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MODELING INFILTRATION INTO A MULTI-LAYERED SOIL DURING AN UNSTEADY RAIN

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SYNOPSIS

The Green-Ampt model is generalized to model infiltration into a multi-layered soil during an unsteady rain. One dimensional numerical simulation scheme of the Richards equation is also developed which is suitable to deal with transfers between two kinds of boundary conditions, that is, water flux control and water head control. Modeling results of the generalized Green-Ampt model are compared with a numerical simulation for cases of two-layered soil and three-layered soil as examples. It is shown that the proposed model matches the Richards equation quite well with computation time less than tenth of the latter, thus promising to play a role in watershed modeling.

INTRODUCTION

Infiltration is an important process in hydrological cycles because it is a basis for determination of surface runoff generation, recharge to groundwater and actual evapotranspiration, etc. A lot of infiltration models have been suggested since the beginning of this century, e.g., the Green-Ampt model (1), the Horton model (2) and the Philip model (3), etc.

Combining the Darcy's law with equation of continuity, Richards (4) derived a partial differential equation for moisture movement in unsaturated soil. This Richards equation has a clear physical basis and it may be most scientific to model infiltration. However, there are limitations to apply it to a large-scale watershed modeling because of heavy computation time and memory requirement and a lot of parameters needed to specify.

The Green-Ampt model has drawn attentions of many researchers in the last three decades because of its simplicity, physically-based characteristic and measurable parameters. Though it was originally a infiltration model for dry and uniform soil under surface ponding, it was extended to model infiltration during a steady rain by Mein & Larson (5), to model infiltration into nonuniform soil during a ponded irrigation by Bouwer (6), to model infiltration during an unsteady rain by Chu (7), and to model infiltration into two-layered soil profiles during steady rains by Moore & Eigel (8). It was also used or referred to in many watershed models, e.g., Kite (9) and Huang et al. (10), etc.

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* This paper includes recent addition in verification of the general model to the paper presented in Volume 41 of Annual Journal of Hydraulic Engineering, JSCE.

In many cases in watershed modeling, vertical heterogeneity of soil can be treated as stratified soil layers, e.g., surface compressing in urbanized areas, surface sealing and cultivation in farmlands, in addition to natural stratification caused by geological processes. However, all forms of the Green-Ampt model mentioned above only consider one of the two factors, namely, soil heterogeneity and rain unsteadiness, and no general formula for a multi-layered soil has been observed even for the case of a steady rain. This research succeeded in extension of the Green-Ampt model to multi-stratified soil with unsteady rain by a general solution for the first time.

DERIVATION OF A GENERAL GREEN-AMPT MODEL

The Green-Ampt Model

Considering infiltration into a vertical uniform soil column when surface is ponded, Green and Ampt (1) proposed an infiltration model by assuming there is a wetting front which separates saturated soil above from soil below and by using the Darcy's law. If the depth of ponding is negligible, then the Green-Ampt model can be written as

$$f = k \cdot (1 + A / F) \qquad F = k \cdot t + A \cdot \ln\left(\frac{A + F}{A}\right) \qquad \textit{with} \quad A = SW(\theta_s - \theta_0) \tag{1}$$

where f = infiltration rate; F = accumulated infiltration; SW = capillary suction at the wetting front; k = hydraulic conductivity in the wetted zone; θ_s = moisture content in the wetted zone; θ_0 = initial moisture content; and t = time.

For infiltration during an unsteady rain, the modified Green-Ampt model (see Chu(7)) can be written as

$$\textit{before ponding} \qquad f = I \qquad F = F_{nb-1} + (t - t_{nb-1}) I \tag{2}$$

$$\textit{after ponding} \qquad f = k \cdot (1 + A / F) \qquad F - F_p = k(t - t_p) + A \cdot \ln\left(\frac{A + F}{A + F_p}\right) \tag{3}$$

$$\textit{with} \quad F_p = \frac{A}{I_p / k - 1}; \quad t_p = t_{n-1} + \frac{F_p - F_{n-1}}{I_p}; \quad A = SW(\theta_s - \theta_0) \tag{4}$$

where F_p = accumulated infiltration at the instant of surface ponding; t_p = time to surface ponding; I_p = rain intensity during the *n*th time step when surface ponding occurs; I = rain intensity; and subscripts n & nb indicate time steps.

