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APPLICATION OF BOUNDARY LAYER THEORY TO PARALLEL FLOWS IN OPEN CHANNELS WITH LATERALLY DIFFERENT FRICTION COEFFICIENTS

by

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SYNOPSIS

This paper presents a mathematical model for the development of a transition diffusion region generated in parallel flows in open channels. First, the governing equations are derived in terms of a boundary layer approximation. Then, with an assumption about the horizontal velocity distribution, the development of transition region is obtained by means of classical integral momentum method. Finally, some examples are illustrated in order to test the performance of this model, in which a set of analytical solutions are compared with the existing experimental data. It is found that both are in good agreement.

INTRODUCTION

In river engineering and hydraulic engineering, engineers always concern with the lateral distributions of flow velocity such as the flow in compound channels, the flow downstream the hydro-projects, etc. Since a transition region is usually generated in this kind of flow, one should, firstly, study the mechanism of the flow in transition region if he wants to investigate the nature of the velocity distribution. This paper aims at studying a typical case of this kind of flow, that is, parallel flows in open channels with laterally different bottom friction coefficients.

Consider a channel with parallel flows of two different velocities separated by a false central wall in the upstream, as shown in Fig. 1. We assume that the bottom friction coefficients of the two sides of the channel are different while the free surface slopes are identical, namely, the bottom friction on the left half of the channel is larger than that on the right half, because the flow velocity in the left half is assumed to be smaller. The two layers of flow with different velocities meet at the point "A", the end of the central wall, and interfere with each other by lateral shear stress. The slow-moving layer of flow exerts a retarding force on the adjacent fast-moving layer of flow and makes it move slower, at the same time, the slow-moving layer of flow itself is accelerated by the fast-moving layer of flow. Since the fluid is incompressible and continuous, some fluid in the fast-moving layer must be displaced into the slow moving layer due to the retardation effect. Therefore, a transition region is generated in the center of the channel. It must be noted that in strict sense the flow velocity on the wall should be zero, and there may be a wake existing behind the point "A". However, when the wall is very thin, compared with the river width, and has a good streamlined configuration, the influence of it is restricted within a small region near the wall. As an approximation, the influence of the wall is not considered in this paper.

Some researches on this subject have been documented (Tanaka et al., 1986; Ishikawa, 1993). Ishikawa (1993) adopted depth-averaged 2-D shallow water equations to represent the flow depicted in Fig. 1. In order to obtain an analytical solution, the shallow water equations were linearized by using a perturbation technique, in which all terms including transverse velocity were neglected.

From the linearized equations and a presumed profile of the streamwise velocity, the width of diffusion layer was obtained by applying a weighted residual method. However, as mentioned by Ishikawa himself, since the approximation was a little crude, the utility of this model is limited. For example, when velocity difference between parallel flows is large, this model is not applicable.

According to the results given by Ishikawa (1993), the flow in lateral diffusion region is found to behave like a boundary layer flow. However, the diffusion layer reaches an equilibrium state at a certain distance far enough from the end of the wall labelled by "A" (see Fig.1). In the equilibrium region, the width of lateral diffusion layer and the velocity distribution remain unchanged, which can not be found in boundary layers of usual parallel flows. The reason why such an equilibrium region can exist is that the flow is affected by the lateral diffusion as well as the bottom friction.

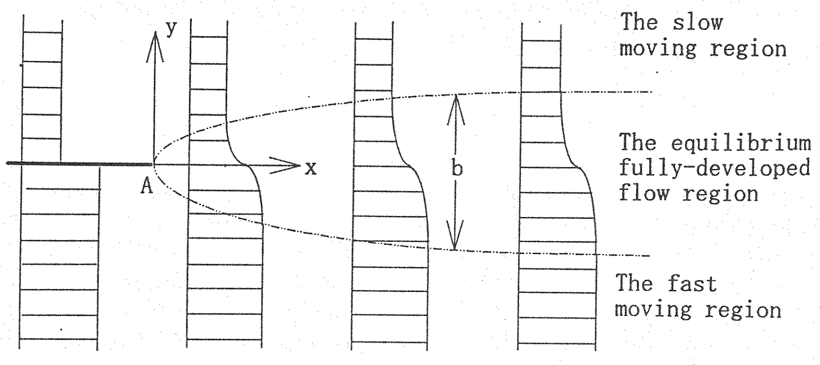


Fig. 1 Sketch of flow field

This paper intends to provide a new mathematical model which can represent the development of lateral diffusion region in open channels. Since the flow in question is analogous to the classical boundary layer flow, we employ Prandtl's boundary layer theory and use the integral momentum method to find its solution.

