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SECOND-ORDER ACCURATE TVD SCHEME FOR ONE-DIMENSIONAL TRANSIENT FREE SURFACE FLOWS

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SYNOPSIS

A second-order accurate explicit TVD scheme is presented for solving one-dimensional transient free surface flows. In the first step, the first-order generalised Courant-Issacson-Rees (GCIR) numerical scheme incorporating flux difference splitting is developed. The flux difference splitting, that is based on the Roe's approximate Jacobian, introduces full upwinding and ensures conservation properties. In the next step, this first-order scheme is converted into a second-order TVD scheme by a suitable modification of the flux. The process of conversion to second-order accuracy is based on the modified flux approach of Harten. The modified flux is so designed that it is conservative and consistent with the energy condition. Numerical results for exacting flow problems are presented to demonstrate shock capturing ability of the proposed scheme.

INTRODUCTION

One-dimensional transient free surface flows in a channel can be described by a set of hyperbolic partial differential equations. Mathematical modeling of these equations has been widely recognised as an important tool for obtaining a solution of transient free surface flow problems (e.g. Mahmood and Yevjevich (18), Abbott (1), Abbott and Basco (2)). The investigation of flow problems involving discontinuities in the form of shock has gained impetus form the shock capturing schemes developed for Euler equations in the field of gas dynamics. Schemes based on flux splitting technique (Beam and Warming (4), Moretti (15), Gabutti (7)) and approximate Riemann solvers (Roe (16)) have been developed that appropriately handle signal propagation and thereby shock propagation. In the field of hydraulics, Fennema and Chaudhry (5, 6) applied Lambda (Moretti (15)) and Gabutti (7) scheme to shock propagation in a channel and the Beam and Warming (4) scheme to dam-break problems. Glaister (9) applied Roe's approximate Riemann solver to dam-break flood wave propagation in a channel of infinite width. Alcrudo et al. (3) extended Glaister's work, based on flux-difference splitting, to channels with a finite cross-section of arbitrary shape. Jha et al. (12, 13) proposed a one-parameter implicit scheme based on flux splitting technique and a modified Beam and Warming scheme based on the concept of an approximate Jacobian.

An undiffused computation of discontinuous flows by finite difference method often requires a second-order accuracy. The conventional second-order schemes invariably develop undesired oscillations that have to be smoothed by using appropriate amount of artificial diffusion. However, the addition of artificial diffusion degrades the desired second-order accuracy. Harten (10, 11) introduced the idea of total variation diminishing

(TVD) schemes for Euler equations. Yee et al. (20) examined implicit high-resolution TVD scheme, a member of a one-parameter family of explicit and implicit schemes developed by Harten, for steady-state computation of one-dimensional hyperbolic conservation laws. These schemes are second-order accurate, consistent with conservation law and energy condition and yield oscillation free results even in the presence of severe discontinuities. Garcia et al. (8) implemented TVD technique onto the MacCormack (14) scheme for one-dimensional open channel flows. Yang et al. (19) presented TVD and essentially nonoscillatory (ENO) schemes for simulating shock propagation in open channels.

In this paper the ideas of Harten (11) and Yee et al. (20) are implemented onto the shallow water equations of open channel flows. The first-order generalised Courant-Issacson-Rees (GCIR) scheme incorporating flux difference splitting based on Roe's (16) approximate Jacobian is converted into a second-order accurate TVD scheme by suitably modifying its flux. The shock capturing ability of the presented scheme is demonstrated through several exacting numerical examples.

GOVERNING EQUATIONS

The governing equations, based on the conservation of mass and momentum, for one-dimensional transient free surface flows in a prismatic channel of arbitrary cross-section can be expressed as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \mathbf{S} = 0 \qquad ; \mathbf{E} = \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} \tag{1}$$

based on the assumption of hydrostatic pressure distribution, incompressibility of water and sufficiently small bottom slope of the channel. The vectors are defined as

$$\mathbf{U} = \begin{bmatrix} A \\ uA \end{bmatrix} \tag{2}$$

$$\mathbf{E} = \begin{bmatrix} uA \\ Au^2 + gF_h \end{bmatrix} \tag{3}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ -gA(S_o - S_f) \end{bmatrix} \tag{4}$$

