). Fig. 5 shows such results for $\alpha=0$ and 3, with rather large yield strain, $\varepsilon_Y=0.02$.

In pratice, since the slender bodies considered as beams have $\bar{\lambda}$ larger than 0. 2, the precise equation as well as its approximations predict more or less same critical stresses. While the critical stresses evaluated with the effect of shear are bounded for any $\bar{\lambda}$, those without shear are not finite at $\bar{\lambda}=0$. Moreover, no buckling is predicted for $\bar{\lambda}$ smaller than approximately 0. 1, by the rigorous theory without shear, because of the extension effect before buckling, as has been also observed in the elastic instability, Fig. 2.

7. DISCUSSIONS

(1) Shear Coefficient

A simple kinematics assumed ; i. e. the displacement field (1), follows from the assumption of the uniform distribution of shear in the cross-section, (2), which violates the boundary condition on the lateral surfaces of a beam where no traction exists unless the distributed load is applied on them. In order to compensate this impropriety, the shear coefficient is introduced in small deformation[14]. With this coefficient, we simply modify the shearing rigidity, GA by GAK, where K is the shear coefficient that depends on both material properties ; i. e. Poisson's ratio, ν, and the shape of the cross-section. This change, accordingly, requires the correction of the definition of α, $(22)_3$, as

$$\alpha=E/(GK). \qquad (52)$$

Although K is calculated in small deformation of an isotropic body, the same expression can be used here only to find an approximate range for α. According to results in Ref. 14, for example, the circular cross-section has the expression as $K=6(1+\nu)/(7+6\nu)$, and for rectangular cross-section, $K=10(1+\nu)/(12+11\nu)$. If we consider E and G as ordinary elastic constants, then, since $E/G=2(1+\nu)$ and since $0<\nu<1/2$, the range of α is obtained as follows ; $7/3<\alpha<10/3$, for circular cross-section ; $12/5<\alpha<35/10$, for rectangular cross-section. Therefore $\alpha=3$ is extensively used in the preceding examples.

(2) Constitutive Laws

As has been physically explained, (28) connects the stress vector to the extension. But, since τ and $(\sigma \sin\phi)$ are completely different quantities on the cross-section in $\boldsymbol{G}_2$-direction, it is not a good idea to relate τ to γ, and $(\sigma\sin\phi)$ to γ by the same coefficient, G. Note that the stress vector is not generally in $\boldsymbol{G}_1$-direction, while the extension is in $\boldsymbol{G}_1$-direction. Because of this non-coaxiality, it is not also recommended to use (28) with E and G constant.

In other words, (28) may be theoretically improved by the refinement as

$$\sigma\cos\phi=E_1(\varepsilon-Y\kappa), \quad \sigma\sin\phi=E_2\gamma, \quad \tau=G\gamma, \qquad (53)$$

where E_1 and E_2 are not necessarily constant and E_1 corresponds to a normal component of Young's modulus on the cross-section, and E_2 to a tangential component. Then (53) and the second-order approximation of the original theory, (A. 2), are essentially equivalent. $(53)_1$ and $(53)_2$ represent a tension-compression law of a fiber which is parallel to the x-axis in the reference configuration, and $(53)_3$ expresses a shearing constitutive relation.

In Ref. 4), it has been also revealed that the different constitutive equations lead to the various predictions of the critical buckling load. The author describes the "stress-strain" relation by that between the stress resultants and their corresponding "generalized" strains of a beam. However we believe that the constitutive relation must be discussed at the level of the stress and strain together with the definitions of stress resultants and "generalized strains" as has been done in the present paper, because the material properties are essentially microscopic.

As long as the slenderness ratio is very large, however, the discrepancies between those constitutive models are negligible.

