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COUPLED MATHEMATICAL MODELING OF ALLUVIAL RIVERS

By

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SYNOPSIS

Most existing mathematical models for alluvial rivers are decoupled ones because the governing equations are oversimplified and asynchronously solved. In these models the strong coupling among water flow, sediment transport, and riverbed evolution is to a certain extent neglected without justification. This paper presents a fully coupled model and investigates the effects respectively of the simplified water-sediment mixture's and global bed material's continuity equations (Exner equation) as well as of the asynchronous solution procedure. River aggradation due to sediment overloading is studied as the test example. It is found that both the simplified continuity equations and the asynchronous solution procedure that have been widely used previously lead to appreciable errors, depending on the changes of water and sediment discharges. And these errors become more pronounced progressively with increasing computational time. Moreover, an asynchronous solution procedure may render the physical process mathematically ill posed under certain conditions. The present finding necessitates synchronous solution of the complete continuity equations for refined and long-term modeling of alluvial rivers.

INTRODUCTION

In alluvial rivers the delicate balance among water flow, sediment transport, and river morphology is often disrupted by man's activities and natural hydrological changes in the watershed. Enhanced understanding of the resulting evolution of the rivers is essential for river training, hydropower generation, flood control and disaster alleviation, water supply, navigation improvement, as well as environment enhancement and ecosystem projects. Computer-based mathematical models of alluvial rivers have been extensively investigated since the 1950's. And softwares based on these models have been aggressively marketed in the last two decades or so. They have been used as one of the primary tools in the research and application communities. It has been recognized that flow, sedimentation, and erosion in alluvial rivers are related to highly variable time and spatial scales. Accordingly, one-, two-, and three-dimensional models have been developed for specific purposes. One-dimensional models are used primarily for long-term and long-reach situations. In the present state-of-the-art, it is a common practice to tune the friction relation (including the parameters thereof) for reconciling the computational results with measurements. However, in existing one-dimensional models, some deficiencies *inter alia* can be identified as follows.

On the one hand, the governing equations are mostly oversimplified. Specifically, the water-sediment mixture (for brevity referred to as "fluid" hereafter) continuity equation is assumed to be identical to that over fixed bed without considering the alluvial riverbed mobility (Chang 1988; Holly and Rahuel 1990), i.e., the fourth term in Eq. (1) is neglected. The effect appears to have been properly addressed only by Correia et al. (1992) and discussed by Rahuel (1993). In discussing Lyn's (1987) work, Stevens (1988) claimed that bed mobility is important for complete coupling of water and sediment. Unfortunately, Stevens' argument was based on a simplified continuity equation for the water phase only (assuming very low sediment concentration) rather than the complete continuity equation for water-sediment mixture. Wormleaton and Ghumman (1994) compared the performance of several simplified models, exclusive of a fully coupled model on a rigorous basis. Therefore the influence of bed mobility has not yet been identified. Further, in the global bed material continuity (hereafter abbreviated to sediment continuity) equation, the temporal change related to sediment discharge is sometimes neglected without any justification (Chang 1988; Zhang and Kahawita 1987), i.e., the second term in Eq. (3) is not considered. Similarly in nonequilibrium models in which bed-load and suspended load are separately considered, the temporal variation associated with bed-load discharge is not accounted for (Holly and Rahuel 1990). In depth-averaged two-dimensional models, the relevant term is almost exclusively ignored, leading to the Exner equation (Kassem and Chaudhry 1998). In reviewing existing models for river width adjustment, the ASCE Task Committee (1998) state that "spatial differences in sediment flux ... determine the evolution of the bed topography via solution of the sediment continuity equation." Actually, the temporal change in relation to sediment discharge is disregarded in the sediment continuity equation. The validity remains unclear.

On the other hand, the governing equations are mostly solved in an asynchronous procedure. The flow continuity and momentum equations are solved first in a time step, assuming negligible bed change rate. Then the sediment continuity equation is solved using the flow variables newly obtained. Models involving the asynchronous solution are usually referred to as decoupled ones. There have been semicoupled models in which the flow and bed equations are solved iteratively in a given time step (Kassem and Chaudhry 1998). Yet the computational cost seems comparable or even higher than the fully coupled models which simultaneously solve the complete set of governing equations. It has been claimed that semicoupled models permit arbitrary sediment discharge formula(s) to be easily incorporated. As a matter of fact, fully coupled models also allow for an expedient use of arbitrary sediment discharge formula(s) as proposed in the present paper.

The asynchronous solution procedure is based on the "fixed bed" and the "quasi-steady flow" assumptions. Often the flow is assumed to be steady when the evolution of riverbed is studied. Alternatively, the riverbed is implicitly assumed to be "fixed" within a time step while the flow over mobile bed is of primary interest. The validity of this feature is determined by the typical time scales or relative magnitude of the characteristic celerities corresponding to the free-surface flow and riverbed evolution respectively. De Vries (1965, 1973) analyzed the relative celerities when the volumetric sediment concentration is negligible. Morris and Williams (1996) confirmed the results of De Vries and extended the analysis to cases with finite sediment concentrations. It has been found that water flow, sediment transport, and riverbed evolution can be considered to be mathematically independent of each other only within very limited ranges of total-load concentration and Froude number. Beyond these ranges, the "quasi-steady state" or "fixed bed" assumption is no longer reasonable. Lyn (1987) identified the multiple time scales of the flow-sediment-riverbed system. He showed that previous models, which reduce the number of conservation equations solved simultaneously from three to two under the "quasi-steady state" or "fixed bed" assumption, are unable to satisfy exactly either a general boundary condition or an arbitrary initial condition. And in situations with highly variable discharge and sediment inputs, the aforementioned assumption is not justified. Despite the studies stated above, the effect of the asynchronous solution procedure is not as yet clearly understood.