BASIC EQUATIONS

It is herein assumed that the width of channel is much larger than the water depth, which allows us to treat the flow as shallow water flow. Therefore, the depth-averaged 2-D equations are adopted to express the flow field. They are given by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial \eta}{\partial x}=\varepsilon(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})-\frac{f}{h}u\sqrt{u^2+v^2} \quad (1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial \eta}{\partial y}=\varepsilon(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})-\frac{f}{h}v\sqrt{u^2+v^2} \quad (2)$$

$$\frac{\partial uh}{\partial x}+\frac{\partial vh}{\partial y}=0 \quad (3)$$

where u, v = the depth-averaged flow velocity components in x and y directions, respectively; η = the elevation of free surface; h = the water depth; g = the gravitational acceleration; ε = the depth-averaged eddy viscosity, which is assumed to be constant in this paper; and f = the bottom friction coefficient. For region with high velocity, f is taken as f_1, and for the slow-moving region, it is specified as f_2. It is apparent that f_2 is larger than f_1.

Since the above equations include advective terms, it is impossible to get an analytical solution without approximation. Hence, the hydraulic variables are decomposed to

$$u = u_0 - u'$$
$$v = v' \tag{4}$$
$$\eta \doteq -s_0 x - h' + H_0$$

where the variables denoted by naught are for the undisturbed parallel flows, and the variables denoted by prime are for the deviations of the variables from the undisturbed state. s_0 = the undisturbed free surface slope; and H_0 = the undisturbed water depth. It is presumed that the deviation terms are very small, i.e.,

$$u', v' << u_0 ; \qquad |h'| << H_0 \tag{5}$$

On the above assumption, Eq. 1 can be simplified in terms of the perturbation technique as Ishikawa (1993) derived as follows:

$$u_0 \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial y^2} - \frac{2f}{H_0} u_0 u \tag{6}$$

However, in most cases, the velocity difference between parallel flows is large, and the lateral gradient of velocity in the transition region is also large. In these cases, the assumption in Eq. 5 does not hold, and we cannot use Eq. 6 to find the solution. We, therefore, have to use another method.

Take u_0 as the characteristic velocity of the problem and introduce dimensionless variables as follows:

$$\tilde{u} = \frac{u}{u_0}; \qquad \tilde{v} = \frac{v}{u_0} \tag{7-a}$$

$$\tilde{h} = \frac{h}{H_0}; \qquad \tilde{\eta} = \frac{\eta}{H_0}; \qquad \tilde{x} = \frac{x}{B}; \qquad \tilde{y} = \frac{y}{B} \tag{7-b}$$

where B is the width of the channel. It is herein assumed that B is much larger than the width of the lateral diffusion layer, b, which leads to

$$y' \sim \delta = \frac{b}{B} << 1 \tag{8}$$

It is therefore assumed that a parameter δ is small (the first order) and $\tilde{u}$, $\tilde{h}$, $\tilde{x}$ are in the zeroth order of magnitude, because the velocity difference between parallel flows is not small. Substituting the dimensionless variables into the continuity equation (Eq. 3), the following equation is derived:

$$\frac{\partial \tilde{u}\tilde{h}}{\partial \tilde{x}} + \frac{\partial \tilde{v}\tilde{h}}{\partial \tilde{y}} = 0 \tag{9}$$

Comparing the orders of all terms in Eq. 9, it is found that $\tilde{v}$ is in the first order of magnitude, i.e., the lateral velocity is small compared with streamwise velocity. It is clear that Eqs. 1 and 2 can also be normalized as follows:

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}+\frac{gH_0}{u_0^2}\frac{\partial \tilde{h}}{\partial \tilde{x}}=\frac{Bgs_0}{u_0^2}+\frac{\varepsilon}{u_0B}(\frac{\partial^2\tilde{u}}{\partial \tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial \tilde{y}^2})-\frac{Bf\tilde{u}}{H_0\tilde{h}}\sqrt{\tilde{u}^2+\tilde{v}^2} \quad (10)$$

$$\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}}+\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}+\frac{gH_0}{u_0^2}\frac{\partial \tilde{h}}{\partial \tilde{y}}=\frac{\varepsilon}{u_0B}(\frac{\partial^2\tilde{v}}{\partial \tilde{x}^2}+\frac{\partial^2\tilde{v}}{\partial \tilde{y}^2})-\frac{Bf\tilde{v}}{H_0\tilde{h}}\sqrt{\tilde{u}^2+\tilde{v}^2} \quad (11)$$