A General Model for a Multi-layered Soil and an Unsteady Rain

Supposing that the wetting front is in the *m*th soil layer, the average conductivity $\bar{k}$ in the transmission zone behind the wetting front can be given by the harmonic mean as follows (see Bouwer(6), Moore & Eigel(8))

$$\bar{k} = \left(\sum_1^m Li\right) / \left(\sum_1^m \frac{Li}{ki}\right) \tag{5}$$

If the surface ponding occurred from the beginning of a rain event and the ponding continues, the Green-Ampt model (Eq.1) will be applicable before the wetting front enters the second soil layer. After it enters the *m*th layer, based on the Darcy's law we have

$$\frac{dF}{dt} = f = \bar{k}\frac{SW_m + L}{L} = \frac{SW_m + L}{\frac{L}{\bar{k}}} = \frac{SW_m + \sum_1^{m-1} L_i + (L - \sum_1^{m-1} L_i)}{\sum_1^{m-1}\frac{L_i}{k_i} + \frac{L - \sum_1^{m-1} L_i}{k_m}} \quad (6)$$

Because $$F = \sum_1^{m-1} L_i \Delta\theta_i + (L - \sum_1^{m-1} L_i)\Delta\theta_m \quad (7)$$

which can be rearranged as $$L - \sum_1^{m-1} L_i = \frac{F - \sum_1^{m-1} L_i \Delta\theta_i}{\Delta\theta_m}$$

Equation (6) can be rewritten as

$$\frac{dF}{dt} = \frac{SW_m + \sum_1^{m-1} L_i + \frac{F - \sum_1^{m-1} L_i\Delta\theta_i}{\Delta\theta_m}}{\sum_1^{m-1}\frac{Li}{k_i} + \frac{F - \sum_1^{m-1} L_i\Delta\theta_i}{k_m\Delta\theta_m}} = k_m \cdot \frac{\sum_1^{m-1}\frac{L_i}{k_i} + \frac{F - \sum_1^{m-1} L_i\Delta\theta_i}{k_m\Delta\theta_m} + (\frac{\sum_1^{m-1} L_i + SW_m}{k_m} - \sum_1^{m-1}\frac{L_i}{k_i})}{\sum_1^{m-1}\frac{L_i}{k_i} + \frac{F - \sum_1^{m-1} L_i\Delta\theta_i}{k_m\Delta\theta_m}} \quad (8)$$

Let $$A = \frac{\sum_1^{m-1} Li + SW_m}{k_m} - \sum_1^{m-1}\frac{L_i}{k_i}; \quad B = \sum_1^{m-1}\frac{L_i}{k_i} - \frac{\sum_1^{m-1} L_i\Delta\theta_i}{k_m\Delta\theta_m}; \quad C = k_m\Delta\theta_m \quad (9)$$

Then Equation (8) can be rewritten as

$$\frac{dF}{dt} = k_m(1 + \frac{A}{B + F/C}) \quad (10)$$

Integration of Equation (10) leads to

$$\int_{F_{m-1}}^{F} dF = \int_{t_{m-1}}^{t} k_m(1 + \frac{A}{B + F/C})dt \Rightarrow \int_{F_{m-1}}^{F}(1 - \frac{AC}{AC + BC + F})dF = \int_{t_{m-1}}^{t} k_m dt \quad (11)$$