While prior studies tended to focus on flow resistance relations, grain sorting, nonequilibrium modules, and numerical techniques (e.g., Holly and Rahuel 1990), this paper seeks to investigate the effects of the simplified continuity equations for fluid and sediment as well as of the asynchronous solution procedure. River aggradation due to overloading is studied as the test case. Numerical calculations are performed for two differently overloaded runs of experiments reported by Soni et al. (1980). To avoid uncertainties stemming from the empirical closure relations for nonequilibrium models (bed sediment entrainment and deposition fluxes and bed-load discharge formulas), the sediment transport capacity model is considered. The total sediment discharge is herein determined by the experimentally validated formula of Soni et al. (1980). It is implied by the concept of transport capacity that total sediment transport adapts to the local hydraulic conditions instantaneously, the spatial and temporal delay effects being negligible. In solving the hyperbolic system of equations, the method of characteristics is used with a third-order interpolation scheme involving first derivative. The first derivative is specified separately according to the characteristic directions to appropriately signalize the up- and downstream flow information. The results demonstrate the importance of the use of complete fluid and sediment continuity equations as well as of the synchronous solution procedure for refined and long-term modeling.

MATHEMATICAL MODEL

Consider sediment-laden flow in an alluvial river channel with idealized rectangular section of constant width. Riverbed is assumed to be composed of erodible sediments with the same particle size as those transported by the water flow. Sediment concentration is assumed to be low (dilute) as is in most natural rivers. The sediment transport capacity model consists of the mass conservation or continuity equations respectively for the fluid and sediment, and the momentum conservation equation for the fluid. The complete governing equations can be derived based on the Reynolds Transport Theorem (Roberson and Crowe 1990), which are as follows

$$\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}+h\frac{\partial U}{\partial x}+\frac{\partial Y}{\partial t}=0 \tag{1}$$

$$\frac{\partial U}{\partial t}+g\frac{\partial h}{\partial x}+U\frac{\partial U}{\partial x}+g\frac{\partial Y}{\partial x}=-gS \tag{2}$$

$$(1-p)\frac{\partial Y}{\partial t}+\frac{\partial}{\partial t}(\sigma\, q_s/U)+\frac{\partial q_s}{\partial x}=0 \tag{3}$$

where t = time; x = streamwise coordinate; h = flow depth; U = cross section-averaged streamwise velocity; Y = bed elevation; g = gravitational acceleration; S = friction slope; $q_s = ChU$ sediment discharge per unit width (unit sediment discharge); C = flux-averaged total-load concentration in volume (time- and location-dependent); p = bed sediment porosity; and σ = modification coefficient denoting the difference between geometrically and flux-averaged sediment concentrations. As its exact value is unavailable, σ is usually assumed to be equal to unity. It is noted that a secondary term representing the influence of bed evolution rate is not included in the momentum Eq. (2) in view of the unavoidable uncertainty of the friction resistance expression that must be incorporated to close the model. This appears to be a common practice followed by many researchers (e.g., Fraccarollo and Armanini 1999).

The unit sediment discharge is determined by

$$q_s = \delta U^{\alpha} h^{\beta} \tag{4}$$

and the friction slope by the relation of Darcy-Weisbach,

$$S=\frac{U^2 f}{8gh} \tag{5}$$

where α, and β = exponents; δ = coefficient; and f = friction factor. In the following calculations, the values of α, β, and δ are specified according to the experimentally validated sediment discharge relation of Soni et al. (1980), while f is set equal to the initial equilibrium value, i.e., $f = f_0$ (Table 1).

To expedite the synchronous solution, it is necessary to rearrange the fluid and sediment continuity equations. From Eq. (4), one can derive the expressions for $\partial(q_s/U)/\partial t$ and $\partial q_s/\partial x$. Substituting these expressions along with Eqs. (1) and (2) into the sediment continuity Eq. (3) and neglecting the terms in the second-order of magnitude of low sediment concentration, one yields (Appendix I)

$$\frac{\partial Y}{\partial t}+\psi Fr^{-2}U\frac{\partial h}{\partial x}+\varphi h\frac{\partial U}{\partial x}+\psi Fr^{-2}U\frac{\partial Y}{\partial x}=-\psi Fr^{-2}US \tag{6}$$

Substituting Eq. (6) into Eq. (1) leads to

$$\frac{\partial h}{\partial t}+(1-\psi Fr^{-2})U\frac{\partial h}{\partial x}+(1-\varphi)h\frac{\partial U}{\partial x}-\psi Fr^{-2}U\frac{\partial Y}{\partial x}=\psi Fr^{-2}US \tag{7}$$

where $Fr = U/\sqrt{gh}$ is Froude number, and

$$\psi=\frac{-(\alpha-1)C}{(1-p)}\,; \qquad \varphi=\frac{-(\beta-1)C}{(1-p)} \tag{8a, b}$$

In the context of mathematical modeling of alluvial rivers, more complicated sediment discharge formulas other than the explicit form as Eq. (4) may be used. Subsequently Eqs. (6) and (7) must be deduced separately to make the synchronous solution procedure tractable. It is straightforward to achieve this on a rigorous basis following the same approach as Eqs. (6) and (7) are derived (Appendix I). Therefore, the present fully coupled model allows for an arbitrary sediment discharge formula to be used.

NUMERICAL SOLUTION PROCEDURE

Eqs. (7), (2), and (6) constitute a third-order hyperbolic system. The traditional method of characteristics is used to solve the flow depth, velocity and bed level. The CFL stability condition must be satisfied, i.e., the Courant number $C_r = \lambda_{max}\Delta t/\Delta x < 1$, where λ_{max} is the maximum characteristic celerity.

As is well known, the interpolation scheme at the foot of characteristic trajectories is crucial to acquiring satisfactory solution particularly when the transport is advection-dominated. Holly and Preissmann (1977) developed a two-node fourth-order scheme involving first derivative at each node. The first derivative is determined by solving the advection equation for it. The evident drawbacks of the Holly-Preissmann scheme are the increased computational cost due to the additional advection equation to be solved and the difficulty in prescribing the boundary value(s) for the first derivative. Schohl and Holly (1991) suggested the use of cubic-spline interpolation to obviate the need to solve the additional advection equation for the first derivative. In this paper, a three-node fourth-order interpolation scheme is adopted as an alternative to the Holly-Preissmann scheme, which involves first derivative only at the mid-node (Appendix II). The first derivative is specified, to the third-order accuracy, distinctly according to the characteristic directions to enforce the correct signalization of flow information from up- and downstreams. To these authors' knowledge, there has been no use of this interpolation as in the present configuration.