From Eqs. 7 and 8, it is easy to find

$$\frac{\partial \tilde{u}}{\partial \tilde{y}}\sim\frac{1}{\delta} \quad (12)$$

which means that there is an appreciable change for streamwise velocity in the diffusion layer. On the other hand, we have

$$\frac{\partial \tilde{u}}{\partial \tilde{x}}\sim 1(\text{zeroth order}) \quad (13)$$

A comparison between Eq. 12 and 13 indicates that the streamwise velocity varies slowly along x-direction compared with that along y-direction, namely, the x-derivative of the streamwise velocity appeared in the diffusion terms is relatively small and can be neglected.

From the above consideration, we can use the thin boundary layer approximation and put simply

$$\frac{\partial \eta}{\partial y}=0 \quad (14)$$

Eq. 14 means that we can assume that the y-derivative of the free surface elevation is constant. Comparing the orders of all terms in the normalized equations (9) and (10), and omitting the terms in higher orders, the governing equations can be simplified. The results are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=gs_0+\varepsilon\frac{\partial^2 u}{\partial y^2}-\frac{f_i}{H_0}u^2 \quad (15)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (16)$$

where i=1, 2. Eqs. 15 and 16 are very similar to the governing equations of Prandtl's classical boundary layer flow, but not the same, because a nonlinear term, i.e., retardation term, is included in Eq. 15. This is why we cannot solve them directly without introducing an assumption about the velocity distribution.

SOLUTION

First, we define that the width b_1 is the expanse from the centerline to the point where the streamwise velocity is

$$u_1|_{y=b_1}=u_s+0.99u_{s1} \quad (17)$$

and the width b_2 is the distance from the centerline to the left limit of the diffusion layer where

$$u_2|_{y=b_2} = u_s - 0.99u_{s2} \tag{18}$$

where u_s = the flow velocity at y=0; $u_{s1} = u_{01} - u_s$; and $u_{s2} = u_s - u_{02}$, in which u_{01} and u_{02} are the faster and the slower undisturbed flow velocities, respectively. They are depicted in Fig. 2. According to the results given by Ishikawa (1993), the velocity distributions in the lateral diffusion region are of exponential type. Ikeda and Izumi (1991) have studied the flow in open channels with pile dikes and found that the velocity distribution in equilibrium region follows the exponential law even if the difference of the undisturbed flow velocity is large. For that case, the governing equations are similar to the equations representing the parallel flow. Therefore, we assume that the flow in the diffusion region takes (not in the strict sense) similar velocity profiles, and the velocity distributions for the fast-moving layer and the slow-moving layer, respectively, are

$$u_1 = u_{01} - u_{s1}\exp(\alpha y / b_1) \tag{19}$$

$$u_2 = u_{02} + u_{s2}\exp(-\alpha y / b_2) \tag{20}$$

where α is a dimensionless numerical factor. Substituting Eq. 17 into Eq. 19 (or substituting Eq. 18 into Eq. 20), we obtain α=4.6052.

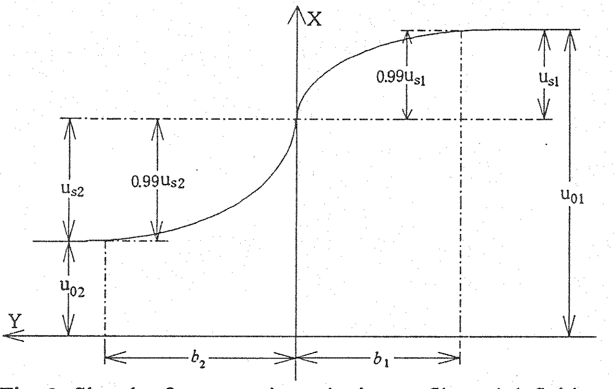


Fig. 2 Sketch of streamwise velocity profile and definition

Since the flow velocity and the velocity gradient must be continuous, they are matched at y=0, the conditions of which are described by

$$u_1|_{y=0} = u_2|_{y=0} \tag{21}$$

$$\frac{\partial u_1}{\partial y}|_{y=0} = \frac{\partial u_2}{\partial y}|_{y=0} \tag{22}$$

