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PROBABILISTIC MODEL OF RAINFALL OF A SINGLE STORM

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SYNOPSIS

We propose a probabilistic model of rainfall of a single storm. Each rainfall event is represented in terms of three characteristic variables: duration, maximum intensity, and total amount. A joint probability density function of the variables and analytical expressions of the moments are derived, including correlation coefficients among the variables. The validity of the proposed model is verified through its application to observed rainfall events. The correlation coefficients between (a) duration and maximum intensity, (b) duration and total amount, and (c) maximum intensity and total amount, and shape indices of the variables, all obtained using this model, showed good agreement with values calculated from historical observations. The distribution function of total rainfall of a single storm is theoretically derived, and is named "SQRT-K distribution," which is a generalized form of the Eagleson's derived distribution.

AIM

It is well known that occurrence and intensity of disasters caused by floodings, debris flows, etc. are accounted for not only by peak rainfall intensity, but also by other characteristic rainfall indices: duration, total depth, etc. A probabilistic model is presented to express the relationship among these rainfall indices.

Eagleson (2) derived the probability distribution function of total depth of a single storm assuming exponentially and independently distributed duration and average intensity, which is a half of peak intensity when a triangular hyetograph is assumed. However, in most cases, there exists a positive correlation between duration and average (or peak) intensity. It may be considered low enough to be negligible, but is of crucial importance for some problems. Córdova and Rodríguez-Iturbe (1) employed a bivariate exponential distribution which had been originally presented by Izawa (4, 5) and applied to hydrology by Nagao and Kadoya (6), and showed an important effect of the correlation on the probabilistic structure of storm surface runoff. Izawa's bivariate Gamma distribution has a significant property in that the regression relation between the two variates is linear. It is unrealistic, however, to assume a linear relationship between peak intensity and duration, since, when the duration of a storm increases, the peak may increase on the average, but the rate of increase may decrease as shown in Fig. 1.

This study aims at presenting a model which can account for both the non-linearity and correlation, and still is simpler than the Izawa's distribution in the expression, while the application of the presented model may be limited within the analyses of the probabilistic structure of peak, duration, and total amount of a hyetograph or a hydrograph.

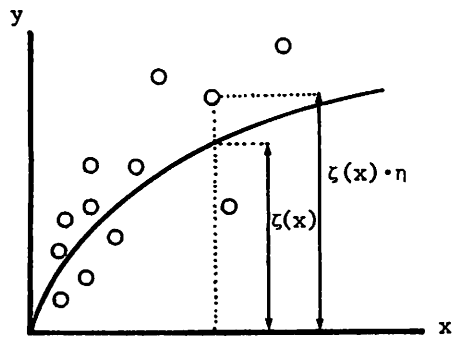

Fig. 1 Conditional distribution of peak intensity y for fixed duration x

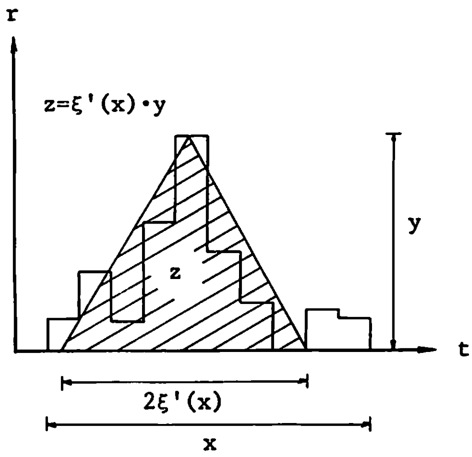

Fig. 2 Symbols

MODELLING CONCEPT

We shall denote the duration, the peak intensity, and the total depth of rainfall of a single storm by x, y, and z, respectively. It is generally observed that peak rainfall intensity y is weakly and nonlinearly correlated to duration x. When the duration of a storm doubles, the average of the peak may increase, but remains less than double. Thus, it may be reasonably assumed that, with storms of longer duration, the average peak intensity increases and the rate of increase gradually decreases.