where A = cross-sectional area of flow; u = mean velocity; g = acceleration due to gravity; S_o = bed slope and S_f = friction slope, assumed to be given by a steady-state formula. F_h = hydrostatic pressure force term expressed as

$$F_h = \int_0^h (h - \eta) W(\eta)\, d\eta \tag{5}$$

where h= depth of flow; η = integration variable indicating distance from channel bottom and $W(\eta)$ = channel width at distance η from the channel bottom and is expressed as

$$W(\eta) = \frac{\partial A}{\partial \eta} \tag{6}$$

Matrix **A** is the Jacobian of **E** with respect **U** given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -u^2 + gA/W & 2u \end{bmatrix} \qquad (7)$$

The hyperbolic nature of the governing equations ensures that matrix **A** has a complete set of independent and real eigenvectors expressed as

$$\mathbf{e}^{1,2} = \begin{bmatrix} 1 \\ u \pm c \end{bmatrix} \qquad (8)$$

where u = flow velocity and c = celerity expressed as $(gh)^{1/2}$. The eigenvalues of **A** are given by

$$\lambda^{1,2} = u \pm c \qquad (9)$$

FINITE DIFFERENCE SCHEME

As the homogeneous part of the governing equations is responsible for most of the problems when transient flows with shocks are modelled numerically, treatment of only the homogeneous part is considered for developing a finite difference scheme. A first-order upwind explicit finite difference scheme can be written as

$$\mathbf{U}_i^{t+1} = \mathbf{U}_i^t - \gamma\left[\Delta\mathbf{E}^{t-}_{i+1/2} + \Delta\mathbf{E}^{t+}_{i-1/2}\right] \qquad (10)$$

where i and t = space and time indices; $\gamma = \Delta t/\Delta x$; Δt = time increment and Δx = finite difference grid size in space (Fig.1). The upwinding is introduced into the scheme through the split flux difference by computing the positive and the negative ΔE by a backward and a forward space differences, respectively. Following Roe (16), the split flux differences can be written as

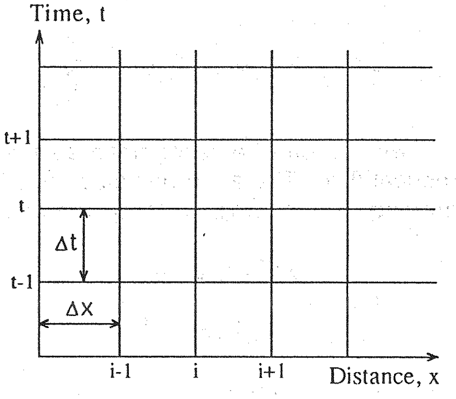

Fig. 1 1-D Finite Difference Grid.

$$\Delta\mathbf{E}^{t-}_{i+1/2} = \sum_{k=1}^{2} \lambda^{k-}_{i+1/2}\,\alpha^{k}_{i+1/2}\mathbf{e}^{k}_{i+1/2} \qquad (11)$$

$$\Delta\mathbf{E}^{t+}_{i-1/2} = \sum_{k=1}^{2} \lambda^{k+}_{i-1/2}\,\alpha^{k}_{i-1/2}\mathbf{e}^{k}_{i-1/2} \qquad (12)$$

where

$$\alpha^{1,2}_{i\pm1/2} = \mathbf{e}^{-1}_{i\pm1/2}\Delta\mathbf{U}_{i\pm1/2} \qquad (13)$$

Inserting Eqs.11 and 12 into Eq.10 and writing the positive and the negative eigenvalues as

$$\lambda^{k+} = 0.5(\lambda^k + |\lambda^k|) \;;\quad \lambda^{k-} = 0.5(\lambda^k - |\lambda^k|) \tag{14}$$

the following is obtained.