From the Eqs. 21 and 22, Ikeda and Izumi (1991) derived the following expression for the equilibrium state:

$$u_{s1} = u_{s2}(1+\chi)^{1/4} \tag{23}$$

where $\chi = f_2 / f_1 - 1$. Since the flow velocity distribution was assumed to be similar, we can use Eq. 23 in the transition region. Substitution of Eqs. 19 and 20 into Eq. 21 gives us:

$$u_{01} - u_{s1} = u_{02} + u_{s2} \tag{24}$$

Combining Eq. 23 and Eq. 24, we can derive

$$u_{s1} = [(1+\chi)^{1/4} - 1](1+\chi)^{1/4} u_{02} \tag{25}$$

$$u_{s2} = [(1+\chi)^{1/4} - 1] u_{02} \tag{26}$$

Let Eq. 16 be multiplied by u, and substitute the product into Eq. 15, then the following equation is derived:

$$\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = g s_0 + \varepsilon \frac{\partial^2 u}{\partial y^2} - \frac{f_i}{H_0} u^2 \tag{27}$$

where the subscript i=1, 2. For the fast-moving layer, integrating Eq. 27 from $y = -b_1$ to y=0 and considering the boundary conditions, i.e., $v_1|_{y=b_1} = 0$ and $(\partial u / \partial y)|_{y=-b_1} = 0$, the following equation is obtained:

$$\frac{\partial}{\partial x} \int_{-b_1}^{0} u_1^2 dy + (u_1 v_1)|_{y=0} = g s_0 b_1 + \varepsilon \frac{\partial u_1}{\partial y}|_{y=0} - \frac{f_1}{H_0} \int_{-b_1}^{0} u_1^2 dy \tag{28}$$

In a similar way, another integrated equation for the slow-moving layer (integration of u_2 from y= 0 to y=b_2) can be obtained as follows:

$$\frac{\partial}{\partial x} \int_{0}^{b_2} u_2^2 dy - (u_2 v_2)|_{y=0} = g s_0 b_2 - \varepsilon \frac{\partial u_2}{\partial y}|_{y=0} - \frac{f_1}{H_0} (1+\chi) \int_{0}^{b_2} u_2^2 dy \tag{29}$$

Since f_2 has been written as $f_1(1+\chi)$ in Eq. 29, hereafter f_1 is written as f for simplicity. Then integrating Eq. 16 from $y = -b_1$ to y=0, we have

$$v_1|_{y=0} = \int_{-b_1}^{0} \frac{\partial u_1}{\partial x} dy \tag{30}$$

In a similar manner, another integrated expression can be obtained as follows:

$$v_2|_{y=0} = -\int_{0}^{b_2} \frac{\partial u_2}{\partial x} dy \tag{31}$$

The expressions (30) and (31) are substituted into Eqs. 28 and 29, respectively, to eliminate v_1 and v_2 from these two equations. The results are

$$\frac{\partial}{\partial x} \int_{-b_1}^{0} u_1^2 dy + (u_{01} - u_{s1}) \int_{-b_1}^{0} \frac{\partial u_1}{\partial x} dy = g s_0 b_1 + \varepsilon \frac{\partial u_1}{\partial y}|_{y=0} - \frac{f}{H_0} \int_{-b_1}^{0} u_1^2 dy \tag{32}$$

$$\frac{\partial}{\partial x} \int_{0}^{b_2} u_2^2 dy + (u_{02} + u_{s2}) \int_{0}^{b_2} \frac{\partial u_2}{\partial x} dy = g s_0 b_2 - \varepsilon \frac{\partial u_2}{\partial y}|_{y=0} - \frac{f}{H_0} (1+\chi) \int_{0}^{b_1} u_2^2 dy \tag{33}$$

Substituting Eqs. 19 and 25 into Eq. 32, the following relation is obtained:

$$c_{11}\frac{db_1}{dx} - c_{12} / b_1 + c_{13} b_1 = 0 \tag{34}$$

where $c_{11} = 3u_{01}u_{s1} - 3u_{s1}^2 / 2$, $c_{12} = \varepsilon u_{s1}\alpha^2$, $c_{13} = f(2u_{01}u_{s1} - u_{s1}^2 / 2) / H_0$.