Further denoting $A_{m-1}=AC$; $B_{m-1}=BC$, we get at last

$$f = k_m \cdot (1 + \frac{A_{m-1}}{B_{m-1} + F})$$
$$F - F_{m-1} = k_m(t - t_{m-1}) + A_{m-1} \cdot \ln(\frac{A_{m-1} + B_{m-1} + F}{A_{m-1} + B_{m-1} + F_{m-1}}) \quad (12)$$

$$A_{m-1} = (\sum_1^{m-1} L_i - \sum_1^{m-1} L_i k_m / k_i + SW_m)\Delta\theta_m$$
with $$B_{m-1} = (\sum_1^{m-1} L_i k_m / k_i)\Delta\theta_m - \sum_1^{m-1} L_i\Delta\theta_i \quad (13)$$
$$F_{m-1} = \sum_1^{m-1} L_i\Delta\theta_i$$

where t_{m-1} = time when the wetting front reached interface of *m*th and *(m-1)*th layers; L = depth of wetting front; L_i = thickness of *i*th soil layer; and $\Delta\theta = \theta_s - \theta_0$. Subscript *i* and *m* represent numbers of soil layers. t_{m-1} can be solved successively from $t_1, t_2, \ldots, t_{m-2}$ to t_{m-1} with t_1 by Eq.1.

If the surface ponding first occurs in the *m*th layer, we will have

$$I_p = f = k_m(1+\frac{A_{m-1}}{B_{m-1}+F_p}) \quad \Rightarrow \quad F_p = \frac{A_{m-1}}{I_p/k_m - 1} - B_{m-1} \tag{14}$$

$$F_p - F_{n-1} = I_p(t_p - t_{n-1}) \quad \Rightarrow \quad t_p = t_{n-1} + \frac{F_p - F_{n-1}}{I_p} \tag{15}$$

According to Equation (12), if the ponding occurs from beginning and continues, the pseudo-time t'_p to infiltrate F_p can be solved from the following equation

$$F_p - F_{m-1} = k_m(t_p' - t_{m-1}) + A_{m-1}\cdot \ln(\frac{A_{m-1}+B_{m-1}+F_p}{A_{m-1}+B_{m-1}+F_{m-1}}) \tag{16}$$

Considering the difference of time coordinate in real situation with the assumed one, Equation (12) should be modified to

$$F - F_{m-1} = k_m\left[t - t_{m-1} - (t_p - t_p')\right] + A_{m-1}\cdot \ln(\frac{A_{m-1}+B_{m-1}+F}{A_{m-1}+B_{m-1}+F_{m-1}}) \tag{17}$$

Combining Equation (16) and (17), we get at last

$$\begin{aligned} f &= k_m\cdot(1+\frac{A_{m-1}}{B_{m-1}+F}) \\ F - F_p &= k_m(t - t_p) + A_{m-1}\cdot \ln(\frac{A_{m-1}+B_{m-1}+F}{A_{m-1}+B_{m-1}+F_p}) \end{aligned} \tag{18}$$

in which the time to surface ponding t_p and the corresponding accumulated infiltration F_p are solved by using Eq.15 and Eq.14 respectively, and the other parameters are same as described above.

Equation (12) and Equation (18) are the generalized Green-Ampt model for a multi-layered soil and an unsteady rain. It can be proved that Equation (12) is also true for the case that the wetting front enters the *m*th layer with the ponding firstly occurred at the *(m-1)*th layer and continued since then, and Equation (18) is also true for the case that the surface is re-ponded after no ponding at the last time step with the wetting front at the *m*th layer.

It can be seen that the Chu 1 layer model (or called as the Mein-Larson model in case of a steady rain) and the Moore-Eigel 2 layer model are special cases of the general model. When $m=1$, $A_0=SW\,\Delta\theta$,, $B_0=0$,, $F_0=0$, Equation (12) becomes the Mein-Larson model; when $m=2$, $A_1=(L_1- L_1\, k_2 / k_1 + SW_2)\,\Delta\theta_2$,, $B_1=L_1\, k_2 / k_1\, \Delta\theta_2 - L_1\,\Delta\theta_1$, $F_1=L_1\,\Delta\theta_1$, it becomes the Moore-Eigel 2 layer model.

NUMERICAL MODEL OF THE RICHARDS EQUATION

For the stratified soil layers, it is reasonable to choose water pressure head or matrix head ψ as a variable, then one-dimensional Richards equation has the form as follows:

$$c(\psi)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}(k(\psi)\frac{\partial\psi}{\partial z}) - \frac{\partial k(\psi)}{\partial z} \tag{19}$$

where $c(\psi) = \partial\theta / \partial\psi$ = specific soil moisture capacity; $k(\psi)$ = soil hydraulic conductivity at pressure head ψ; θ = volumetric soil moisture content; z = distance below the surface; and t = time.

