

## **DETERMINATION OF SEDIMENT CONVEYANCE RATE IN ALLUVIAL STREAMS BY USING NEURAL NETWORKS**

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### **SYNOPSIS**

The concept of artificial neural networks (ANN) is an advanced topic that provides hydraulic and environmental engineers with a strong tool for estimating the missing information to be used for design purposes and management practice. In this study, we demonstrate how neural networks can be used to estimate sediment discharge in rivers by simple extrapolation of reliable data collected from other sources. The problems in selecting appropriate data, training algorithms and neural network structure were addressed using a constructive algorithm called Back-Propagation Algorithm (BPA). Sensitivity analyses based on flow and sediment parameters were also performed. The sediment concentrations predicted from the neural networks model were in good agreement with measured values. Error analysis was used to confirm the accuracy of results from this novel approach using data with wide ranges from several rivers. Compared with the other conventional methods for calculating sediment discharge, the neural networks based model generally gave better results among the well-known previous published methods for calculating sediment discharge.

### **INTRODUCTION**

The movement of sediment in streams and rivers is a complex process that is dependent on the interplay of several variables and parameters. Several approaches have been proposed to estimate the sediment discharge by using the similarity principle (5), dimensional analysis (2) and analytic power models (15), etc. Because of the nature of their discrete formulations, some effective parameters should be disregarded, and consequently the accuracy of the predicted results will decrease. Recently, the computer science known as artificial neural network (ANN) has found many applications in several engineering fields. However, as far as we can assess, this computer-based technique has not yet been used to study practical fluvial engineering problems such as sediment transport.

In this study, we have evaluated the applicability of a neural networks approach to examine the environmental problem of sediment transport utilizing the back propagation algorithm (12). The water and sediment parameters are decided by using the previous conventional dynamic analysis. Several trials are conducted to decide the effective input parameters and to design the suitable

structure of the network. The model is trained by real field data, and its parameters are adjusted to produce the most accurate results. While our computer-based model did provide a more accurate prediction of sediment discharge than the conventional models currently in use, this study is more far reaching in that it spotlights how existing global database systems can be used to study important hydroinformatic and related engineering issues.

## NEURAL NETWORKS MODEL

### Network Architecture

An artificial neural network represents a net of simple local memory units called neurons or nodes. Such units are connected by unidirectional links that carry data. The semilinear feed forward net (12) has been found to be an effective system for discriminating patterns from a body of examples. Node outputs from one layer are transferred to nodes in another layer through links that amplify or inhibit data outputs through weighting factors. Except for the input layer nodes, the net input to each node is represented by the sum of the weighted node outputs in the previous layer. Each node is activated in accordance with the node inputs, the node activation function, and the node bias.

Figure 1 shows the general feed-forward multilayer net model, including a hidden layer,  $j$ . The input pattern constitutes the inputs to the nodes in the input layer,  $i$ , representing a set of variables  $(x_1, x_2, \dots, x_n)$ . For the model, node outputs in the layer are taken to be equal to node inputs. Alternatively, the inputs can be normalized or scaled to fall between the values of 0 and +1. The output layer,  $k$ , in the model generally consists of multiple nodes  $(o_1, o_2, \dots, o_m)$ , although a single node is sometimes used. Figure 2 shows the simulated structure of the node. The node sums the product of outputs from nodes of the previous layer and connection weights, and then limits it by a nonlinear threshold function.

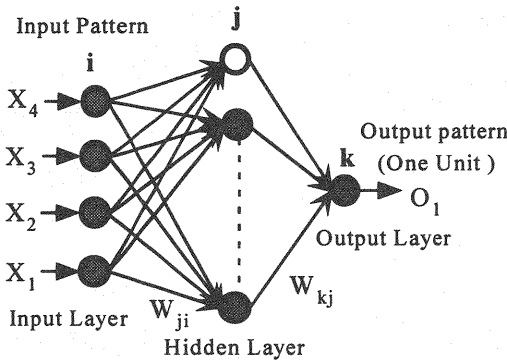


Fig. 1 Feed-forward multilayer network.