In the synchronous solution procedure, Eqs. (7), (2), and (6) are solved simultaneously. It is easy to prove that the second flow-related characteristic λ_2 is always negative, irrespective of the Froude number and sediment concentration. Recognizing this feature is essential to implementing the boundary conditions described later.

In the asynchronous solution procedure, the flow Eqs. (7) and (2) are solved first in a given time step assuming negligible bed level change, and then the sediment continuity Eq. (6) is solved using the newly obtained flow variables. In contrast to the synchronous solution procedure as stated above, λ_2 may be positive as the Froude number is greater than unity, weakly dependent on sediment concentration. Therefore, the asynchronous solution procedure necessitates an additional upstream boundary condition for supercritical incoming flow. When no additional boundary condition is appropriate, the physical problem becomes mathematically ill posed purely due to the asynchronous solution procedure. This explains why decoupled solution is unachievable for the test Case A (Table 2). As the boundary conditions are model and test case specific, they are depicted later.

TEST EXAMPLE: AGGRADATION DUE TO OVERLOADING

Natural rivers are subject to significant changes in water and sediment discharges. Sediment overloading often results in river aggradation, which may be caused by either reduction of water discharge or increase of sediment supply. Severe aggradation may aggravate flood disasters and cause undesirable impact on the environment. Aggradation due to overloading has been extensively studied by experiments, analytical and mathematical models (Soni et al. 1980; Zhang and Kahawita 1987; Bhallamudi and Chaudhry 1991). But most suffer from oversimplifications and asynchronous solution procedure of the governing equations as already identified.

The flume experiments of Soni et al. (1980) have been referred widely in prior investigations. The laboratory channel was 0.2 m wide and 30 m long. The mean sediment particle diameter was 0.32 mm. Bed sediment porosity was 0.4. The parameters for the unit sediment discharge formula were $\alpha = 5.0$, $\beta = 0$, and $\delta = 0.00145$ m^2/s. An equilibrium state was first formed. Then overloading was activated by feeding additional sediment at the upstream end while water discharge remained unaltered. These measured data are utilized to verify and test the present numerical models. Two experimental cases to different extent of overloading are considered. The flow depth h_0, velocity u_0, bed slope S_0, and Darcy-Weisbach friction factor f_0 under the initial equilibrium state as well as the overloading parameter $\Delta q_s / q_{s0}$ in terms of the ratio of increased sediment discharge Δq_s to initial value q_{s0} are listed in Table 1. The spatial and time steps through out the present calculations are $\Delta x = 1.0$ m, and $\Delta t = 0.6$ s. Numerical tests show that this spatial step is appropriate as there is no appreciable difference among the computational results when $\Delta x = 0.5$ m, and 0.1 m are respectively used (the time step changed according to stability requirement).

Table 1. Summary of Test Cases (Soni et al. 1980)

Case	h_0 (cm)	u_0 (m/s)	S_0	$\Delta q_s / q_{s0}$	f_0
A	5.0	0.4	0.00356	4.0	0.0872
B	7.5	0.473	0.00363	1.0	0.0954

The models studied are summarized in Table 2 according to the governing equations and the numerical solution procedures used. FCM denotes fully coupled model that employs the complete equations and synchronous solution procedure. PCM means partially coupled model as the equations are simplified compared to FCM (but are solved simultaneously). DCM indicates decoupled model characterized by the asynchronous solution procedure utilized (the equations are complete in DCM1 and simplified in DCM2 and DCM3). It is recognized that a fully coupled model, strictly speaking, should use the complete governing equations with synchronous solution procedure and further incorporate the effects of grain sorting and changed friction resistance as well as the delay effects of sediment transport etc. This must be reserved as a topic of future research.

Table 2: Summary of Models Studied

Model	Governing Eqs.	Solution procedure	Application case(s)	Notes
FCM	Complete Eqs. (1)–(3)	Synchronous	A, B	
PCM1	$\frac{\partial Y}{\partial t}$ neglected in Eq. (1)	Synchronous	A, B	
PCM2	$\frac{\partial (hC)}{\partial t}$ neglected in Eq. (3)	Synchronous	A, B	
DCM1	Complete Eqs. (1)–(3)	Asynchronous	B	Unachievable for Case A because there is no way to specify the boundary condition as $Fr > 1$ at the upstream end.
DCM2	$\frac{\partial Y}{\partial t}$ neglected in Eq. (1)	Asynchronous	B	
DCM3	$\frac{\partial Y}{\partial t}$ and $\frac{\partial (hC)}{\partial t}$ neglected respectively in Eqs. (1) and (3)	Asynchronous	B	

Denote the negative characteristic by $\lambda_2 < 0$, while the positive characteristics by λ_1 and $\lambda_3 > 0$. For the fully and partially coupled models FCM, PCM1, and PCM2, the boundary conditions are as follows. At the upstream end (i.e., sediment feeding section, $x = 0$ m, node number i=1),

$Uh = u_0 h_0$;

λ_2–based characteristic relation; and

$q_s = q_{s0} + \Delta q_s$ (after cell integration of Eq. (3), Appendix III).

At the computational downstream end, which is set sufficiently far away from the sediment feeding section (100 m in the present computations),

$Y + h$ = initial free surface level;

λ_1–based characteristic relation; and

λ_3–based characteristic relation.