$$\mathbf{U}_i^{t+1} = \mathbf{U}_i^t - \frac{\gamma}{2}\sum_{k=1}^{2} (\lambda_{i+1/2}^k - |\lambda_{i+1/2}^k|)\, \alpha_{i+1/2}^k \mathbf{e}_{i+1/2}^k$$

$$+ \frac{\gamma}{2}\sum_{k=1}^{2} (\lambda_{i-1/2}^k + |\lambda_{i-1/2}^k|)\, \alpha_{i-1/2}^k \mathbf{e}_{i-1/2}^k \tag{15}$$

which can be rearranged as follows;

$$\mathbf{U}_i^{t+1} = \mathbf{U}_i^t - \gamma\left[\mathbf{E}_{i+1/2}^N - \mathbf{E}_{i-1/2}^N\right] \tag{16}$$

with the following numerical fluxes

$$\mathbf{E}_{i\pm1/2}^N = 0.5(\mathbf{E}_i + \mathbf{E}_{i\pm1}) - 0.5\sum_{k=1}^{2} \left|\lambda_{i\pm1/2}^k\right| \alpha_{i\pm1/2}^k \mathbf{e}_{i\pm1/2}^k \tag{17}$$

Eq. 17 can be recognised as the generalised Courant-Issacson-Rees (GCIR) numerical flux. The average values at (i+1/2) and (i-1/2) are computed by the Roe's (16) averaging. The average velocity and celerity are given by

$$u_{i\pm1/2} = \frac{A_{i\pm1}^{1/2} u_{i\pm1} + A_i^{1/2} u_i}{A_{i\pm1/2}^{1/2} + A_i^{1/2}} \tag{18}$$

$$c_{i\pm1/2}^2 = g\frac{\Delta(F_h)_{i\pm1/2}}{\Delta A_{i\pm1/2}} \tag{19}$$

It has been shown by Roe and Pike (17) that the above are unique averages satisfying conservation law and consistency with the governing equations.

CONVERSION TO SECOND-ORDER TVD SCHEME

A three-point first-order explicit scheme can be converted into a five-point second-order TVD scheme by suitably modifying the flux in the first-order scheme (Harten (10)). A second-order TVD modified flux for Eq. 17 can be written as (Yee et al. (20))

$$\mathbf{E}_{i\pm1/2}^N = 0.5(\mathbf{E}_i + \mathbf{E}_{i\pm1}) + 0.5\sum_{k=1}^{2} (F_i^k + F_{i\pm1}^k)\, \mathbf{e}_{i\pm1/2}^k$$

$$- 0.5\sum_{k=1}^{2} \psi(\lambda_{i\pm1/2}^k + \xi_{i\pm1/2}^k)\, \alpha_{i\pm1/2}^k \mathbf{e}_{i\pm1/2}^k \tag{20}$$

where

$$F_i^k = \text{minmod}\left[\sigma(\lambda_{i+1/2}^k)\alpha_{i+1/2}^k,\ \sigma(\lambda_{i-1/2}^k)\alpha_{i-1/2}^k\right] \quad (21)$$

$$\xi_{i+1/2}^k = \left(F_{i+1}^k - F_i^k\right)/\alpha_{i+1/2}^k \quad ; \quad \alpha_{i+1/2}^k \neq 0 \quad (22a)$$

$$= 0 \quad ; \quad \alpha_{i+1/2}^k = 0 \quad (22b)$$

The minmod function in Eq. 21 controls the second-order terms so that a smooth result is guaranteed even in the presence of discontinuities. The terms of Eq. 21 are expressed as

$$\sigma(\lambda_{i+1/2}^k) = 0.5\left[\psi(\lambda_{i+1/2}^k) + \gamma\,(\lambda_{i+1/2}^k)^2\right] \quad (23)$$

$$\psi(z) = 0.5\,(z^2 + \delta^2)/\delta \quad ; \quad |z| < \delta \quad (24a)$$

$$= |z| \quad ; \quad |z| \geq \delta \quad (24b)$$

where z = dummy variable. Eqs. 24a and 24b are meant for satisfying energy inequality condition with δ as a small positive number.

NUMERICAL RESULTS

Numerical results of several exacting flow problems with sharp discontinuities are presented to demonstrate applicability and shock-capturing ability of the second-order TVD scheme. All results in the following are for a 4000 m long, horizontal and frictionless channel with 401 grid points.