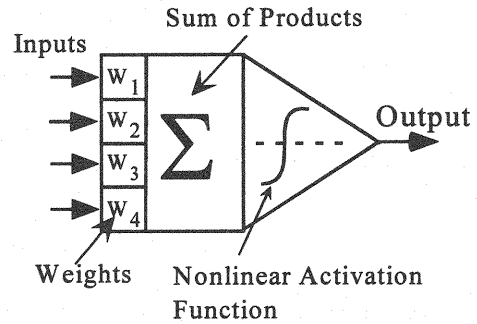


Fig. 2 Structure of the node.

The net input to and output from the  $j$ 'th node of the layer  $j$  are

$$net_j = \sum_i w_{ji} o_i \quad (1)$$

$$o_j = f(net_j) \quad (2)$$

where  $o_i$  = output from the  $i$ 'th node of the previous layer,  $i$ ;  $w_{ji}$  = connection weight between the nodes  $i$  and  $j$ ; and  $f$  = activation function. In calculating a node output, the activation function

may be considered as a threshold function, in which generation of a node output occurs once a threshold level is reached.

### General Delta Rule ( GDR )

In the learning phase of training such a net, the pattern,  $x_p$ , is the input parameter, where  $p$  is the pattern number. The weights in all connecting links are adjusted accordingly. Once this adjustment has been accomplished, another pair of  $x_p$  and  $t_{pk}$  is presented, and the net learns that association too, where  $t_{pk}$  is the target value for  $x_p$ . In general, actual output,  $o_{pk}$ , in the output layer,  $k$ , will differ from the target or desired value,  $t_{pk}$ . For each pattern, square of average error is taken to be

$$E_p = \frac{1}{2} \sum_k (t_{pk} - o_{pk})^2 \quad (3)$$

The derivative of the error function,  $E$ , with respect to any weight in the network is in proportional to the incremental weight changes. For general delta rule (GDR), the change of weight for the pair from  $j$ 'th to  $i$ 'th nodes can be set as

$$\Delta w_{ji} = -\varepsilon \frac{\partial E}{\partial w_{ji}} = \varepsilon \delta_j o_i \quad (4)$$

where  $\varepsilon$  = learning rate;  $\delta_j = -\partial E / \partial o_j f'(net_j)$ ; and  $f'(net_j) = \partial o_j / \partial net_j$ .

The deltas at internal node can be evaluated in terms of the deltas at the upper layer. In particular,  $o_j$  is represented by a sigmoid function (7) as

$$o_j = \frac{1}{1 + \exp[-\alpha (\sum_i w_{ji} o_i - \theta_j)]} \quad (5)$$

where  $\alpha$  = shaping ratio of function  $f$ ; and  $\theta_j$  = threshold or bias. Then, the following expressions may be presented, for output and hidden layers, respectively.

$$\delta_{pk} = (t_{pk} - o_{pk}) o_{pk} (1 - o_{pk}) \alpha \quad (6)$$

$$\delta_{pj} = o_{pj} (1 - o_{pj}) \alpha \sum_k \delta_{pk} w_{kj} \quad (7)$$

### Back-propagation Algorithm

Using the back-propagation procedure, the net calculates  $\Delta_p w_{ji}$  for all  $w_{ji}$  in the net for the particular  $p$ . This procedure is repeated for all the patterns in the training set to yield the resulting  $\Delta w_{ji}$  for all the weights for that one presentation. The correction to the weights is made and the outputs are again evaluated in feed-forward manner. Discrepancies between actual and target output values again result in evaluation of weight changes. After complete presentation of the all patterns in the training set, a new set of weights is obtained and new outputs are again evaluated in feed-forward manner. This is repeated until a specific error level is obtained.

