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AN APPROXIMATE ANALYSIS FOR ENTRAINMENT RELATION OF INCLINED PLUMES BY THE GALERKIN METHOD

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SYNOPSIS

A purely mathematical approach is undertaken to the *E-Ri* relation of a two-dimensional inclined plume flowing over a frictionless slope. This is an idealized mathematical experiment to gain knowledge of the basic characteristics of the relation. The Galerkin method is applied to Fukushima's equations (1988), a set of dimensionless boundary layer equations including the effect of buoyancy, to obtain efficiently the *E-Ri* relation under conditions of systematically changed hydraulic parameters. The resulting *E-Ri* relation shows the same tendency as an entrainment formula empirically proposed by Turner(1986) in which *E* decreases rapidly as *Ri* approaches unity. This characteristic of the *E-Ri* relation is physically explained by a consideration of the TKE budget in a plume.

INTRODUCTION

An inclined plume, a layer of fluid little heavier than surroundings flowing along a sloping ground, appears in many forms at many situations, such as a turbidity current in the ocean, cold down-wind from mountains at night, flood water intrusion into a reservoir. The density currents of this kind play an important role in heat and mass transport in natural environment.

The entrainment coefficient *(E)*, a ratio of the velocity of inflow across the density interface to the velocity of the underflow, is an important factor to describe the motion of density currents(Ellison and Turner, (3)). A lot of experiments have been conducted in laboratory flumes in order to obtain a relation between the coefficient and the Richardson number *(Ri)*, a parameter indicating the stability of density currents.

Several empirical formulae of "the entrainment law" have been proposed for inclined plumes on the basis of flume data. Most of them can be classified into two categories: One is a group of power functions(Egashira(2), Fukushima(5)) and its modifications(Parker et al. (8)), which show straight lines for large Ri in a logarithmic plot. The other is a group of fractional functions(Akiyama et al. (1) and Turner(9)) with which E drops sharply as Ri approaches some value (~ 1) even in a logarithmic plot. In this paper, the former is called as Type-1 formula, and the latter as Type-2 formula.

It is not concluded yet which expression would be more appropriate to describe the entrainment relation for inclined plumes because of the following problems of laboratory experiment: The range of experimental parameters such as bottom slope, relative density difference and Reynolds number are limited due to the scale of facilities and the available working fluids. As a result, the data are piled in a restricted area of E-Ri map, and most of the empirical formulae pass through the cloud of data and then diverge from one another. However, information on entrainment rate at conditions out of the cloud becomes significant when we estimate the scale of plumes appearing in natural environment.

To break this deadlock, two new approaches different from laboratory experiment are being taken recently. One is a field experiment on an inclined plume in Lake Ogawara far larger than laboratory flumes(Nagao and Ishikawa, (7)). Its results support

Type-2 formula, providing entrainment coefficients quite smaller than those predicted by Type-1 formula at Ri of around 1. The other is a mathematical approach using turbulent model equations (Fukushima (6)). Fukushima investigated the E-Ri relation by using k-ε model equations based on a dynamic similarity of inclined plumes. The results of his study showed the tendency similar to Type-2 formula. However, because the condition for numerical calculation was not controlled systematically, the results were not in a line but a fragment of a twisted band, and himself did not refer to the analogy of his results with Type-2 formula.

In this paper, a mathematical approach is presented as an extension of Fukushima's method: the Galerkin method is applied to Fukushima's equations to obtain a series of analytical solutions under systematically changed flow conditions. A unique curve for the E-Ri relation showing the same tendency as Type-2 formula is obtained under the condition of negligible bottom friction. A remarkable feature of Type-2 formula, the existence of upper limit for Ri, is theoretically illustrated by energy budget consideration.

STRATEGY

Empirical relations for entrainment rate of density currents including inclined plumes have been proposed based on flume data. Many of them are expressed by the following power function:

$$E \propto E_0 \cdot Ri^{-n} \tag{1}$$

where E_0 and n = constants. Egashira(2) recommended $E_0 = 0.0015$ and $n = 1$, while Fukushima(5) gave $E_0 = 0.003$ and $n = 1.5$. On the other hand, Parker et al. (8) proposed a modified power function which has a finite E at Ri=0,as different from a simple power function.

$$E = 0.075/(1 + 718 \cdot Ri^{2.4})^{0.5} \tag{2}$$

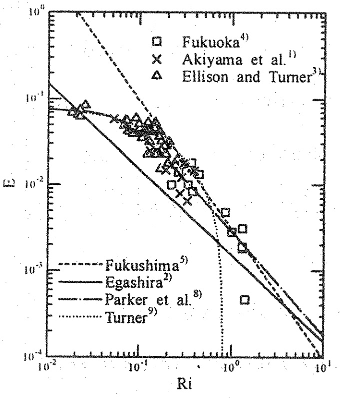

Fig.1. Comparison between flume data and emperical relations for the dependence of *E* on *Ri*.