As the decoupled models DCM1, DCM2, and DCM3 are concerned, part of the boundary conditions for coupled models are no longer applicable. First, $\lambda_2 > 0$ when $Fr > 1$ (e.g., Case A, see Fig. 3a for velocity evolution). Therefore, an additional upstream boundary condition is needed to replace the λ_2–based characteristic relation in the coupled models. Unhappily, there is no way for this condition. As a result, the physical problem is rendered ill posed as already described. Second, λ_3 becomes non-positive in the decoupled solution procedure. Specifically, when the complete sediment continuity Eq. (3) or (6) is used (DCM1 and DCM2), $\lambda_3 < 0$, recalling the concept of the asynchronous solution procedure. Thus, at the downstream end, the bed level must be specified. This is resolvable by fixing the bed level at its initial elevation. The prerequisite for this treatment is that the computational downstream end is adequately far away from the sediment feeding section. If the second term in Eq. (3) is ignored (DCM3), Eq. (6) is no longer applicable. From Eqs. (3) and (4) it is easy to derive $\lambda_3 = 0$. The sediment continuity is simply a reaction-type equation. No downstream boundary condition for bed level is needed. The bed level at the upstream end is computed using cell integration of the simplified sediment continuity equation [i.e., (A17) is eliminated in Appendix III].

VERIFICATION OF FULLY COUPLED MODEL: FCM

Fig. 1 shows the bed and free surface profiles of the fully coupled model FCM compared to measured data of Soni et al. (1980) for Case A. The agreement between computations and measurements is reasonably good. The satisfactory performance of FCM is further demonstrated in Fig. 2, in which the dimensionless bed profiles are shown to be in excellent agreement with measured data. In Fig. 2, Y_0 is the increase of bed level at x=0 m, and k is the aggradation coefficient (Soni et al. 1980, Eq. 17). There is slight overestimation of bed level and subsequently of free surface elevation (Fig. 1). This in fact arises from (a) the friction resistance relation; and (b) the boundary condition at the upstream end. Numerical experiments have been performed to make sure that reduced friction (e.g., $f = 0.9 f_0$) does lead to aggradation to a smaller extent (not shown). It is revealed that the flow resistance is reduced due to increased sediment load. Most plausibly, turbulence is suppressed by sediment particles and the modulation of turbulence is aggravated by increased sediment load. Moreover, in the physical experiments, a small fraction of sediment deposited upstream the feeding section (Soni et al. 1980) whereas it is not accounted for in the present computations (Appendix III). Inevitably this contributes to the overestimation of aggradation. In principle, this overestimation is unimportant as the primary purpose of the present study is to investigate the effects of the simplified continuity equations and asynchronous solution procedure rather than calibrating the model by tuning the friction factor.

More evidence of support for the fully coupled model is provided in Fig. 3 for computational velocity (Case A). Physically, a new equilibrium state will be established ultimately corresponding to the overloaded yet constant sediment input ($\Delta q_s + q_{s0}$) and water discharge. The velocity under the new equilibrium state can be predicted according the sediment discharge relation in the form of Eq. (4). For Case A (Table 1), this theoretical velocity is 0.552 m/s. It is seen from Fig. 3(a) that the computational velocity at x=0 m assumes a constant value 0.550 m/s about 15 minutes after the addition of overloading sediment. This is in good agreement with the theoretical value predicted from Eq. (4). Further, no spurious oscillation of the longitudinal velocity profiles is found (Fig. 3b) related to the method of characteristics along with the interpolation involving first derivative (Appendix II). An attempt was made to solve the hyperbolic system composed of Eqs. (7), (2) and (6) using the MacCormack scheme following Bhallamudi and Chaudhry (1991). The resulting velocity profiles were found to exhibit significant numerical oscillations for Case A due to advection-dominance immediately after the addition of overloading sediment (results not shown). This was not reported by Bhallamudi and Chaudhry (1991).

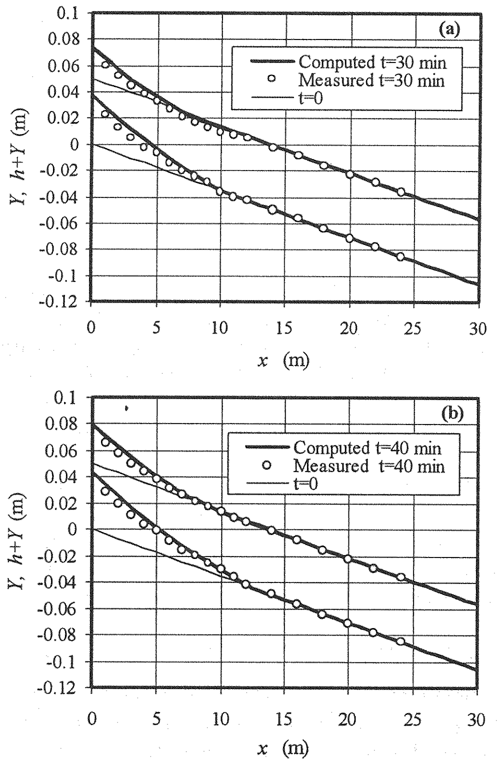

FIG. 1. Computed Bed and Free Surface Profiles Compared to Measured Data for Case A