## ESTIMATION OF SEDIMENT DISCHARGE

The pertinent variables in river hydraulics are the water discharge per unit width,  $q$ , water depth,  $h$ , longitudinal bed slope,  $S$ , channel width,  $B$ , bed shear stress,  $\tau$ , total sediment discharge per unit width,  $q_t$ , median diameter,  $d_{50}$ , sediment and fluid densities,  $\rho_s$  and  $\rho$ , kinematic viscosity,  $\nu$ , acceleration gravity,  $g$ , and fall velocity,  $w_0$ . The parameters  $\rho_s$  and  $\rho$  are constants. In previous studies, effective parameters for sediment discharge problems are often presented in dimensionless forms. The most acceptable and wide-use forms of these parameters are presented in the following expression as

$$C_s = f(\psi, \phi, w_0/u_*, S, h/d_{50}, F_r, R_{e*}, h/B) \quad (8)$$

where  $C_s = q_t/q$  = total sediment concentration parameter;  $\psi = hS/sd_{50}$  = dimensionless bed shear stress;  $s$  = submerged specific weight of sediment ( $= 1.65$ );  $\phi = u_m/u_*$  = velocity ratio;  $u_m$  = mean velocity;  $u_* = \sqrt{ghS}$  = shear velocity;  $w_0/u_*$  = dimensionless suspended sediment parameter;  $h/d_{50}$  = water depth scale ratio;  $F_r = u_m/\sqrt{gh}$  = Froude number;  $R_{e*} = u_*d_{50}/\nu$  = shear velocity Reynolds number; and  $h/B$  = stream width scale ratio. The net is set up with eight parameters of Eq. 8 as the input pattern and the sediment concentration,  $C_s$ , as the output pattern. The network is trained with well-shuffled data. Input layer contains eight neurons, while output layer contains one. Between them, there is another hidden layer with suitable number of neurons that are under investigation.

### Data for Learning and Verification

Measuring the total sediment discharge in rivers is difficult under most natural conditions. Ideally such measurements should be conducted at some locations with natural or even artificial contraction in the cross section of the river, where the total load is converted to suspended load. This is not easy done for natural rivers and streams where a wide range of physical situations is clearly at play. Therefore, for this study the available data sets for flow and sediments in natural streams, which comprise wide range of situations and contain the total load discharge, are those of the Niobrara River (3), the Middle Loup River (6), the Hii River (13), and the small streams (1).

Other published data that are unreliable in bed load are excluded in the analysis. The data group in this study consists of 161 sets. Half of the data were used for the learning process, while the other half were used for prediction. Dimensionless parameters and their numerical ranges are summarized in Table 1.

Table 1 Range of data for learning and verification.

Variables	Range	Variables	Range
Bed shear stress $\psi$	0.10 ~ 3.68	Froude number $F_r$	0.15 ~ 0.56
Velocity ratio $\phi$	4.10 ~ 15.0	Reynolds number $R_{e*}$	4.36 ~ 135.5
Suspension parameter $w_0/u_*$	0.13 ~ 2.39	Stream width ratio $h/B$	0.002 ~ 0.10
Longitudinal bed slope $S$	.00041 ~ .00287	Sediment concentration $C_s$	10 ~ 3240
Water depth ratio $h/d_{50}$	152 ~ 6242	(ppm)	

### Calibration of Neural Networks Parameters

The model is constructed with 81 shuffled data sets (patterns). The original target output data are first used during learning process, then discarded. New results are obtained for the 81 data sets. Number of neurons in the hidden layer, the parameters,  $\alpha$  and  $\varepsilon$ , are determined by calibration

through several computer run tests. The parameter,  $\varepsilon$ , is recommended to be 0.04 to 0.10. The best fitting is shown in Fig. 3, where number of neurons in the hidden layer is 12,  $\varepsilon$  is 0.075 and  $\alpha$  is 12. After the network is well trained, the weights as well as the bias are adjusted, and the pre-mentioned parameters are calibrated. Then the other half of 80 patterns is used for prediction without target outputs,  $C_s$ . Estimated values for sediment concentration,  $C_s$ , were compared with the measured ones. As seen in Fig. 4, the values obtained from the computer network model favorably agreed with the measured ones.