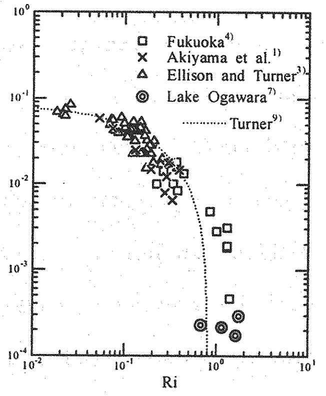

Fig.2. Comparison of field data in Lake Ogawara(7) with flume data and an entrainment relation proposed by Turner(9).

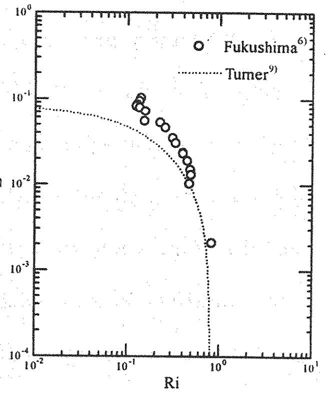

Fig.3.Numerical calculation results by Fukushima(6) and Turner's relation(9).

This relation almost coincides with an ordinary power function when $Ri>0.1$. This group of formulae called as Type-1 formula herein.

On the other hand, Turner(9) proposed the following empirical relation based on the experimental results obtained by Ellison and Turner(3).

$$E = (0.08 - 0.1Ri)/(1 + 5Ri) \qquad (3)$$

A feature of this expression is that *E* decreases to *0* rapidly when *Ri* approaches *0.8*. Akiyama et al. (1) modified the constants in Eq.(3) being based on their own experiments. These kinds of formulae are called as Type-2 formula herein.

Eqs.(1)~(3)are compared with flume data in Fig. 1. It should be noted that Eq(2) of Type-1 and Eq.(3) of Type-2 coincide with each other in a region of $Ri<0.6$ where most of flume data concentrate. However, they diverge to one another rapidly outside of the region. As mentioned earlier, information on entrainment rate at large *Ri* or small *E* becomes significant when we estimate the scale of plumes appearing in natural environment. In order to obtain data in this area of *E-Ri* map, it is necessary to carry out an experiment on a very mild slope. However, it becomes more difficult to keep the flow in well-developed turbulent state as the slope is set milder in laboratory experiment of small size. After all, an experiment of larger scale than ordinary laboratory ones is required for judging which type of relation is more appropriate.

Nagao and Ishikawa (7) carried out a detailed field measurement of an inclined plume which was generated in Lake Ogawara by the density difference between lake water and sea water intruding into the lake. This plume was fully turbulent; the layer thickness was about 1 m, the flow velocity was 20-40 cm/sec and the flow continues over a distance of 1.5 km on the slope of 0.25 degree. (It is usually thought that field measurements are less accurate than laboratory experiments. In this field measurement, however, the entrainment rate was estimated from the spatial variation of salinity over a distance of 1 km which is far larger than the length of usual laboratory flumes so that the accuracy of estimation is satisfactory. Please refers to the original paper on this point.) The result of the *E-Ri* relation obtained in Lake Ogawara is shown in Fig.2. The field and laboratory data clearly suggests that Eq.(3) of Type-2 describes the entrainment relation better than Type-1 formula.

On the other hand, Fukushima(6) made an epoch-making analytical study to estimate the entrainment relation by using k-ε model equations. At first, he assumed all physical quantities in a plume, say S_i, as the following form being based on a consideration of dynamic similarity.

$$S_i \propto x^p f_i(\eta) \tag{4}$$

where $\eta = y/x$; x = the distance along the sloping ground; y = the distance perpendicular to it. Substituting them into the k-ε model equations, and determining the exponents p s from dimensional argumentation, he derived a set of dimensionless ordinary differential equations for $f_i(\eta)$ s. Finally, he obtained the *E-Ri* relation numerically for a lot of slope angles with some variation of a coefficient C_3 in k-ε model. The results are plotted in Fig.3 in which E drops sharply as Ri approaches 1. It should be noted that Fukushima's study was purely analytical without any help of experimental evidence. (However, as mentioned earlier, himself did not refer to the analogy of his results to

Type-2 formula.)