When infiltration during rain is simulated by using Equation (19), two kinds of boundary conditions will be encountered. One is the condition of pressure head control when surface is ponded:

$$\begin{array}{lll} \psi = \psi_0 & & t = 0, \quad z \geq 0 \\ \psi = \psi_0 & when & t > 0, \quad z \rightarrow \infty \\ \psi = 0 & & t > 0, \quad z = 0 \end{array} \tag{20}$$

Another is the condition of flux control when surface is not ponded:

$$\begin{array}{lll} \psi = \psi_0 & & t = 0, \quad z \geq 0 \\ \psi = \psi_0 & when & t > 0, \quad z \rightarrow \infty \\ -k(\psi)\dfrac{\partial \psi}{\partial z} + k(\psi) = I(t) & & t > 0, \quad z = 0 \end{array} \tag{21}$$

In Eqs.20 and 21, ψ_0 = the initial water pressure head in soil and I(t) = rain intensity at time t.

Equation (19) is a second-order non-linear partial differential equation. The implicit finite volume scheme is chosen to convert it into algebraic equations and Tri-Diagonal Matrix Algorithm (TDMA) method is used to solve them. To deal with non-linear parameters $c(\psi)$ and $k(\psi)$, iteration is conducted and under-relaxation factor is adopted as 0. 5. For the boundary condition of flux control (Eq.21), the water balance in the first half grid is used to convert it into the first algebraic equation. Because $c(\psi)$ and $k(\psi)$ are changed greatly with the variation of ψ, and two kinds of boundary conditions switch from one to another, the implicit finite volume scheme and iteration method are used to ensure the convergence and stability of computation.

VERIFICATION OF THE GENERAL MODEL

To evaluate the generalized Green-Ampt model, it is applied to infiltration into uniform soil profile, two-layered soil profile and three-layered soil profile during an unsteady rain, and compared with numerical simulation of the Richards equation which is assumed to give a correct result.

For simplicity, unsteady rain intensity I is expressed by a sine function of time t, i.e., $I(t) = I_m \sin(\pi t/T)$ where I_m is the maximum rain intensity, and T is the rain duration. I_m =0. 3cm/min and T=300min are assumed in this study.

Two relations are needed for the Richards equation. One is the relation between soil moisture content θ and suction S (negative pressure head $-\psi$) which is expressed as follows (see Haverkamp (13))

$$\frac{\theta - \theta_r}{\theta_s - \theta_r} = \frac{\alpha}{\alpha + (\ln S)^{\beta}} \tag{22}$$

where θ_s = saturated moisture content; θ_r = residual moisture content; and α, β = constants.

The other is the relation between unsaturated hydraulic conductivity $k(\theta)$ and soil moisture content θ which is expressed as follows (see Mualem (14))

$$k(\theta) = k_s\left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^n \qquad with \quad n = 0.015 \int_{S=15atm}^{S=0} \gamma_w S \cdot d\theta + 3.00 \tag{23}$$

where k_s = saturated hydraulic conductivity; and γ_w = specific weight of water.

Five kinds of soil, that is, sand, loam, clay, Kanto loam and urbanized Kanto loam are considered in this study. The parameters θ_s, θ_r, α, β, n & k_s of these soils are referred to Herath (11), Chung and Horton (12) and are listed in Table 1. The soil moisture and suction relation curves of the five kinds of soil are shown in Fig. 1 which is based on Equation (22) and the parameters in Table 1.

Many researchers have studied the parameters of the Green-Ampt model, e.g., Bouer (6), Mein & Larson (5), Neuman (15) and Rawls et al. (16). Following the Neuman method (15), the moisture content in wetted zone is considered to be saturated soil moisture θ_s and the capillary suction SW at wetting front is calculated as follows:

$$SW = \int_0^{S_0} k_r(\theta) \cdot ds \qquad (24)$$

where $k_r(\theta)=k(\theta)/k_s$, relative hydraulic conductivity; and S_0 = initial suction in soil.