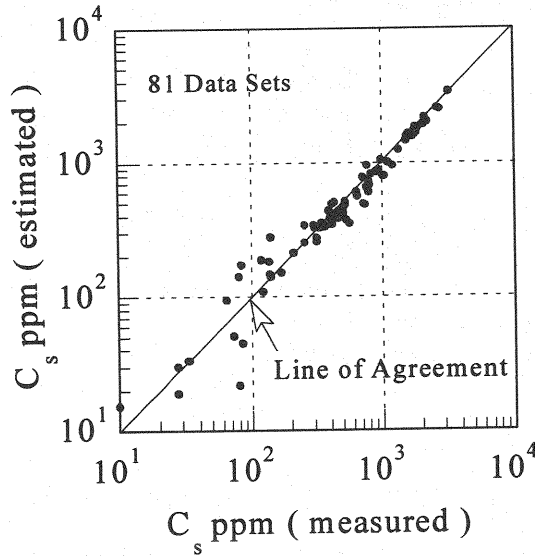


Fig. 3 Comparison between measured and estimated concentrations,  $C_s$ .

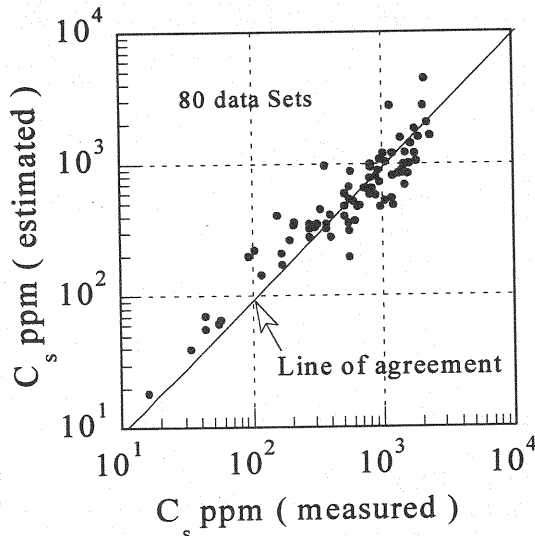


Fig. 4 Verification of the present model.

## SENSITIVITY ANALYSIS FOR WATER AND SEDIMENT PARAMETERS

Several experiments were conducted to examine the sensitivity of the provided sediment parameters in each pattern. With fixed model parameters, the first run was carried out with eight input variables that are mentioned above. Then, each parameter was eliminated by turn from the group. Statistical analysis was used for determining the accuracy of the results.

A discrepancy ratio,  $D_r = C_c / C_m$ , was used for comparison, where  $C_c$  represents the calculated total load concentration, and  $C_m$  is the measured one. The mean value,  $\bar{D}_r$ , and the standard deviation,  $\sigma$ , are expressed as

$$\bar{D}_r = \frac{\sum_{i=1}^N D_{ri}}{N} \quad (9)$$

and

$$\sigma = \sqrt{\frac{\sum_{i=1}^N (D_{ri} - \bar{D}_r)^2}{N-1}} \quad (10)$$

Also, the ranges of  $\pm 25\%$ ,  $\pm 50\%$  and  $\pm 75\%$  of the predicted concentrations are presented.

From Table 2, it can be concluded that the most important dimensionless parameters in the group are the six of  $\psi$ ,  $w_0/u_*$ ,  $h/d$ ,  $F_r$ ,  $R_{e*}$  and  $h/B$ . The parameters which have effect less than 10% may be neglected without fear of accuracy, such as,  $\phi$ , and  $S$ . Then the functional form of new group is

$$C_s = f(\psi, w_0/u_*, h/d_{50}, F_r, R_{e*}, h/B) \quad (11)$$

The new group of parameters is tested again after eliminating non-effective ones.