From the above evidences, the authors consider as follows: ① Type-2 formula proposed by Turner(9) and Akiyama et al. (1) is probably more appropriate than Type-1 formula which is rather widely accepted now. ② Analytical verification of Type-2 formula will be possible by using turbulent models such as k-ε model.

The strategy of this study is basically same as Fukushima's: a set of dimensionless ordinary differential equations is derived from k-ε model equations based on dynamic similarity assumption and dimensional argumentation. The solutions of the equations for different slope angles are plotted on *E-Ri* map to discuss the entrainment relation. However, the analysis in this study is different from Fukushima's on the following three points: The first is to take account the term of additional pressure gradient appearing from the change of coordinate scale into calculation which Fukushima eliminated from his analysis. The second is the mathematical technique to solve the ordinary differential equations: Fukushima solved the equations numerically by using the finite difference method. This method takes a lot of time for iteration, and quantities sometimes do not converge enough because of the number of unknown variables, high nonlinearity of the equations and the boundary conditions assigned at two different places. Accordingly, the Galerkin method is adopted in this study to obtain a series of approximate solutions for many different slope angles under the systematical change in a coefficient C_3, which is the only uncertain factor in k-ε model. The Galerkin method is one of approximate solvers of macroscopic point of view, which obtains not the detailed profiles of variables but their magnitude or amplitude by minimizing the error of equations in average or integral. This method is considered to be suitable for the purpose of the present study, since we do not need the detailed profile of an inclined plume but the entrainment rate which is a macroscopic quantity relating to the volume integral. The third point is that the friction or drag force on the sloping ground is assumed to be negligible. The reason for this assumption is rather philosophical: There are two sources of turbulence for mixing in inclined plumes, i.e., the shear layer in halocline and the bottom boundary layer. If the latter contributes to the mixing with significant amount, the entrainment rate must be affected by the bottom roughness. This means that the entrainment coefficient *(E)* is a function not only of the Richardson number *(Ri)* but also of some dimensionless parameter indicating the effect of bottom roughness, such as the drag coefficient (C_f). In other words, in order to investigate the dependence of E on Ri, it is necessary to specify the bottom roughness or to assume conditions in which the bottom friction is negligible.

ANALYSIS

(1) Basic Equations

The basic equations for analysis are the continuity equation, conservation equations for momentum in *x*-direction, buoyancy, turbulent kinetic energy and dissipation rate, and an equation of assumption for kinetic eddy viscosity. Using "boundary layer approximations", we obtain.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5}$$

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=eg\sin\theta-g\cos\theta\frac{\partial}{\partial x}\int_y^\infty edy+\frac{\partial}{\partial y}(\nu_t\frac{\partial u}{\partial y}) \tag{6}$$

$$u\frac{\partial(eg)}{\partial x}+\mathrm{v}\frac{\partial(eg)}{\partial y}=\frac{\partial}{\partial y}(\frac{\nu_t}{\sigma_g}\frac{\partial(eg)}{\partial y}) \tag{7}$$

$$u\frac{\partial k}{\partial x}+\mathrm{v}\frac{\partial k}{\partial y}=\frac{\partial}{\partial y}(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial y})+\nu_t(\frac{\partial u}{\partial y})^2+\cos\theta\frac{\nu_t}{\sigma_g}\frac{\partial(eg)}{\partial y}-\varepsilon_d \tag{8}$$

$$\begin{aligned}u\frac{\partial\varepsilon_d}{\partial x}+\mathrm{v}\frac{\partial\varepsilon_d}{\partial y}&=\frac{\partial}{\partial y}(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial\varepsilon_d}{\partial y})\\&+C_1\frac{\varepsilon_d}{k}\{\nu_t(\frac{\partial u}{\partial y})^2+(1-C_3)\cos\theta\frac{\nu_t}{\sigma_g}\frac{\partial(eg)}{\partial y}\}-C_2\frac{\varepsilon_d^{\,2}}{k}\end{aligned} \tag{9}$$

$$\nu_t=C_\mu\frac{k^2}{\varepsilon_d} \tag{10}$$

where x = the distance along the slope; y = the distance perpendicular to the slope surface; e = relative density difference; u, v = flow velocity in the x and y direction; k = turbulent kinetic energy; ε_d = dissipation rate; g = gravity acceleration; θ = slope angle; ν_t = kinetic eddy viscosity. The values of model parameters except C_3 are set as the standard ones,

$$\sigma_g=1/1.2;\quad \sigma_k=1.0;\quad \sigma_\varepsilon=1/0.77;\quad C_\mu=0.09;\ C_1=1.44;\ C_2=1.92$$

The parameter C_3, the only uncertain parameter in k-ε model, is varied in the range from 0 to 1 in order to examine its influence on the calculation result.