Table 1. Soil parameters

	soil types	θs	θr	α	β	n	k_s (cm/ min)	cal. SW (cm)	fitted SW (cm)	S_0 (cm)	θ_0	L_i (cm)
(1) uniform soil layers	sand	0.400	0.077	1.75E10	16.95	3.37	0.150	39.5	22.0	68.5	0.174	500
	loam	0.422	0.104	6451	5.56	3.97	0.042	38.5	26.0	68.5	0.321	500
	clay	0.394	0.120	6.579E7	9.00	4.38	0.012	66.9	67.0	68.5	0.392	500
	*KL	0.707	0.598	72.8	3.92	3.11	0.060	10.1	3.5	68.5	0.620	500
	*UKL	0.420	0.300	1244	4.41	4.28	0.0083	33.2	30.0	68.5	0.382	500
(2) fine over coarse												
	loam	0.422	0.104	6451	5.56	3.97	0.042	38.5	26.0	68.5	0.321	10
	sand	0.310	0.077	1.75E10	16.95	3.37	0.125	39.5	30.0	68.5	0.174	490
(3) fine over coarse over coarser												
	clay	0.394	0.120	6.579E7	9.00	4.38	0.012	66.9	67.0	68.5	0.392	10
	loam	0.340	0.104	6451	5.56	3.97	0.025	38.5	45.0	68.5	0.321	10
	sand	0.320	0.077	1.75E10	16.95	3.37	0.050	39.5	38.0	68.5	0.174	490

(Note: KL = Kanto loam, UKL = urbanized Kanto loam)

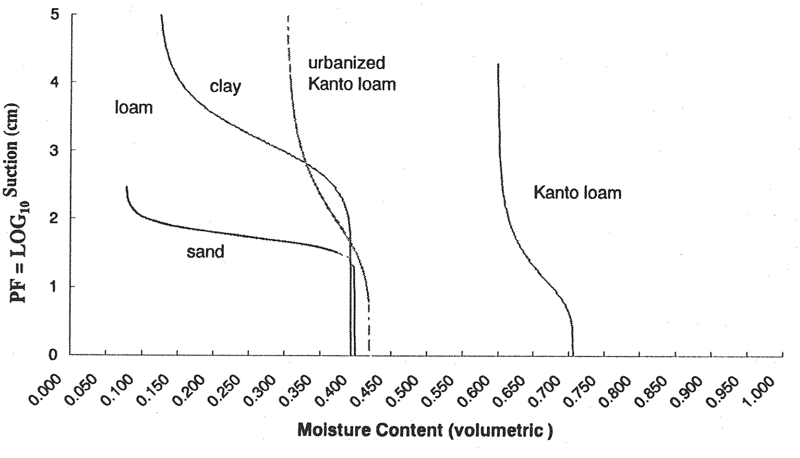

Fig. 1 Soil moisture and suction relation

The assumed and deduced parameters θ_0, S_0 & SW are also shown in Table 1. All of soil layers are assumed to have a same initial suction S_0 so as to ensure modeling results comparable. The initial suction of 68.5cm is assumed which is correspondent to the sand of 30% saturation. Though the capillary suction SW can be calculated from Eq.24, it can also be fitted. The difference of calculated SW and fitted SW is believed to be caused by the assumption of saturation in the wetted zone. For multi-layered soil profiles, soil parameters can be used directly in the case when coarse layers over fine ones. However, in the case of fine over coarse, the coarse lower layers can not reach saturation because the surface layer restricts infiltration, the parameters of k_s, θ_s and SW in lower layers are different from those of uniform soil but can be obtained by trial and error (see Moore (8)). In addition, the assumed thickness L_i of every soil layer is also shown in Table 1.