Table 2 Effect of flow and sediment parameters on results accuracy.

Inputs of flow and sediment parameters	Number of data sets	Discrepancy Ratio				
		Mean	Standard Deviation	Percent of Data in Range		
				0.75~1.25	0.5~1.5	0.25~1.75
1.The full parameters in Eq. 8	161	1.03	0.40	65	87	94
2. Eliminating " $\psi$ "		1.30	1.45	58	80	87
3. Eliminating " $\phi$ "		1.01	0.60	60	85	92
4. Eliminating " $w_0/u_*$ "		1.57	2.50	53	78	86
5. Eliminating " $S$ "		1.05	0.56	59	81	89
6. Eliminating " $h/d_{50}$ "		1.85	2.40	55	69	78
7. Eliminating " $F_r$ "		1.29	1.47	58	78	87
8. Eliminating " $R_{e*}$ "		1.20	1.25	63	82	89
9. Eliminating " $h/B$ "		1.03	1.74	42	78	87

## COMPARISON WITH THE PREVIOUS STUDIES USING TOTAL LOAD DATA

The present model was compared with seven previously published studies; see (14 and 15). The same numbers of data sets were used for prediction for all tested models. The analysis is shown in Table 3. Figure 5 shows a comparison between the best fit for results of the present model and the most acceptable results of Brownlie formula (2). The legitimacy of our model is therefore fulfilled.

Table 3 Accuracy of formulas for total sediment concentration, { field data }.

Method	Number of data sets	Discrepancy Ratio				
		Mean	Standard Deviation	Percent of Data in Range		
				0.75~1.25	0.5~1.5	0.25~1.75
1. Present ANN model, Eq. 11	80	1.04	0.42	58	78	93
2. Engelund and Hansen (1967)		2.34	1.69	14	35	45
3. Ackers and White ( $d_{50}$ ) (1973)		1.10	1.45	48	75	90
4. Yang ( $d_{50}$ ) (1973)		1.30	0.81	51	69	81
5. Brownlie (1981)		1.04	0.67	56	76	93
6. Shen and Hung (1972)		1.26	0.62	44	70	84
7. Laursen (1958)		0.55	0.89	8	20	60
8. Toffaleti (1968)		0.41	0.46	5	20	66

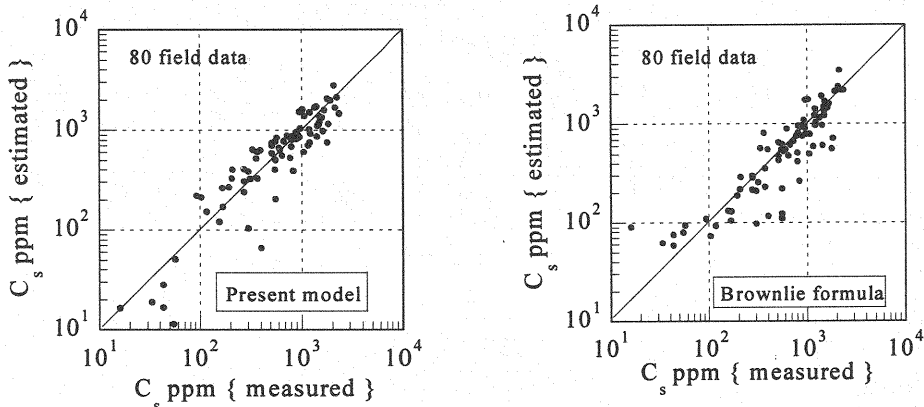


Fig. 5 Comparison between the present model and Brownlie formula.