Dam-Break Flood Wave Propagation

Dam-break problem with a reservoir depth of 10 m and tailwater depth to reservoir depth ratio equal to 0.05 is considered. The downstream end is kept closed. The discontinuity in the depths at 1000 m from downstream end of the channel is given as the initial condition that simulates sudden and complete collapse of a dam. Computed water

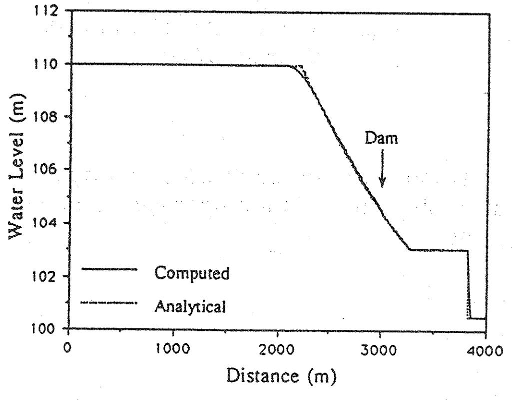

Fig.2 Dam-Break Flood Wave Propagation.

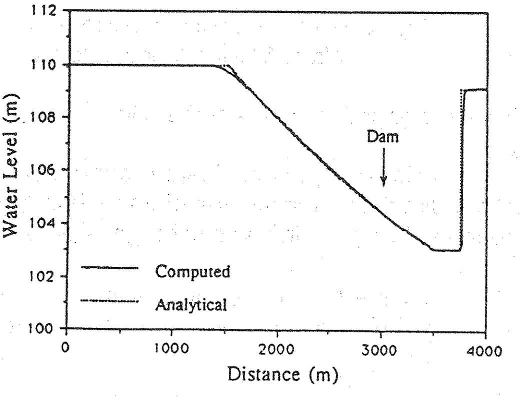

Fig.3 Dam-Break Flood Wave with Reflected Wave.

surface profiles at 80 seconds and 150 seconds are shown with analytical solutions in Fig. 2 and Fig. 3, respectively. At 80 seconds, the wave front has not reached the closed downstream end. Fig. 3 shows the reflected wave from the downstream wall. The steep wave fronts are resolved within three nodes in both cases. The negative wave is also well simulated by the model.

Sudden Release of Steps of Water

This example considers sudden release steps of water. The initial condition is described as

$h(x,0) = 10$ m ; $0 \le x \le 1500$m
$h(x,0) = 5$ m ; $1500 < x \le 3000$m
$h(x,0) = 0.5$ m ; $3000 < x \le 4000$m

Water surface profile along the channel at 70 seconds is shown in Fig. 4 together with the analytical solution. At this time wave fronts due to two steps of water are moving independently of each other. The computed result compares very well with the analytical solution. At 100 seconds the wave front from the upper step runs over the negative wave from the lower step and forms a complex structure (Fig. 5). The computed profile remains very smooth and discontinuities are resolved with good accuracy.

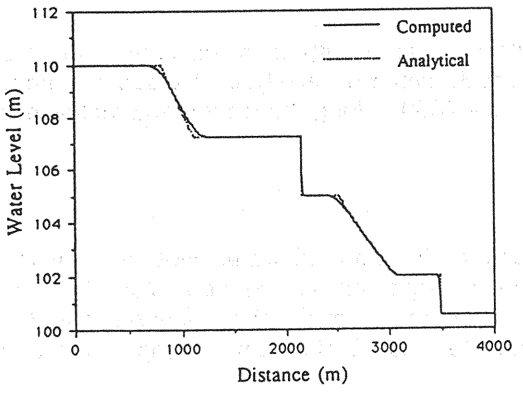

Fig. 4 Wave Fronts Due to Steps of Water Moving Independently.