Comparison of the general model with the Richards equation is shown in Fig.2 to Fig.4. Infiltration into uniform soil during the assumed unsteady rain is shown in Fig.2. Infiltration into two-layered soil during the unsteady rain is shown in Fig.3, which includes two cases, that is, loam over sand and sand over loam. Infiltration into three-layered soil during the unsteady rain is shown in Fig.4. While Fig.4 (a) is the case of natural stratification, that is, sand over loam over clay or clay over loam over sand, Fig.4 (b) shows the case of urbanization-caused stratification (surface effect). The so called wet Kanto loam layer means that the initial moisture content in this layer (0.653) is higher than that of other Kanto loam layers (0.620). The reason why they are separately considered is that the initial moisture content has a quite big impact on the infiltration process. From these figures we can see that the modeling results of the general model match quite well with the numerical simulation of the Richards equation. The computation time by using the general model is less than tenth of that by using the Richards equation.

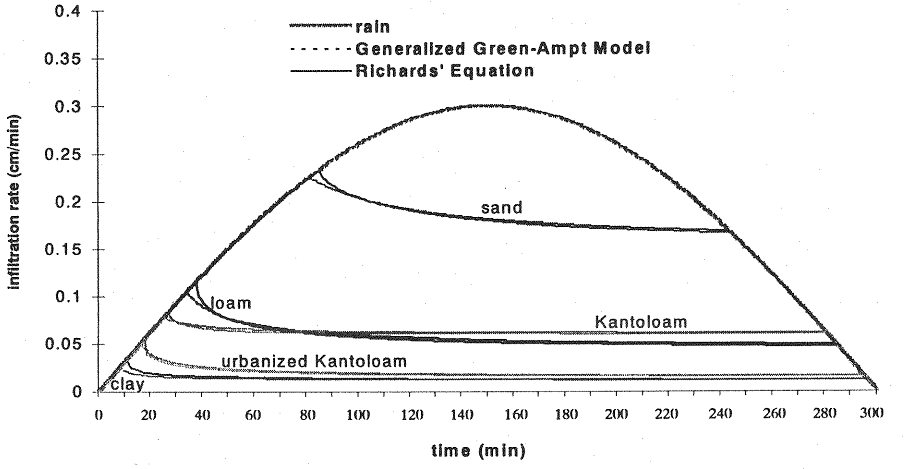

Fig.2 Infiltration into uniform soil

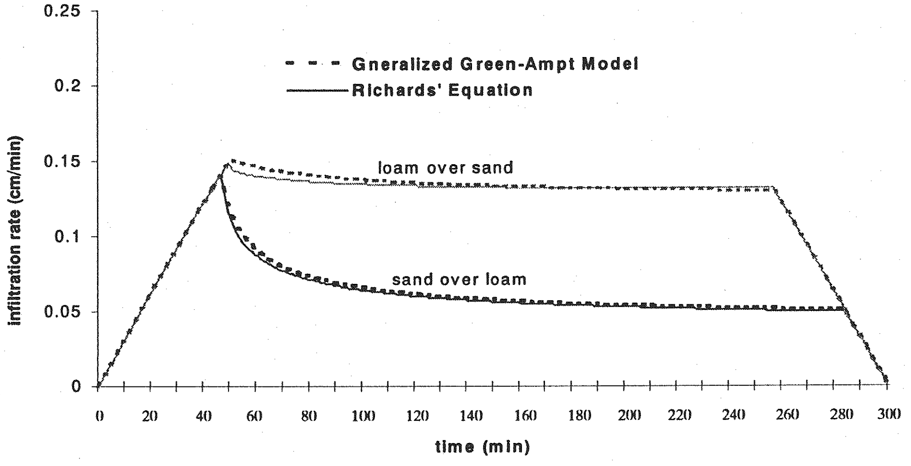

Fig.3 Infiltration into two-layered soil

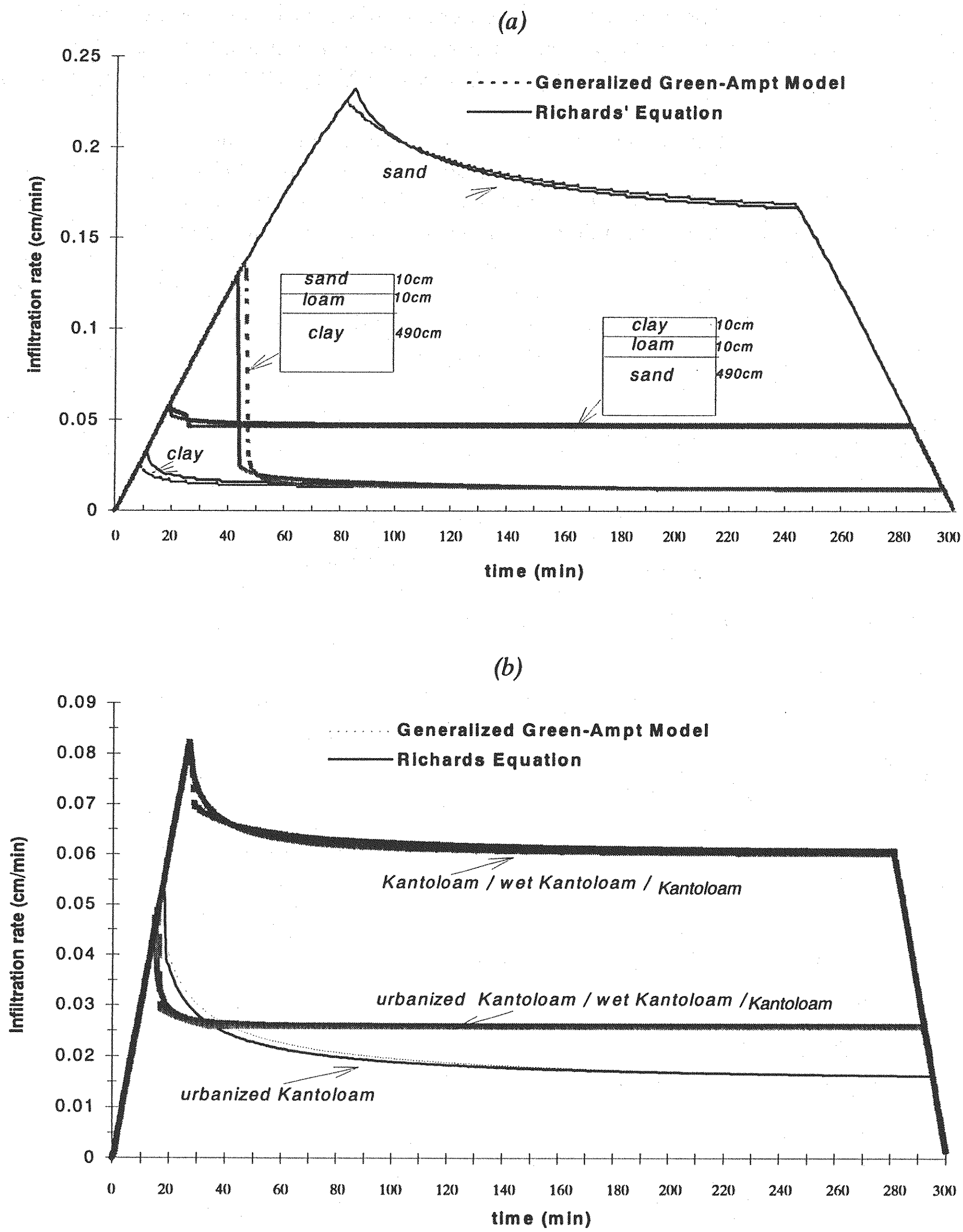

Fig.4 Infiltration into three-layered soil: *(a) natural stratification; (b) surface effect*

The soil moisture profiles are shown in Fig.5 for (a) the case of uniform Kanto loam and (b) the case of multi-layered soil (urbanized Kanto loam over wet Kanto loam over Kanto loam). It can be seen that the wetting front assumption is approximately correct, though the three-layered case is not as good as the uniform soil case. Because the wet front assumption may not be applied to more complicated cases, the general model may has a limitation for numbers, thickness and property differences of soil layers. However, three-layered model is usually enough to consider the vertical heterogeneity of soils in practical watershed modeling.