### EVALUATION OF THE MODEL USING SUSPENDED SEDIMENT DATA

The ANN model was also verified using suspended sediment data from another group of 485 data sets collected from the Rio Grande River (4, 10 and 11), the Mississippi River (8), and the Sacramento River (9). While the suspended load for these three rivers have been measured, the bed load concentration,  $C_b$ , have not. We, therefore, used the Meyer-Peter and Muller formula (14) to calculate the bed load of these three rivers. Simple summation of the measured suspended load concentration,  $C_{sus}$ , and the calculated bed load concentration,  $C_b$ , would then yield a value of the total concentration,  $C_s$ . The range of variables is shown in Table 4. However, because some variables of the Mississippi and the Sacramento Rivers have values with ranges wider than those of the used in trained pattern, their sediment concentration cannot be extrapolated. Therefore, one half of the Mississippi River data were used as new training patterns. Training results were used to then estimate sediment concentration in the other half of the Mississippi River as well as the Sacramento River. When comparing with the results obtained using other formulas, the ANN model generally produced the best results. Furthermore, the ANN model's accuracy has improved significantly when the training patterns contain data with wider numerical ranges of variables. Table 5 shows the error analysis for results in all tested rivers using the ANN model. As can be seen, ANN was better than 5 out of 10 tested formulas (14 and 15) for total sediment discharge. The  $\bar{D}_r$  value of Nordin-Beverage group is rather large because most of sets have variables with extremely larger values than the trained ones; see Tables 1 and 4.

Table 4 Hydraulic and sediment data for the tested rivers.

River Variables	Rio Grande R. (Nordin)	Rio Grande R. (Nordin et al.)	Rio Grande R. (Culberston)	Mississippi R. (Jordan)	Sacramento R. (Nakato)
Num. of data	58	234	139	34	20
$\psi$	0.29~2.34	0.08~5.98	0.5~4.46	0.29~2.39	0.25~2.98
$w_0/u_*$	0.28~1.06	0.17~2.79	0.16~0.67	0.17~1.36	0.36~1.58
$h/d_{50}$	583~4735	107~8388.2	1016~14696	10855~56693	3220~18770
$F_r$	0.24~0.68	0.11~0.58	0.225~0.79	0.084~0.196	0.11~0.21
$R_{cs}$	8.29~33.61	5.9~396.54	6.36~32.67	6.29~94.85	18.6~149
$h/B$	.0017~.042	0.002~0.078	.0014~.066	0.01~0.031	0.017~.078
$C_s$ (ppm)	130~4236	10~9186	285~6773	13~271	23~242
$C_b/C_s$	0.50	0.31	0.11	0.093	0.30

Table 5 Accuracy of methods for different rivers data.

Method	Number of data sets	DISCREPANCY RATIO				
		Mean	Standard deviation	Percent of Data in Range		
				0.75~1.25	0.5 ~ 1.5	0.25 ~ 1.75
Rio Grande River Data (by Nordin)						
Present ANN model, Eq. 11	58	0.998	0.46	41	76	88
Engelund and Hansen		0.96	0.44	37	72	91
Ackers and White ( $d_{50}$ )		0.80	0.48	33	71	84
Yang ( $d_{50}$ )		0.57	0.25	26	66	88
Brownlie		0.88	0.48	31	69	85
Shen and Hung		0.74	0.41	31	62	86
Rio Grande River Data (by Nordin and Beverage)						
Present ANN model, Eq. 11	234	1.55	6.0	31	67	84
Engelund and Hansen	234	2.9	8.3	31	39	54
Ackers and White ( $d_{50}$ )	233	0.96	0.8	33	62	82
Yang ( $d_{50}$ )	233	1.37	1.40	35	56	77
Brownlie	234	1.53	1.15	35	64	84
Shen and Hung	234	1.21	0.91	36	62	79
Rio Grande River Data (by Culberston and Dawdy)						
Present ANN model, Eq. 11	139	0.93	0.43	45	76	92
Engelund and Hansen		0.95	0.43	46	75	93
Ackers and White ( $d_{50}$ )		1.37	0.74	32	57	75
Yang ( $d_{50}$ )		0.62	0.28	24	68	93
Brownlie		1.21	0.56	37	67	84
Shen and Hung		0.95	0.44	45	77	93
Sacramento River Data (by Nakato)						
Present ANN model, Eq. 11	20	1.01	0.43	40	85	95
Engelund and Hansen		2.48	2.01	10	30	40
Ackers and White ( $d_{50}$ )		1.0	0.76	35	55	75
Yang ( $d_{50}$ )		1.08	0.75	45	75	80
Brownlie		1.62	1.31	35	70	75
Shen and Hung		1.23	1.46	40	55	70
Mississippi River Data (by Jordan)						
Present ANN model, Eq. 11	34	0.98	0.33	53	88	100
Engelund and Hansen		1.68	0.90	35	44	59
Ackers and White ( $d_{50}$ )		1.09	0.70	35	50	82
Yang ( $d_{50}$ )		0.75	0.50	35	50	88
Brownlie		1.40	0.59	29	56	68
Shen and Hung		0.59	0.43	15	53	76