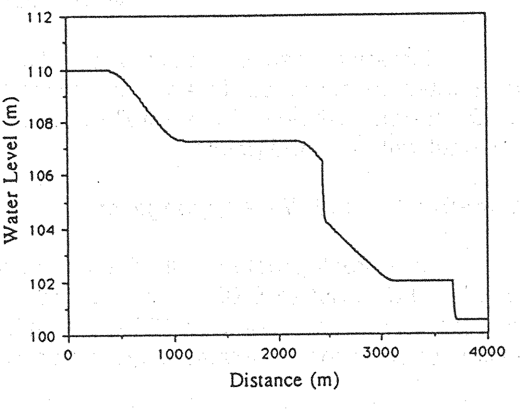

Fig. 5 Interaction of Positive and Negative Wave.

Bore Propagation over Another Bore

This example considers propagation of one bore over the other bore moving in the same direction. Initially the channel has 1 m deep still water. At the upstream boundary, sudden change in inflow discharge is prescribed as

$Q(0,t) = 50$ m^3/s per unit width ; $0 \le t \le 100$ s
$Q(0,t) = 100$ m^3/s per unit width ; $t > 100$ s

Fig. 6 shows the water profile of the bore at 100 seconds, just before a new bore is introduced at the upstream boundary. The computed bore is very accurate when compared with the analytical solution. The resolution of discontinuity is also very high.

Results at 130 seconds and 270 seconds are shown in Fig. 7. At 130 seconds, the second bore is seen travelling over the first bore. The second bore has higher celerity than the first one and at 270 seconds the first bore is completely overtaken by the second bore. After this time the both bores travel as a single bore. The solution of this highly discontinuous flow is very smooth with high resolution of the discontinuities. This complex phenomenon is very accurately simulated.

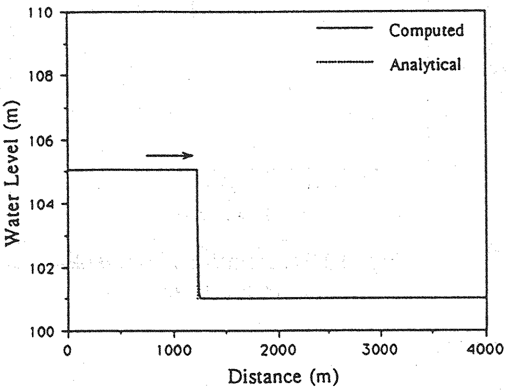

Fig. 6 Bore Propagation Due to Sudden Inflow.

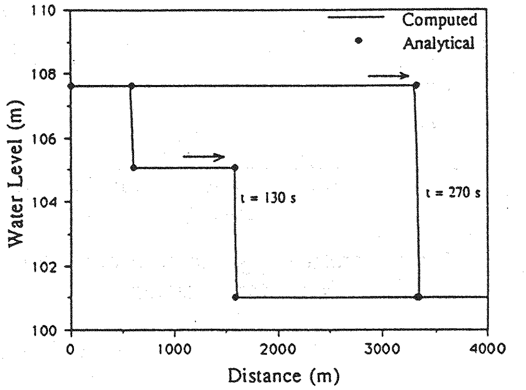

Fig. 7 Propagation of One Bore Over Another Bore.

Interaction of Bores

This last example considers another exacting flow problem. The channel has an initial uniform flow with depth 6 m and velocity 3.125 m/s. At the initiation of the computation a zero outflux condition is imposed at the downstream boundary that simulates sudden closure of gate. A surge is formed which travels upstream leaving still water behind. Also at time equal to zero, a sudden inflow of 100 m^3/s per unit width of the channel is imposed at the upstream boundary which is maintained throughout the computation. The upstream inflow condition forms another surge that travels downstream. Result at 150 seconds is shown in Fig. 8. It shows two waves travelling in the opposite direction facing each other. Both waves are accurately computed by the model and the steep fronts are correctly captured. A head-on collision of the two wave fronts occurs at 176 seconds at 2758 m from the upstream end. Water surface profile 24 seconds after this collision is shown in Fig. 9. A new bore is formed that travels downstream and eventually hits the downstream wall. Reflection from the wall results in yet another higher bore that travels upstream leaving still water behind (Fig. 10). In all stages of the interaction of bore, the TVD scheme yields very smooth results with high resolution of bores.