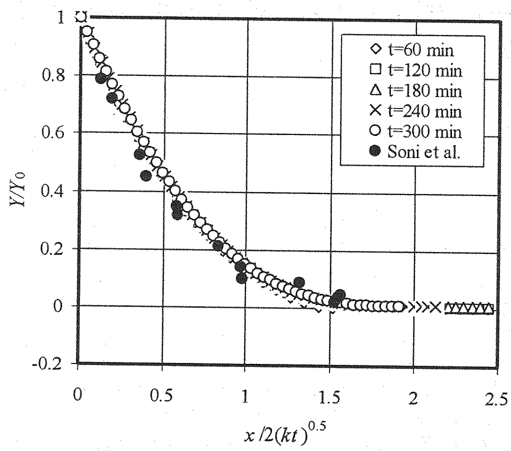

FIG. 2. Computed Dimensionless Bed Profiles Compared to Measured Data for Case A

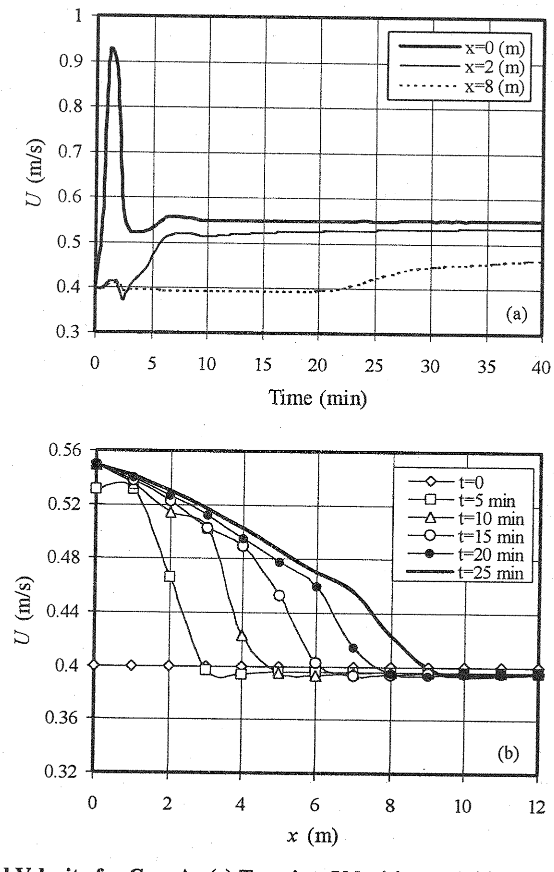

FIG. 3. Computed Velocity for Case A. (a) Transient Velocities; and (b) Longitudinal Velocity Profiles

EFFECTS OF SIMPLIFIED CONTINUITY EQUATIONS: PCM1 AND PCM2

Figs. 4 shows the bed and free surface profiles from the fully (FCM) and partially (PCM1, PCM2) coupled models for Case A. The differences between FCM and PCM1 or PCM2 are indistinguishable at t=1 hr, but significant at t=5 hrs. This can also be found in Fig. 5 that plots the bed and free surface levels at the upstream end x=0 m. Evidently the differences between FCM and PCM1 or PCM2 become more pronounced steadily with increasing computational time due to error accumulation. For Case B with a smaller overloading than Case A, the differences between FCM and PCM1 or PCM2 are less appreciable as can be anticipated (Fig. 6). Comparatively, the effect of PCM2 is a little smaller than PCM1 for both cases considered, although it is not so distinguishable in Figs. 4 through 6.

To avoid misunderstanding the effects of the simplified continuity equations (PCM1 and PCM2), it is necessary to note that there is no appreciable difference between FCM and PCM1 or PCM2 within about 1 hr immediately after overloading is activated (Figs. 4 and 5). Therefore the overestimation by FCM (Fig. 1) compared to measured data as interpreted earlier must not be attributed to the different governing equations used in FCM and PCM1 or PCM2.

In natural rivers, both water and sediment discharges are invariably subject to substantial changes. It follows that it is necessary to use the complete fluid and sediment continuity equations when long-term and refined modeling is pursued. A closer examination of prior studies shows that only Correia et al. (1992) reported quantitatively the importance of the bed evolution rate in the fluid continuity Eq. (1), see also Rahuel (1993) for discussions. To these authors, the influence of the simplified sediment continuity equation (Exner equation) has not been properly addressed previously.

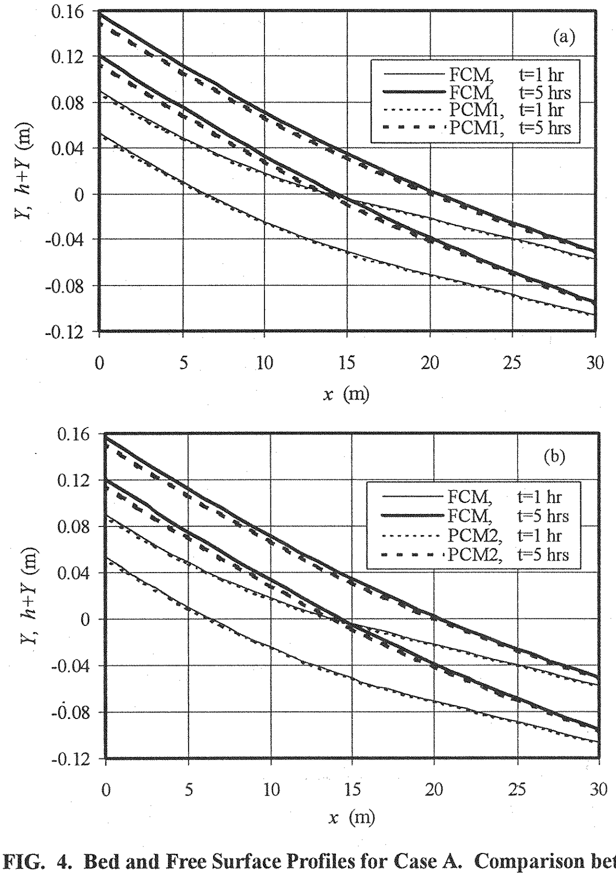

FIG. 4. Bed and Free Surface Profiles for Case A. Comparison between (a) FCM and PCM1; and (b) FCM and PCM2

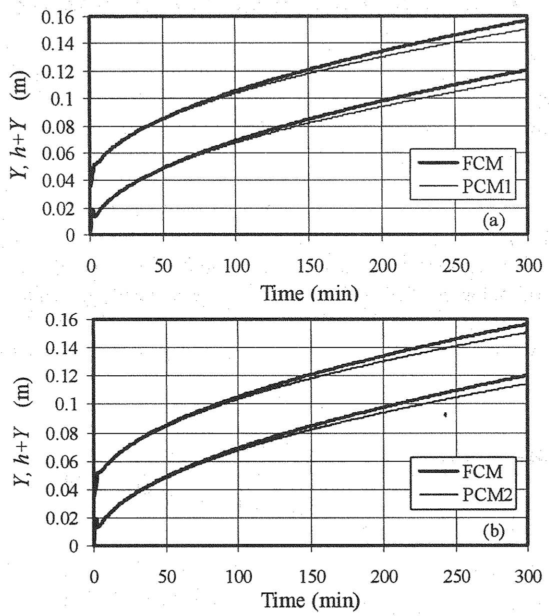

FIG. 5. Bed and Free Surface Levels at x=0 m for Case A. Comparison between (a) FCM and PCM1; and (b) FCM and PCM2