(2). Dimensionless equations

Let us assume that physical quantities can be expressed by Eq.(4). Substituting them into Eqs.(5)~(10), we can determine the dependence of each valuable on x from dimensional argumentation:

$$u=U_0 f_u(\eta) \tag{11a}$$

$$\mathrm{v}=V_0 f_\mathrm{v}(\eta) \tag{11b}$$

$$eg=E_0 x^{-1} f_e(\eta) \tag{11c}$$

$$k=K_0 f_k(\eta) \tag{11d}$$

$$\varepsilon_d=\varepsilon_{d0} x^{-1} f_\varepsilon(\eta) \tag{11e}$$

$$\nu_t=\nu_{d0} x f_\nu(\eta) \tag{11f}$$

where $\eta = y/x$; f_u, f_v, f_e, f_k, f_ε=dimensionless functions for the profiles of u, v, e, k, ε, respectively. Then, the following dimensionless equations are derived.

$$\eta f'_u - f_\mathrm{v}' = 0 \tag{12}$$

$$\eta f_u f'_u - f_\mathrm{v} f'_u + f_e - \underline{\cot\theta\,\eta f_e} + (f_\nu f'_u)' = 0 \tag{13}$$

$$f_e f_u + \eta f_u f'_e - f_\mathrm{v} f'_e + \frac{1}{\sigma_g}(f_\nu f'_e)' = 0 \tag{14}$$

$$\eta f_u f'_k - f_v f'_k + \frac{1}{\sigma_k}(f_v f'_k)' + f_v f'^2_u + \frac{1}{\sigma_g}(\cot\theta\, f_v f'_e) - f_\varepsilon = 0 \qquad (15)$$

$$f_u f_\varepsilon + \eta f_u f'_\varepsilon - f_v f'_\varepsilon + \frac{1}{\sigma_\varepsilon}(f_v f'_\varepsilon)' + C_1 \frac{f_\varepsilon}{f_k} f_v f_u'^2 + C_1(1-C_3)\frac{\cot\theta}{\sigma_g} f_v \frac{f_\varepsilon}{f_k} f'_e - C_2 \frac{f_\varepsilon^2}{f_k} = 0 \qquad (16)$$

where ' (prime) = differentiation by η ; f_v = dimensionless eddy viscosity defined by

$$f_v = C_\mu f_k^2 / f_\varepsilon \qquad (17)$$

The term underlined in Eq.(13)= an additional pressure gradient which appears from the change of coordinate scale from y to η. This term was disregarded in Fukushima's analysis, but it is correctly taken into account in the present study. The six scale factors in Eqs. (11a)~(11f), $U_0 \sim \nu_{d0}$,can be defined arbitrarily. The following five relations are assumed among them in the derivation of Eqs.(12) –(17).

$$\frac{E_0 \sin\theta}{U_0^2} = 1;\ \frac{\nu_{d0}}{U_0} = 1;\ \frac{K_0}{U_0^2} = 1;\ \frac{\varepsilon_{d0}}{U_0^3} = 1 \qquad (18)$$

It should be noted that there still remains a freedom to set one of the scale factors arbitrarily because the number of above relations is five while those of the scale factors are six.

When $\eta \to \infty$, all the variables except ν must become 0. Namely,

$$f_u=0;\quad f_e=0;\quad f_k=0;\quad f_\varepsilon=0 \quad \text{at } \eta=\infty \qquad (19)$$

At the bottom surface, the fluxes of momentum, buoyancy, turbulent kinetic energy and dissipation rate must be 0 as well as vertical velocity because the frictionless condition is assumed in this study. Then,

$$f'_u=0;\quad f_v=0;\quad f'_e=0;\quad f'_k=0;\quad f'_\varepsilon=0 \quad \text{at } \eta=0 \qquad (20)$$