8. CONCLUDING REMARKS

Two theories based on two different constitutive models have been presented, where the second theory is formally identical to a first-order approximation of the first one. A simple buckling problem is solved, and the effects of shear as well as extension prior to buckling are examined. In the elastic buckling of a column, the original theory predicts much smaller critical stresses in the range of small values of the slenderness ratio than the Bernoulli-Euler beam theory. On the other hand, the alternative theory shows the same asymptotes as the classical buckling stress, although there exists the definite reduction of the critical stresses. As long as the elastic buckling is concerned, a beam with the slenderness ratio larger than 10π can be treated as a Bernoulli-Euler beam, because the difference between theories becomes less than about 3% for such relatively longer beams.

The slenderness ratio, where these differences become eminent, are so small that, in practice, the inelastic analyses are necessary. Using a phenomenological model which is based on the deformation-theory-type plasticity, we have shown similar differences between those theories as the elastic case. A further approximation on the extensibility of the axis of beams before buckling has been adopted to find the agreement with the pre-existing solutions, and to examine them. It then follows that the original theory formulated in the second section yields the most reasonable results from a theoretical point of view. The most interesting result is that the critical stress, which used to be considered "more accurate" in Ref. 9), is not obtained by the rigorous theory, but by its approximation.

From a practical viewpoint, however, columns have the unavoidable imperfection to some extent, which produces non-uniaxial stress state before the instability, and therefore the numerical calculation is required. But, in order to utilize this theory with shear more effectively, it will be necessary to develop a more reasonable constitutive model in such a general stress state.

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APPENDIX. SECOND-ORDER APPROXIMATION

As is clear from the results for elastic buckling, Fig. 3, the second term in the integrand of the definition of shear, $(12)_3$, plays an important role for shorter beams. The alternative theory, discussed in section 4, absorbs this term within the constitutive law, (28), and thus its contribution can not emerge explicitly. In the approximated theory, however, this term can be taken into account within the assumption of *small strain* as follows.

Substitution of $(19)_2$ and $(19)_3$ into $(12)_1$ and $(12)_2$ yields

$$N(X) \simeq \int_A \sigma\, da, \quad V(X) \simeq \int_A \tau da + \gamma \int_A \sigma da. \qquad \text{(A. 1)}$$

If the constitutive relation, (14), together with the approximation, $(19)_1$, is substituted into (A. 1), then the second term in $(\text{A. 1})_2$ drops because it becomes the higher order term with respect to strain. But, before such manipulation, the direct substitution of $(\text{A. 1})_1$ into $(\text{A. 1})_2$ with (14) results in the following nonlinear expression for shear ;

$$V = (GA+N)\gamma. \qquad \text{(A. 2)}$$

Apparently, $(GA+N)$ acts like an "effective" shearing rigidity which can be reduced by compression, and this is the reason for the considerable difference in predicting buckling loads. Then the field equations for this approximation are expressed by the following system of 6 nonlinear ordinary differential equations :

$$\begin{aligned}
&\dot{z}_1 = -q_1, \quad \dot{z}_2 = -q_2, \\
&\dot{z}_3 = -\{\beta^2 y_1 + (1+\alpha\beta^2 y_1)^{-1}\} y_2 - q_3, \\
&\dot{z}_4 = (1+\beta^2 y_1)\cos z_6 - \alpha\beta^2 y_2 (1+\alpha\beta^2 y_1)^{-1} \sin z_6 - 1, \\
&\dot{z}_5 = (1+\beta^2 y_1)\sin z_6 + \alpha\beta^2 y_2 (1+\alpha\beta^2 y_1)^{-1} \cos z_6, \\
&\dot{z}_6 = z_3,
\end{aligned} \qquad \text{(A. 3)}$$

in *linear elasticity*, where the *uniform cross-section* and $B=O$ are assumed. The first-order approximation or the alternative theory, (23), is obtained by the Taylor expansion of $(1+\alpha\beta^2 y_1)^{-1}$ and neglect of the higher order terms with respect to $\alpha\beta^2$, provided $\alpha\beta^2$ is very small.

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