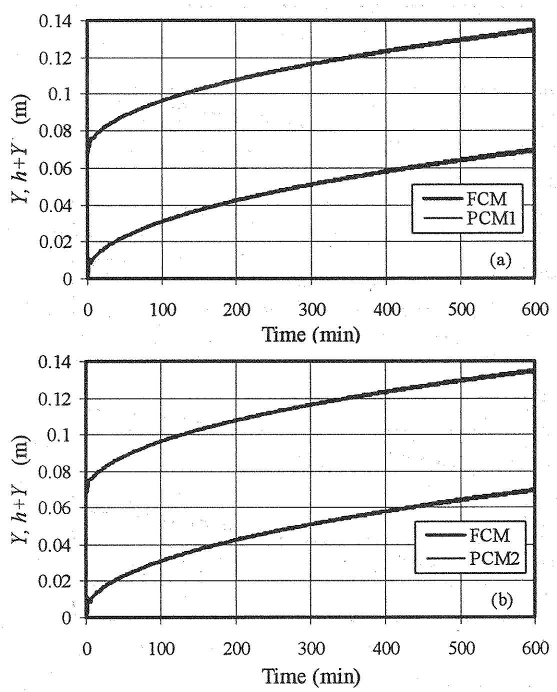

FIG. 6. Bed and Free Surface Levels at x=0 m for Case B. Comparison between (a) FCM and PCM1; and (b) FCM and PCM2

ASYNCHRONOUS SOLUTION PROCEDURE: DCM1, DCM2, AND DCM3

As described earlier, decoupled solution for Case A is hampered by unavailability of the additional upstream boundary condition when *Fr* >1 at *x*=0 m immediately after overloading is started (see Fig. 3a for velocity evolution of FCM). Purely this is caused by the asynchronous solution procedure. The following discussion in this section pertains to Case B only. In the literature there has been no reporting of the quantitative behavior of the decoupled models as described in the following, except the analysis of celerity and multiple time scales stated earlier.

Fig. 7 shows the free surface and bed levels at *x*=0 m, while Fig. 8 illustrates the bed and free surface profiles at *t*=2 hrs and 10 hrs after overloading is imposed. It is obvious from Figs. 7 and 8 that the inaccuracy of the decoupled models compared to FCM becomes appreciable roughly after 4 hrs of overloading, which ever increases progressively with increasing computational time. There is substantial errors of the decoupled models at *t*=10 hrs. Moreover, the inaccuracy due to the asynchronous solution procedure is found to be greater than that resulting from purely simplifying the continuity equations (but following the synchronous solution procedure, PCM1 and PCM2), see Figs. 6 and 7. Because of the small disturbance (overloading) in Case B, only slight differences are found between DCM1, DCM2, and DCM3 each other. This is consistent with what indicated by PCM1 and PCM2 (Fig. 6).

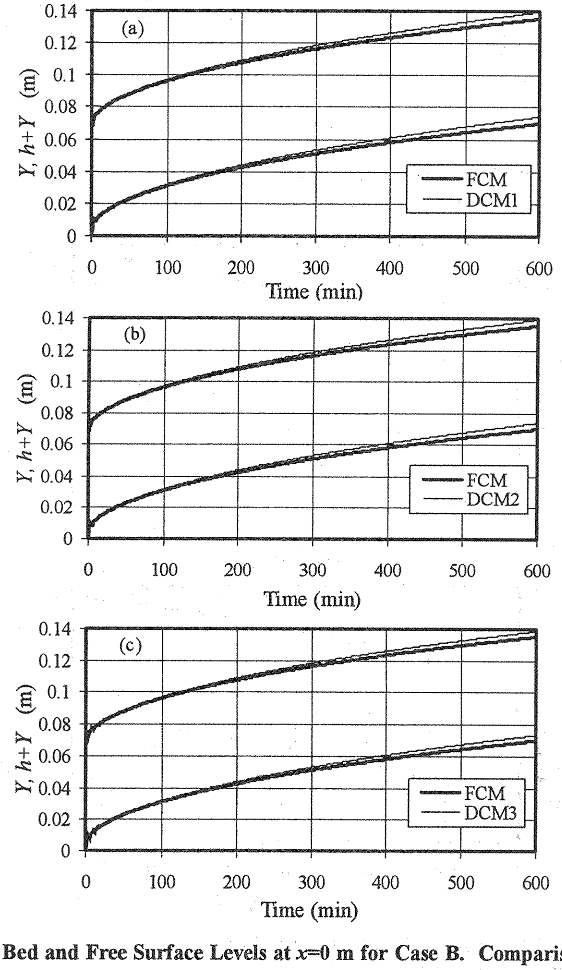

FIG. 7. Bed and Free Surface Levels at *x*=0 m for Case B. Comparison between (a) FCM and DCM1; (b) FCM and DCM2; and (c) FCM and DCM3