(2) Galerkin Equations

The trial functions of f_u to f_v are prepared on the basis of Fukushima's results and another numerical experiments by the authors which will be published in another place:

$$f_u = u_0(1-\xi) \qquad (21a)$$

$$f_v = v_0 \xi \tag{21b}$$

$$f_e = e_0(1-\xi) \tag{21c}$$

$$f_k = k_o f(\xi) \tag{21d}$$

$$f_\varepsilon = \varepsilon_0 f(\xi) \tag{21e}$$

$$f_\nu = \nu_0 f(\xi) \tag{21f}$$

where u_0, v_0, k_0, ε_0, ν_0 =constants; $\xi = \eta / \delta$; δ = the dimensionless plume layer thickness; $f(\xi)$ = a function expressed by

$$f(\xi) = \begin{cases} 2\xi & (0 < \xi < \frac{1}{2}) \\ 2(1-\xi) & (\frac{1}{2} < \xi < 1) \end{cases} \tag{22}$$

In the Galerkin method, which is one of weighted residual methods, the weight functions are taken to be the same form as the trial functions. The correspondence of variables to equations, such that Eq.(8) corresponds to the conservation of k, leads to the following combinations of weight functions:

The weight function of Eq(12) → ξ

The weight function of Eq.(13)and (14) → $(1-\xi)$

The weight function of Eq.(15) and (16) → $f(\xi)$

Multiplying weight functions to Eqs.(12)-(16) with above mentioned combinations, and integrating them over the range of $\eta = 0 \sim \infty$, we obtain the Galerkin equations as follows:

$$\int_0^\infty (\eta f'_u - f'_v) w_v d\eta = 0 \tag{23}$$

$$\int_0^\infty (\eta f_u f'_u - f_v f'_u + f_e - \cot\theta \, \eta f_e) w_u d\eta - \int_0^\infty f_v f'_u d\eta + \left| f_v f'_u w_u \right|_0^\infty = 0 \tag{24}$$

$$\int_0^\infty (f_e f_u + \eta f_u f'_e - f_v f'_e) w_e d\eta - \frac{1}{\sigma_g} \int_0^\infty f_v f'_e w'_e d\eta + \frac{1}{\sigma_g} \left| f_v f'_e w_e \right|_0^\infty = 0 \tag{25}$$

$$\int_0^\infty (\eta f_u f'_k - f_v f'_k + f_v f'^2_u + \frac{\cot\theta}{\sigma_g} f_v f'_e - f_\varepsilon) w_k d\eta$$
$$- \frac{1}{\sigma_k}\int_0^\infty f_v f'_k w'_k d\eta + \frac{1}{\sigma_k} \left| f_v f'_k w_k \right|_0^\infty = 0 \tag{26}$$

$$\int_0^\infty \{ f_u f_\varepsilon - \eta f_u f'_\varepsilon - f_v f'_\varepsilon + C_1 \frac{f_\varepsilon}{f_k} f_v f_u'^2$$
$$+ C_1(1-C_3)\frac{\cot\theta}{\sigma_g} f_v \frac{f_\varepsilon}{f_k} f'_e - C_2 \frac{f_\varepsilon^2}{f_k} \} w_\varepsilon d\eta \tag{27}$$
$$- \int_0^\infty f_v f'_\varepsilon w'_\varepsilon d\eta + \frac{1}{\sigma_\varepsilon} \left| f_v f'_\varepsilon w_\varepsilon \right|_0^\infty = 0$$

Substituting the trial functions of Eqs.(11.a-f) into the Galerkin equations, and executing the integration under the boundary conditions of Eqs.(19) and (20), we obtain the following algebraic equations:

$$\frac{\mathrm{v}_0}{u_0} = -\frac{2}{3}\delta \tag{28}$$

$$\frac{\nu_0}{u_0} = \frac{\sigma_g}{9}\delta^2 \tag{29}$$

$$\frac{e_0}{u_0^2} = \frac{7+2\sigma_g}{3(4-\cot\theta\delta)} \tag{30}$$

$$-\frac{1}{9}(1+2\frac{\sigma_g}{\sigma_k})(\frac{k_0}{u_0^2}) + \frac{\sigma_g}{27} - \frac{\cot\theta}{27}(\frac{e_0}{u_0^2})\delta - \frac{3C_\mu}{\sigma_g}(\frac{k_0}{u_0^2})^2 \frac{1}{\delta^2} = 0 \tag{31}$$

$$-\frac{1}{2\sigma_g}(1-4\frac{\sigma_g}{\sigma_\varepsilon})(\frac{k_0}{u_0^2}) + \frac{C_1}{3} - \frac{1}{3}C_1(1-C_3)\frac{\cot\theta}{\sigma_g}(\frac{e_0}{u_0^2})\delta$$
$$-27\frac{C_2 C_\mu}{\sigma_g}(\frac{k_0}{u_0^2})^2 \frac{1}{\delta^2} = 0 \tag{32}$$