For the slow-moving flow region, substituting Eqs. 20 and 26 into Eq. 33, the following relation can be derived:

$$c_{21}\frac{db_2}{dx} - c_{22} / b_2 + c_{23} b_2 = 0 \tag{35}$$

where $c_{21} = 3u_{02}u_{s2} + 3u_{s2}^2 / 2$, $c_{22} = \varepsilon u_{s2}\alpha^2$, $c_{23} = f(2u_{02}u_{s2} + u_{s2}^2 / 2)(1+\chi) / H_0$.

Using the boundary condition, $b_1 = 0$ at $x = 0$, the solution of Eq. 34 is found to be

$$b_1^2 = b_{01}^2[1 - \exp(-2\gamma x)] \tag{36}$$

where $\gamma = f(4u_{01} - u_{s1}) / [H_0(6u_{01} - 3u_{s1})]$, $b_{01}^2 = 2\varepsilon\alpha^2 H_0 / [f(4u_{01} - u_{s1})]$. In a similar manner, Eq. 35 can be solved, and the solution is

$$b_2^2 = b_{02}^2[1 - \exp(-2\beta x)] \tag{37}$$

where $\beta = f(4u_{02} + u_{s2})(1+\chi) / [H_0(6u_{02} + 3u_{s2})]$ and $b_{02}^2 = 2\varepsilon\alpha^2 H_0 / [f(1+\chi)(4u_{02} + u_{s2})]$.

In Eqs. 36 and 37, b_{01} and b_{02} are the width components of the fully-developed diffusion layer for the fast-moving layer and the slow-moving layer, respectively. It is easy to find that b_{01} and b_{02} are in proportion to the square root of the eddy viscosity.

EFFECT OF BOTTOM FRICTION COEFFICIENT

We first give two examples to show the effect of the bottom friction coefficient on the diffusion width. The result of example 1 is depicted in Fig. 3-a. It shows the effect of the ratio of friction coefficients on the widths b_{01}, b_{02}, and $b_0 = b_{01} + b_{02}$, in which the former two are defined in the above section. In this example, the constant parameters are determined according to the experimental conditions conducted by Ikeda et al. (1994, Run 1), in which the water depth is 6 *cm*, the river bed slope is 0.001, resistance coefficient f is 0.007. The ratio of the bottom friction coefficients is taken as a varying parameter, and the eddy viscosity, which is calculated by using the empirical expression proposed by Ikeda et al. (1994), is varied consequently. It is found that there is a critical value for χ; when χ is smaller than it, b_0 increases as χ increases, when χ exceeds this critical value, the width decreases as χ increases. Furthermore, b_{01} and b_{02} also have their own critical values. The reason why the critical values appear could be found from Eqs. 36 and 37. In these equations b_{01} and b_{02} are directly proportional to the eddy viscosity, ε, but ε varies likewise as the widths as χ increases (Ikeda et al., 1994), as shown in Fig. 3-b.

In example 2, the resistance coefficient is varied, but the water depth and the river bed are still kept to be the same as in the example 1. The calculated results are illustrated in Fig. 3-c. It is found that the diffusion width b_0 decreases as the friction coefficient increases. This indicates that the friction suppresses the development of the diffusion width.

Fig. 4 shows an example of the effect of eddy viscosity on the diffusion width. In this example, the basic parameters of calculation are the same as those in the examples shown in Fig. 3. Here, χ is taken to be 10.2 according to the parameters of Run 1 performed by Ikeda et al. (1994). However, the eddy viscosity is varied from 1.5 cm^2/s to 3.0 cm^2/s to test the effect of eddy viscosity on the development of diffusion layer. Corresponding to the variation of ε, the diffusion width of fully-developed region varies from 10.62 to 15.03 *cm*. The diffusion width is calculated to be 13.35 *cm* for $\varepsilon = 2.37 cm^2/s$ which is determined from the empirical relation proposed by Ikeda et al. (1994). The measured value for Run 1 is 13.30 *cm*, which indicates that the agreement is very good.

Fig. 4 also reveals that for different eddy viscosity the developments of diffusion layers are structurally similar. When the distance x is less than 150 *cm*, the widths of the diffusion layers increase quickly as x increases, and when x is larger than 250 *cm*, the diffusion layers nearly stop increasing. This means that the development of the lateral diffusion layer along x-direction is independent of ε, the reason of which is that only the lateral eddy-diffusion term, i.e., the y-derivative term of u, is retained in Eq. 15 and the diffusion in x-direction has been neglected. As a result, both γ and β in Eqs. 36 and 37, respectively, are not related to ε.