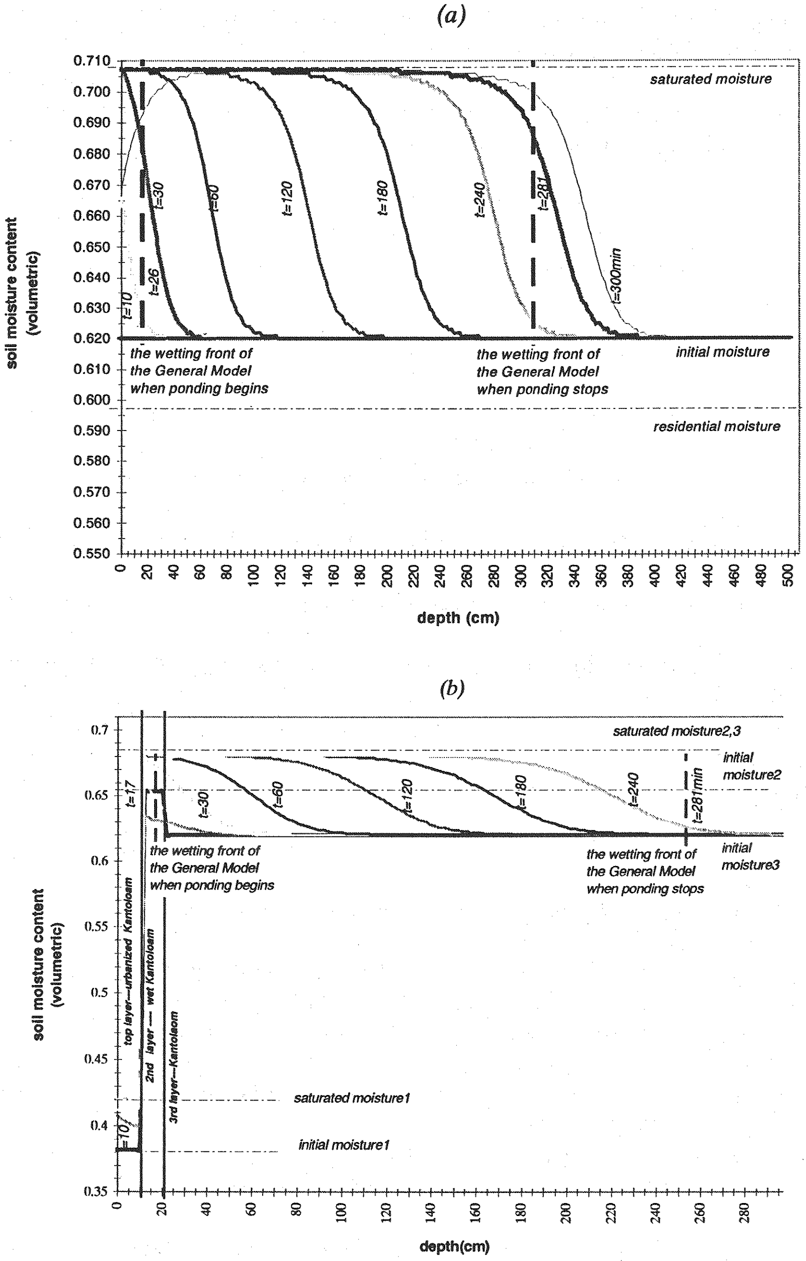

Fig.5 Moisture content profiles :*(a)uniform Kanto loam; (b) multi-layered soil*

CONCLUDING REMARKS

From the above analysis, it can be concluded that:

1. The Green-Ampt Model can be extended to model infiltration into a multi-layered soil during an unsteady rain. A general form has been proposed for the first time.
2. The proposed model has been verified by favorable comparison with the Richards equation, though it needs further verification by experiments and observations.

3. It is promising to apply the general form of the Green-Ampt model to watershed modeling because of its higher computation efficiency than the Richards equation.
4. However, in application, it should be mentioned that there may be certain limitations concerning numbers, thickness and property differences of soil layers, because the assumption of wetting front may not be applicable in very complicated cases.

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