There are six unknowns δ, u_0, v_0, e_0, k_0 and ε_0, while the number of equations are five. The fact means that one of them can be determined freely because we assumed only five relations among six arbitrary scale factors in Eqs.(18). By setting u_0=1, we find

$$\mathrm{v}_0 = -\frac{2}{3}\delta \tag{33}$$

$$\nu_0 = \frac{\sigma_g}{9}\delta^2 \tag{34}$$

$$e_0 = \frac{7+2\sigma_g}{3(4-\cot\theta\delta)} \tag{35}$$

$$-\frac{1}{9}(1+2\frac{\sigma_g}{\sigma_k})k_0 + \frac{\sigma_g}{27} - \frac{\cot\theta}{27}e_0\delta - \frac{3C_\mu}{\sigma_g}\frac{k_0^2}{\delta^2} = 0 \tag{36}$$

$$-\frac{1}{2\sigma_g}(1-4\frac{\sigma_g}{\sigma_\varepsilon})k_0 + \frac{C_1}{3} - \frac{1}{3}C_1(1-C_3)\frac{\cot\theta}{\sigma_g}e_0\delta - 27\frac{C_2C_\mu}{\sigma_g}\frac{k_0^2}{\delta^2} = 0 \tag{37}$$

We can easily obtain the five unknowns for specified slope angle θ by iteration with some numerical technique.

The definition of Ri proposed by Ellison and Turner(3) can be expressed by the dimensionless valuables in this study. Substituting the expression of Eq.(11.a) and Eq.(21.a) into the integrand gives the characteristic velocity defined by Ellison and Turner:

$$V = \frac{\int_0^\infty u^2 dy}{\int_0^\infty u dy} = \frac{2}{3}U_0 \tag{38}$$

In the same way, the flux of relative buoyancy is obtained as,

$$A = \int_0^\infty egudy = \frac{E_0 e_0 U_0 \delta}{3} \tag{39}$$

Therefore, the Richardson number and the entrainment coefficient are expressed by

$$Ri = \frac{A\cos\theta}{V^3} = \frac{9}{8}\cot\theta\,\delta\,e_0 \tag{40}$$

$$E = \frac{1}{V}\frac{d}{dx}\left(\int_0^\infty udy\right) = \frac{4}{3}\delta \tag{41}$$

RESULTS AND DISCUSSIONS

(1) The Entrainment Relation

As mentioned before, the most proper value of C_3 in the k-ε model has not been recommended yet. Therefore, calculations are made for several values of C_3 ranging

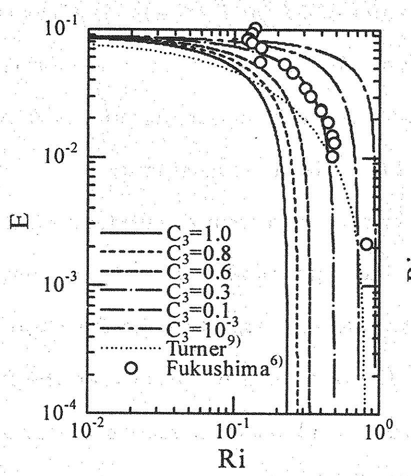

Fig.4. Comparison between the present analysis and Fukushima's numerical calculation for the dependence of *E* and *Ri*.

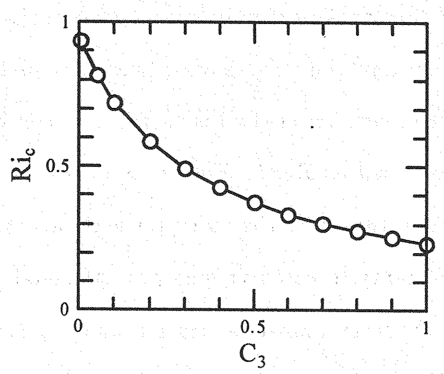

Fig.5. Dependence of the critical Richardson number on C_3.

from 0 to 1. The result of *E-Ri* relation is shown in Fig.4,and compared with Turner's relation(9) and Fukushima's numerical results(6). It is seen from the figure that all curves become a constant value (=0.085) as *Ri* approaches *0*. This value corresponds to the entrainment coefficient for jets and vertical plumes in which no densimetric effect exists on entrainment. On the other hand, as *Ri* increases, *E* rapidly drops showing the same tendency as Turner's relation and Fukushima's results. It should be noted that Fukushima's results are plotted almost along one of the curves, though the value of C_3 is not always same as each other. (In Fukushima's analysis, C_3 is changed case by case.) But, the decreasing rate of the curves of present study is larger than that of Turner's relation. A successive study by one of authors, which will be published near future, found that the decreasing rate of theoretical curves become milder when the bottom friction is taken into account.