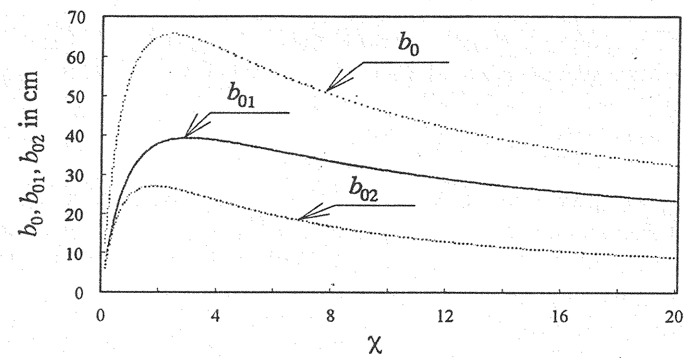


Fig. 3-a Effect of the ratio of bottom friction coefficients on the diffusion widths

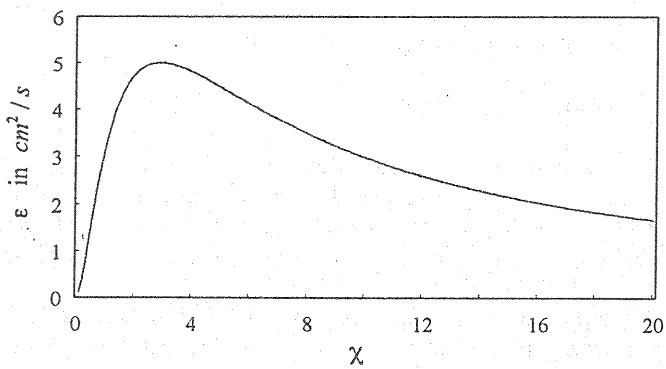


Fig. 3-b Effect of the ratio of bottom friction coefficients on the eddy viscosity

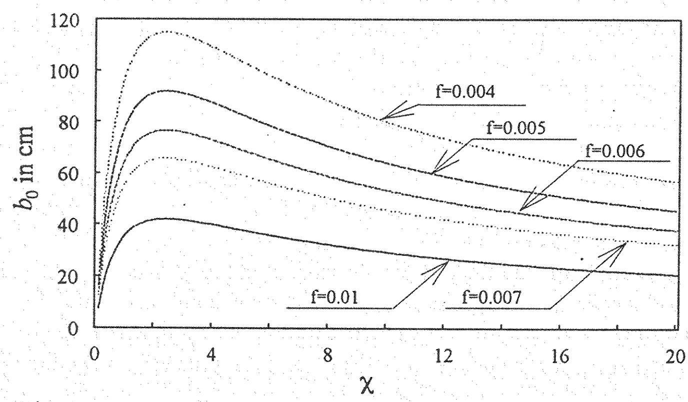


Fig. 3-c Effect of bottom friction coefficient on the diffusion width

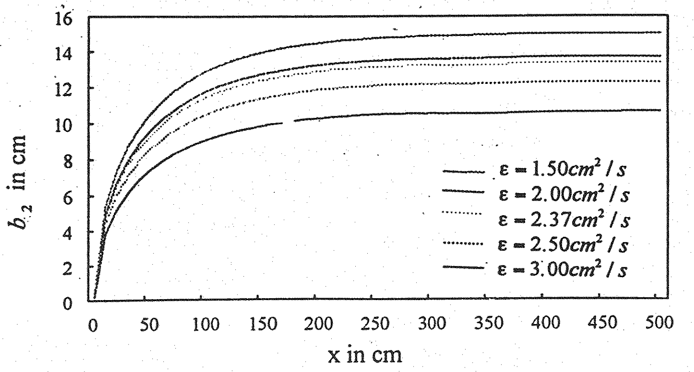
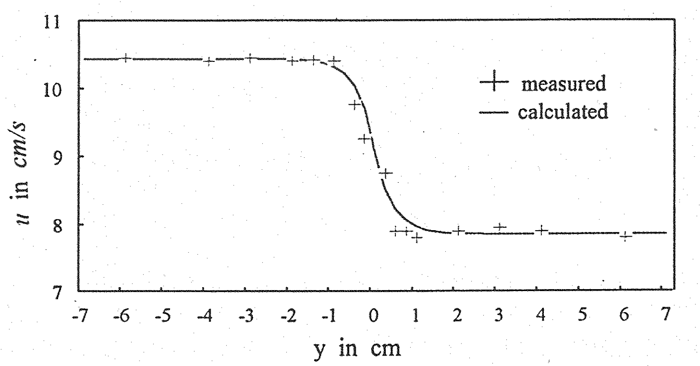