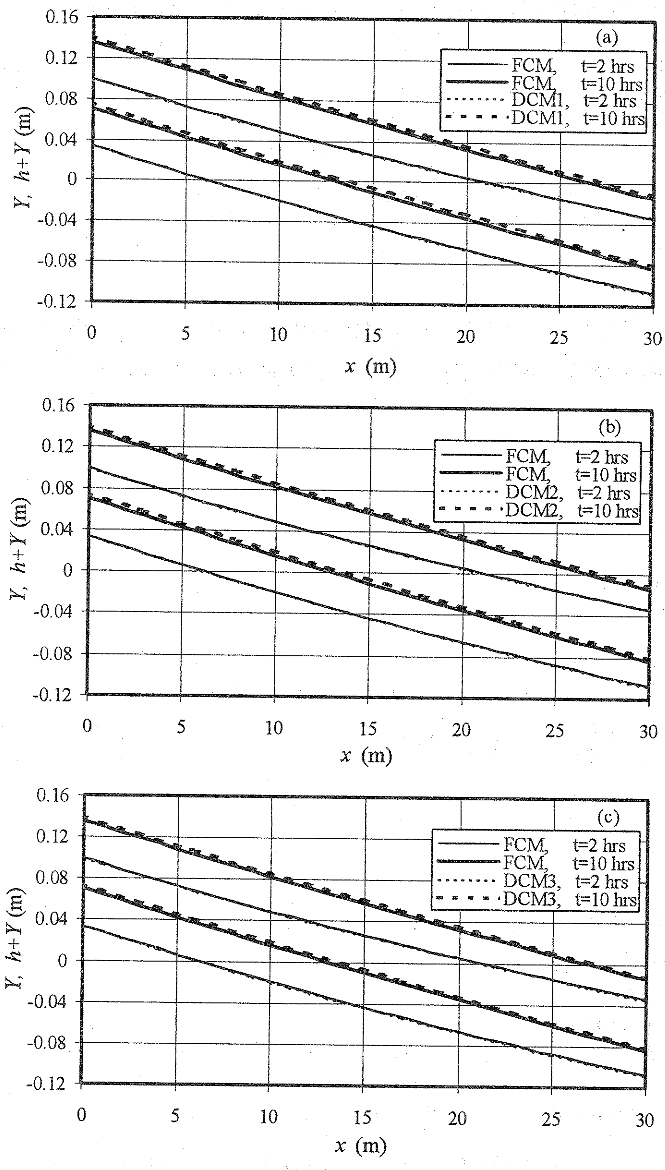

FIG. 8. Bed and Free Surface Profiles for Case B. Comparison between (a) FCM and DCM1; (b) FCM and DCM2; and (c) FCM and DCM3

CONCLUSIONS

Natural alluvial rivers are typical of bed boundary mobility and subject to significant changes of water and sediment discharges. In mathematical models for such rivers, simplifying the fluid continuity equation (as that over fixed bed) and sediment continuity equation (as the Exner equation) leads to substantial inaccuracy. This inaccuracy becomes more pronounced progressively with increasing computational time. The asynchronous solution procedure either renders the physical problem ill posed or results in considerable errors that increase steadily with computational time. Recall that one-dimensional mathematical models are used primarily for long-term and long-reach simulation purposes, and the reaction time of the river system to constructions and other human activities may go on for a long period of time (centuries for large rivers!). Consequently, it is necessary to use the complete fluid and sediment continuity equations and also synchronous solution procedure for refined modeling of alluvial rivers. A model of this type is presented and preliminarily verified in this paper.

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APPENDIX I — DEDUCED CONTINUITY EQUATIONS FOR SYNCHRONOUS SOLUTION

To make synchronous solution tractable, the continuity equations need to be rearranged. Let

$$q_s = q_s(h, U, ...) \tag{A1}$$

Then from Eq. (A1)

$$\frac{\partial q_s}{\partial x} = \frac{\partial q_s}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial q_s}{\partial U}\frac{\partial U}{\partial x} \tag{A2}$$

$$\frac{\partial}{\partial t}\left(\frac{q_s}{U}\right) = U^{-1}\frac{\partial q_s}{\partial h}\frac{\partial h}{\partial t} + U^{-2}\left(U\frac{\partial q_s}{\partial U} - q_s\right)\frac{\partial U}{\partial t} \tag{A3}$$

Substituting Eqs. (1), (2) into (A3), and then the resulting equation along with Eq. (A2) into Eq. (3), neglecting the terms of the second-order of magnitude of low sediment concentration, one yields

$$\frac{\partial Y}{\partial t} + \Theta_1 U\frac{\partial h}{\partial x} + \Theta_2 h\frac{\partial U}{\partial x} + \Theta_1 U\frac{\partial Y}{\partial x} = -\Theta_1 US \tag{A4}$$

where

$$\Theta_1 = \frac{-g(U\partial q_s/\partial U - q_s)}{(1-p)U^3}\ ; \qquad \Theta_2 = \frac{-(h\partial q_s/\partial h - q_s)}{(1-p)Uh} \tag{A5a, b}$$

From Eq. (A4), Eq. (1) is rewritten as

$$\frac{\partial h}{\partial t} + (1-\Theta_1)U\frac{\partial h}{\partial x} + (1-\Theta_2)h\frac{\partial U}{\partial x} - \Theta_1 U\frac{\partial Y}{\partial x} = \Theta_1 US \tag{A6}$$

It is easy to prove that Eqs. (A4) and (A6) reduce to (6) and (7) when the explicit formula Eq. (4) is used. Otherwise, the values of $\partial q_s/\partial h$ and $\partial q_s/\partial U$ in Eqs. (A5a, b) can be determined from the specific sediment discharge formula. While bed shear velocity or Shields parameter is involved as in a number of sediment discharge formulas, a relationship of friction resistance (or bed shear velocity) is needed to derive $\partial q_s/\partial h$ and $\partial q_s/\partial U$.