It must be noted that the value of *Ri* at the position of falling, say the critical Richardson number Ri_c, depends on C_3. Ri_c corresponds to the limit where the slope angle approaches 0; i.e. no mixing on a horizontal bottom. Fig.5 shows the dependence of Ri_c on C_3, in which Ri_c takes the maximum value of *0.94* at $C_3=0$. This fact means that *k-ε turbulence model has no solution for inclined plumes of Ri~>1 in the condition of no bottom friction.* Accordingly, it is suggested that there is the maximum value of *Ri* for an inclined plumes on a frictionless boundary, and the value is at most unity.

The existence of the limitation of *Ri* can be explained from a consideration of TKE balance based on Eq.(8). The second term on the right hand side of Eq.(8) shows the production rate, and the third term shows the rate of energy transfer from turbulent kinetic energy to potential energy. Although the latter can be larger than the former locally, the former must be larger than the latter in total in a cross section, otherwise the dissipation rate would become negative. Then, the following inequality must be satisfied:

$$\nu \frac{U^2}{\delta^2} > \frac{\nu}{\sigma_g} \frac{Eg}{\delta} \qquad \Rightarrow \qquad Ri'(=\frac{Eg\delta}{U^2}) < \sigma_g \tag{42}$$

where U is the characteristic velocity in a section, E is the total relative buoyancy and δ the plume layer thickness. Note that the definition of the Richardson number in Eq.(42) is not strictly same as that of Eq.(40). Then the above estimation is an approximate one in a style of order estimation. A detailed calculation based on Eq.(40) leads to the following result, though the calculation process is omitted here because of the restriction of space.

$$Ri < \frac{9}{8}\sigma_g = 0.94 \tag{43}$$

The above mentioned results depend on the assumption of trial functions and weight functions, of course. However, the basic characteristics of solutions are considered to be universal because the Galerkin method is a mathematical technique to optimize the amplitude of trial function so that the total error in a section becomes the smallest, and its solution is not sensitive to minor changes of the functions.

(2) Vertical Profiles of Quantities

To examine the adequacy of solutions in detail, the dimensionless profiles of all

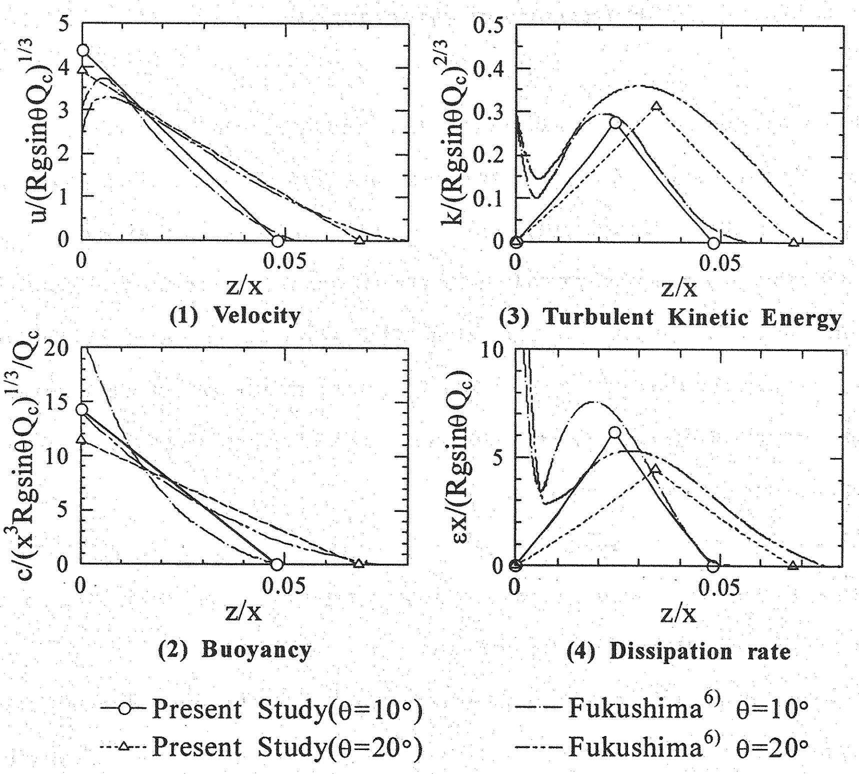

Fig.6. Dimensionless profiles of physical quantities in a cross section.
(θ= 10° , 20°)