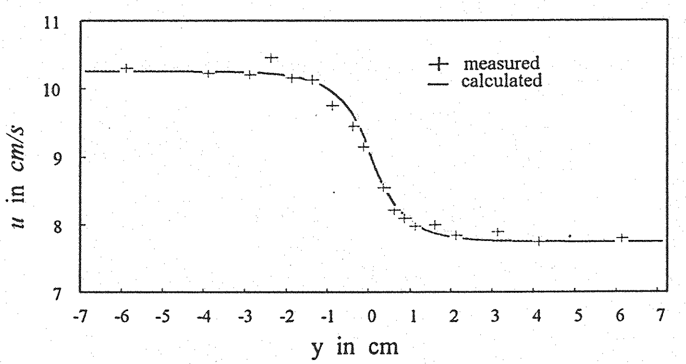
Fig.4 Effect of eddy viscosity on the development of diffusion layer

COMPARISON WITH THE EXISTING DATA

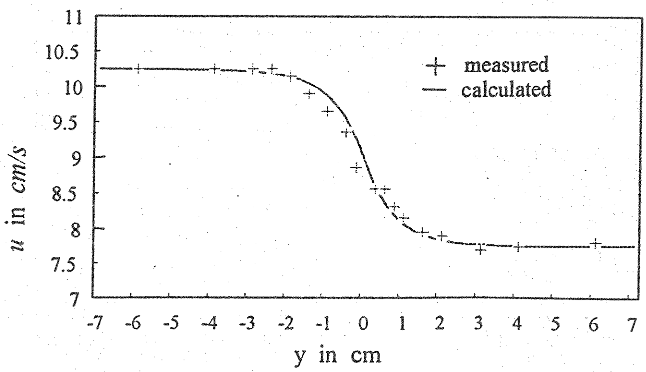
The utility of the derived equation is verified in the following example. The condition for calculation is adopted from the experiment made by Kimura (1985), in which the water depth is 0.85 cm; bottom friction $f_1 = 0.007$, $f_2 = 0.018$; the bed slope is 0.0084. The velocity distributions calculated from Eqs. 36 and 37 are compared with the existing experimental data in Fig. 5, and the comparison of the development of diffusion layer is shown in Fig. 6. The comparisons of the velocity profiles are performed at x=15, 50 and 80 cm, respectively. Figs. 5 and 6 reveal that the equations perform well at every location of the laboratory test.



(a) x=15cm



(b) x=50cm



(c) x=80cm

Fig. 5 Comparisons between the analytical results and the existing data for the depth-averaged streamwise velocity

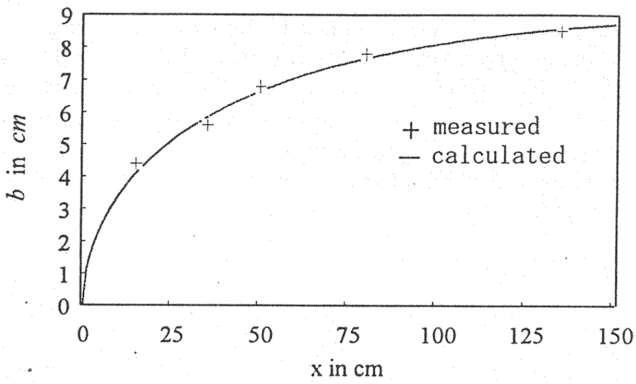


Fig. 6 Verification of the development of diffusion layer

CONCLUSIONS

This paper has developed a mathematical model to represent the development of flow in the lateral diffusion layer, which is generated in open channels with parallel flows due to laterally different bottom friction coefficients. The solution is obtained by using the integral momentum method. The result indicates that the model can represent the diffusion flow field successfully even if the velocity difference is relatively large.

The lateral distribution of streamwise flow velocity calculated is compared with the existing experimental data, and the present model is found to reproduce the actual flow field well. This indicates that the similarity assumption on the streamwise velocity performs well. The verification of the development of diffusion layer reveals that this mathematical model can represent the parallel flows with different bottom frictions.

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