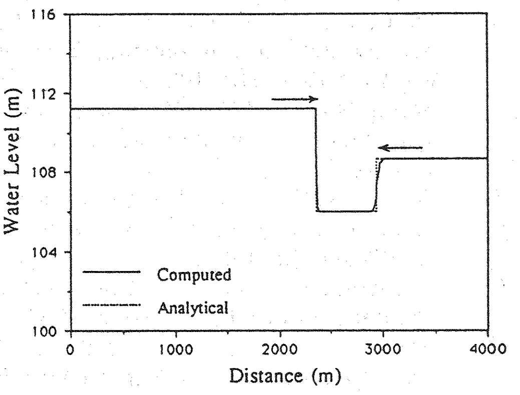

Fig. 8 Bore Propagating in Opposite Directions.

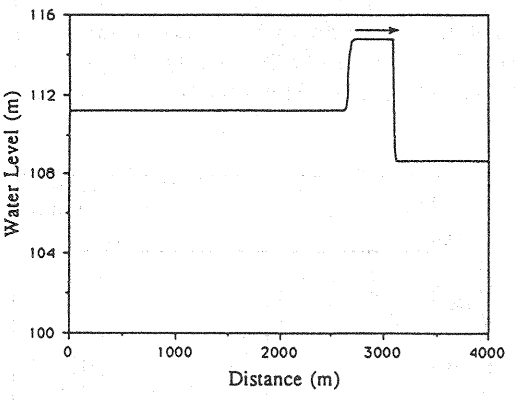

Fig. 9 Interaction of Bores Propagating in Opposite Directions.

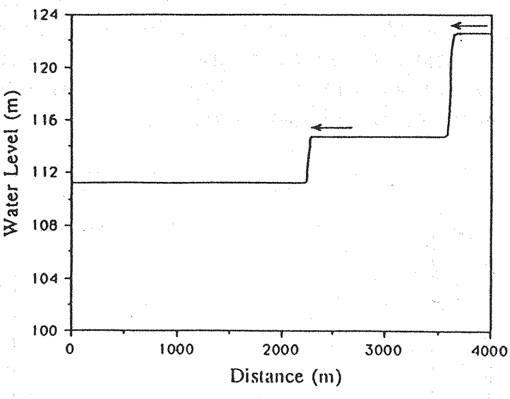

Fig. 10 Interacting Wave Reflected from the Wall.

CONCLUSIONS

A second-order accurate TVD scheme has been presented for simulating one-dimensional transient free surface flows. The first-order generalised Courant-Issacson-Rees (GCIR) numerical scheme incorporating flux difference splitting has been converted into a second-order TVD scheme by the modified flux approach of Harten. Roe's averaging has been used which satisfies conservation law and is consistent with the governing equations. Numerical results for exacting flow problems have been presented to demonstrate shock capturing ability of the proposed scheme. Computed results compare well with analytical solutions and discontinuities are captured with high resolution in all cases.

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APPENDIX - NOTATIONS

The following symbols are used in this paper:

$\mathbf{A}$ = Jacobian of $\mathbf{E}$ with respect to $\mathbf{U}$;
A = cross-sectional area of flow;
B = base width of the channel;
c = celerity;
$\mathbf{E}$ = flux matrix;
E^N = numerical flux;
e = matrix of eigenvectors;
F = modification to the flux;
F_h = hydrostatic pressure force term;
g = acceleration due to gravity;
h = flow depth;
i = grid location in space;
k = wave number;
Q = discharge;
$\mathbf{S}$ = matrix containing source terms;
S_f = friction slope;
S_o = bed slope;
t = index for time;

U	= vector for flow variables;
u	= velocity;
W(η)	= channel width at distance h from channel bottom;
x	= distance along channel.
z	= dummy variable;
α	= wave strength;
Δ	= operator, i.e. $\Delta f_{i+1/2} = f_{i+1} - f_i$;
δ	= small positive number;
γ	= Δt/Δx;
η	= integration variable indicating distance from channel bottom;
λ	= eigenvalues of **A**;
σ	= a function of λ;
ξ	= modification to the eigenvalue; and
ψ	= a function of modified eigenvalue.