## CONCLUSIONS

The present study illustrates one particular aspect of hydroinformatics as an application of the neural networks to sediment transport problems. Armed only with river sediment data records obtained from various sources and ANN, we were able to accurately estimate the sediment discharge in the several rivers. Several trials were done to determine the appropriate input parameters (optimized to give accurate results) and network structure (trained by use of real field data). The sediment discharge obtained from the computer-base method compared favorably with those obtained using other well-known methods based on sediment concentration determinations.

The main attractiveness of the ANN model is that, unlike other approaches, it can be applied regardless of the uncertainty or stochastic nature of the sediment movement. Increasing input patterns for learning with wide range values that come from well-established database system will increase the accuracy of the model-estimated values. In conclusion, the future implications of computer generated models for the study of important water- based engineering and environmental issues are both exciting and far-reaching.

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## APPENDIX - NOTATION

The following symbols are used in this paper:

$B$	= channel width;
$C_b$	= bed load concentration;
$C_c$	= calculated total load concentration;
$C_m$	= measured total load concentration;
$C_s$	= total sediment load concentration;
$C_{sus}$	= suspended load concentration;
$d_{50}$	= particles median diameter;
$D_r$	= statistical discrepancy ratio;
$E$	= error analysis function;
$f$	= activation function;
$F_r$	= Froude number;
$g$	= acceleration gravity;
$h$	= water depth;
$h/B$	= stream width scale ratio;
$h/d_{50}$	= water depth scale ratio;
$i, j, k$	= names of network layers, respectively;
$N$	= number of examined data;
$o$	= variable representing outputs;
$o_{pk}$	= output for $x_p$ ;
$q$	= water discharge per unit width;
$q_t$	= total sediment discharge per unit width;
$Re^*$	= shear velocity Reynolds number;
$s$	= submerged specific weight of sediment = 1.65;
$S$	= longitudinal bed slope;
$t_{pk}$	= target value for $x_p$ ;
$u_m$	= mean velocity of flow;
$u_*$	= shear velocity;
$w_{ji}, w_{kj}$	= weights on the network connections;
$w_0$	= particles fall velocity;
$w_0/u_*$	= dimensionless suspended sediment parameter;
$x$	= variable representing inputs;
$\alpha$	= shaping ratio of function $f$ ;
$\Delta w$	= correction of the weight;
$\varepsilon$	= learning rate;

$\theta_j$	= bias value;
$\nu$	= kinematic viscosity;
$\rho_s, \rho$	= sediment and fluid densities;
$\sigma$	= statistical standard deviation;
$\tau$	= bed shear stress;
$\phi$	= velocity ratio; and
$\psi$	= dimensionless bed shear stress.

### Subscripts

$i, j, k$	= positive integer indices; and
$p$	= pattern number.

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