quantities are compared with those of Fukushima's results. Figs.6 (1)~(4) show the comparisons for slope angles of 10 ° and 20° . In this calculation, the conditions are set in the same manner as Fukushima's; C_3 is 0.6, and the additional pressure term (the underlined term in Eq.(6)) is ignored. The definition of dimensionless parameter is also made in the same way as his definition. These figures show the good agreements of the plume thickness and the amplitude of all quantities are seen between the two analyses, although there are discrepancies in the region of wall boundary layer, the existence of which is ignored in this analysis. This fact suggests that the approximate analysis of present study provides efficiently the *E-Ri* relation.

SUMMARY AND CONCLUSION

Previous studies on inclined plumes have proposed two kinds of entrainment relations(E-Ri relation); a group of power functions (Type-1) and a group of fractional functions (Type-2). Although they have very different mathematical characteristics from each other, it has not been concluded yet which one is more adequate, because both were empirical relations based on laboratory experiments of same kind: i.e. both were drawn to get through the same cloud of data. However, information of entrainment rate at conditions out of the existing data becomes significant when we assume the scale of plumes appearing in natural environment.

This study presented purely analytical approach to E-Ri relation modifying and extending Fukushima's analysis, where k-ε model equations were used as the governing equations of "boundary layer analysis".

The most distinctive feature of the present analysis is the application of Galerkin method to convert the dimensionless k-ε model equations, which are high nonlinear differential equations, into a set of algebraic equations. As a result, the E-Ri relation for the condition of no bottom friction is obtained systematically and continuously for various values of C_3, an uncertain parameter of k-ε model, with great time saving. The results clearly show the same tendency as Type-2 formula proposed by Turner(9) and Akiyama et.al. (1).

However, the agreement of the present result with Turner's relation, and the flume data of Ellison and Turner, is not perfect yet. Some possible reasons for the disagreement can be considered as follows: One might be the problems of k-ε model in a sense that no turbulent model can be perfect. The other might rise from the assumption that the bottom friction would be negligible. As mentioned before, the entrainment rate (E) must be considered as a function not only of the Richardson number (Ri) but also of some dimensionless parameter indicating the effect of bottom friction such as the drag coefficient (C_f), if the bottom friction is not negligible. One of the authors has extended the analysis taking in to account the effect of the bottom friction, and the most recent results show a good agreement with the existing flume data and field data quantitatively.

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APPENDIX - NOTATION

The following symbols are used in this paper:

A, V	=	the flux of relative buoyancy and the characteristic velocity defined by Ellison and Turner respectively;
C_1, C_2, C_3, C_μ	=	model parameters respectively;
C_f	=	the drag coefficient
E	=	the entrainment coefficient;
E_0, n	=	constants for Eq. (1) respectively;
e	=	relative density difference;
f_i	=	dimensionless function for the profiles;
$f_u, f_v, f_e, f_k, f_\varepsilon, f_\nu$	=	dimensionless functions for the profiles of u, v, e, k, ε and ν

respectively;

$f(\xi)$	=	a function expressed by Eq. (22);
g	=	gravity acceleration;
k	=	turbulent kinetic energy;
Ri	=	the Richardson number;
Ri_c	=	the critical Richardson number;
S_i	=	physical quantities in a plume;
$U_0, V_0, E_0, K_0, \varepsilon_{d0}, \nu_{d0}$	=	constants for Eq. (11) respectively;
$u_0, v_0, e_0, k_0, \varepsilon_0, \nu_0$	=	constants for Eq. (21) respectively;
u	=	flow velocity in the x direction;
v	=	flow velocity in the y direction;
$w_u, w_v, w_e, w_k, w_\varepsilon$	=	the weight functions for u, v, e, k and ε respectively;
x	=	the distance along the sloping ground;
y	=	the distance perpendicular to the slope surface;
δ	=	the dimensionless plume layer thickness;
ε_d	=	dissipation rate;
η	=	dimensionless distance($= y/x$);
θ	=	slope angle;
ν_t	=	kinetic eddy viscosity;
$\sigma_g, \sigma_k, \sigma_\varepsilon$	=	model parameters; and
ξ	=	dimensionless distance ($= \eta/\delta$).

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