APPENDIX II — INTERPOLATION SCHEME

Consider scalar $\Omega(x)$, the following three-node cubic interpolation polynomial $p(x)$ is used to determine the values at the foot of characteristic trajectories, which reads

$$p(x) = L(x) - (x - x_{i-1})(x - x_i)(x - x_{i+1})[\partial\Omega_i/\partial x + (\Omega_{i-1} - \Omega_{i+1})/2\Delta x]/\Delta x^2 \tag{A7}$$

where the three-node Lagrangian interpolation polynomial $L(x)$ is

$$L(x) = [(x - x_i)(x - x_{i+1})\Omega_{i-1} - 2(x - x_{i-1})(x - x_{i+1})\Omega_i + (x - x_{i-1})(x - x_i)\Omega_{i+1}]/2\Delta x^2 \tag{A8}$$

The interpolation scheme $p(x)$ can be proved to satisfy the following conditions, i.e.,

$$p(x_j) = \Omega_j, \qquad j = i\text{-}1,\ i,\ i\text{+}1 \tag{A9}$$

$$p'(x_i) = \partial\Omega_i / \partial x \tag{A10}$$

The interpolation is fourth-order accurate (Tsinghua Univ. and Peking Univ. 1985) if the value of the first derivative $\partial\Omega_i / \partial x$ can be accurately specified. In the present study, the first derivative is approximated distinctly according to the characteristic directions in order to enforce the correct transmission of flow information respectively from the up- and downstreams. It is third-order accurate as follows.

$$\frac{\partial\Omega_i}{\partial x} = \frac{2\Omega_{i+1} + 3\Omega_i - 6\Omega_{i-1} + \Omega_{i-2}}{6\Delta x}, \qquad \text{for } \lambda > 0 \tag{A11}$$

$$\frac{\partial\Omega_i}{\partial x} = -\frac{2\Omega_{i-1} + 3\Omega_i - 6\Omega_{i+1} + \Omega_{i+2}}{6\Delta x}, \qquad \text{for } \lambda < 0 \tag{A12}$$

At the nodes (i = 2, N-1) immediately next to the up- and downstream boundary nodes ($i = 1$ and N respectively), the following relations are complemented empirically,

$$\frac{\partial\Omega_2}{\partial x} = \frac{\Omega_3 + \Omega_2 - 2\Omega_1}{3\Delta x}, \qquad \text{for } \lambda > 0 \tag{A13}$$

$$\frac{\partial\Omega_{N-1}}{\partial x} = -\frac{\Omega_{N-2} + \Omega_{N-1} - 2\Omega_N}{3\Delta x}, \qquad \text{for } \lambda < 0 \tag{A14}$$

It is not necessary to specify the first derivative at the boundary nodes as opposed to some previous schemes, which is often a difficult task. The characteristic relations thereof can be discretized using the interpolations respectively at nodes i=1, 2, and 3 (upstream) or at i=N, N-1, and N-2 (downstream).

APPENDIX III — CELL INTEGRATION OF SEDIMENT CONTINUITY EQUATION

The boundary condition $q_s = q_{s0} + \Delta q_s$ for sediment discharge at the upstream end must be translated into the form by which bed elevation can be determined in conjuction with the flow boundary conditions thereof. This is achieved by integrating the sediment continuity Eq. (3) within the cell adjacent to the upstream end (Fig. 9), i.e.,

$$\iint_{cell} [(1-p)\frac{\partial Y}{\partial t} + \frac{\partial}{\partial t}(q_s/U) + \frac{\partial q_s}{\partial x}]dxdt = 0 \tag{A15}$$

By using trapezoidal integration to the second-order accuracy in time and space, one yields

$$\iint_{cell} (1-p)\frac{\partial Y}{\partial t}dxdt = (1-p)\int_0^{\Delta x}(Y^{n+1} - Y^n)dx$$

$$= 0.5\Delta x(1-p)(Y_1^{n+1} + Y_2^{n+1} - Y_1^n - Y_2^n) \tag{A16}$$

$$\iint_{cell} \frac{\partial}{\partial t}(q_s/U)dxdt = \int_0^{\Delta x}[(q_s/U)^{n+1} - (q_s/U)^n]dx$$

$$= 0.5\Delta x[(q_s/U)_1^{n+1} + (q_s/U)_2^{n+1} - (q_s/U)_1^n - (q_s/U)_2^n] \tag{A17}$$

$$\iint_{cell} \frac{\partial q_s}{\partial x}dxdt = \int_{n\Delta t}^{(n+1)\Delta t}(q_{s2} - q_{s1})dt = 0.5\Delta t(q_{s2}^{n+1} + q_{s2}^n - q_{s1}^{n+1} - q_{s1}^n)$$

$$= 0.5\Delta t(q_{s2}^{n+1} + q_{s2}^n - 2q_{s0} - 2\Delta q_s) \tag{A18}$$

Substitution of Eqs. (A16), (A17), and (A18) into Eq. (A15) renders the formulation for determination of bed elevation Y_1^{n+1} at the upstream end at $t = (n+1)\Delta t$. When the second term in Eq. (3) is neglected as in PCM2 and DCM3, the cell integration gives a simpler formulation (Eq. (A17) is eliminated). In the foregoing equations, the values at node i=2 can *priori* be determined by the characteristic method.

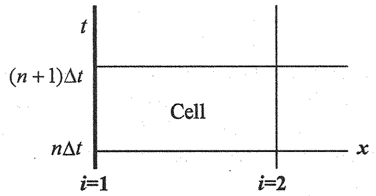

FIG. 9. Sketch of Cell Integration

APPENDIX IV — REFERENCES

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APPENDIX V — NOTATION

The following symbols are used in this paper:

C = flux-averaged sediment concentration in volume;
d = sediment particle diameter;
Fr = Froude number;
f = friction factor;
f_0 = friction factor under initial equilibrium state;
g = gravitational acceleration;
h = flow depth;
h_0 = flow depth under initial equilibrium state;
k = aggradation coefficient;
p = bed sediment porosity;
q_s = sediment discharge per unit width;
q_{s0} = sediment discharge per unit width under initial equilibrium state;
S = friction slope;
S_0 = bed slope under initial equilibrium state;
t = time;
U = cross section-averaged streamwise velocity;
u_0 = streamwise velocity under initial equilibrium state;
x = streamwise coordinate;
Y = bed elevation;
Y_0 = increase of bed elevation at upstream end;
Δq_s = increment of sediment discharge per unit width due to overloading;
Δx = time step;
Δx = spatial step;
α, β, δ = parameters in sediment discharge formula;
ψ, φ = parameters related to total-load concentration;
$\lambda_{1,2}$ = flow-related characteristic celerities;
λ_3 = bed-related characteristic celerity; and
σ = modification coefficient.

***Subscripts and superscripts*:**
i= node number in x coordinate; and
n= step number in time.

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