We shall express the nonlinear regression between x and y by:

$$\hat{y} = \zeta(x) \quad (\hat{y} > 0,\ x > 0) \tag{1}$$

where $\hat{y}$ is the expectation of y for a given value of x, and y distributes around $\hat{y}$. The function ζ is a monotonically-increasing function whose rate of increase drops as x increases. We shall introduce a random variable η to express the distribution of y. Then,

$$y = \hat{y} \cdot \eta = \zeta(x) \cdot \eta \quad (\eta > 0) \tag{2}$$

where η is independent of x, and $E(\eta)=1$.

It is convenient to employ the expression of the product of $\hat{y}$ and η as shown in Eq. 2 instead of the sum, because, like rainfall intensity, the product is always positive. If we employ the sum, η which represents the deviation of y from $\hat{y}$ should distribute both in positive and negative domains, since the expectation must be zero. And if η is independent of x, y may take a negative value for a smaller x. If η is conditioned to avoid the negative y corresponding to the value of x, the assumption of independency between x and η does not hold, which makes the following derivation complicated and intractable.

Fundamentally, the function of $\zeta(x)$ must be determined on the basis of the probabilistic model of a rainfall intensity series during a single rain storm.

We shall employ the following approximate expression for total rainfall depth of a single storm:

$$z = \xi'(x) \cdot y = \xi(x) \cdot \eta \tag{3}$$

where $\xi(x)=\xi'(x)\cdot\zeta(x)$ (see Fig. 2). Eq. 3 indicates that depth z can be approximated by the product of the peak intensity and an appropriate function of the duration. Then, x, y, and z are all expressed in terms of two independent random variables, x and η. The model is less common in the respect that only two random variables are involved to express the relation among three random variables. Usually three random variables are required to express every random component of a model with three original probabilistic variables. However, if the degree of freedom (standard deviation) of z is sufficiently small under the condition that x and y are fixed, it is practical to employ a probabilistic model with fewer random variables, as in the present study. As shown in a later section, the correlation coefficients between z and x, and z and y are relatively high, which implies less freedom of z for fixed x and y; this can be the basis of a model with two independent random variables. The simplification neglecting the residual random component of z is considered justifiable since it makes the model much more tractable both in the analyses and practical applications.

GENERAL EXPRESSIONS OF JOINT PROBABILITY DENSITIES AND MOMENTS

First, we shall deduce the general expression of the joint probability density of duration and peak intensity in view of Fig. 3.

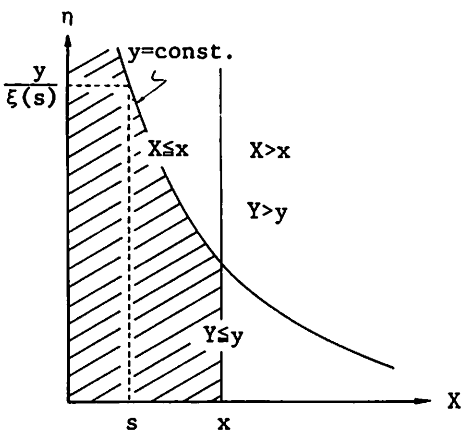

Fig. 3 The domain of integral to derive the joint p.d.f. of x and y

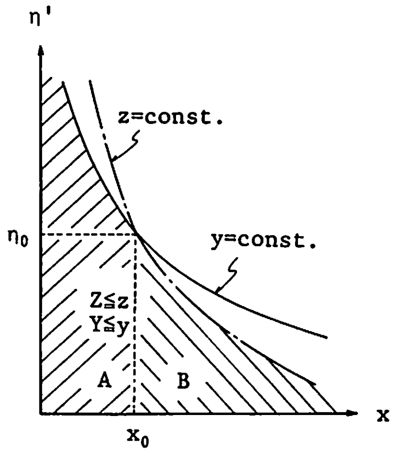

Fig. 4 The domain of integral to derive the joint p.d.f. of y and z

$$F_5(x,y) = \mathrm{Prob}\{(X \leq x) \cap (Y \leq y)\} = \int_0^x \int_0^{y/\zeta(s)} f_1(s) \cdot f_2(\eta) \cdot d\eta ds \qquad (4)$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are the probability density functions (p.d.f.) of x and η, and $F_5(x, y)$ the joint cumulative distribution function (c.d.f.) of x and y.*

By partially differentiating F_5 with respect to x and y,

$$f_5(x,y) = \partial^2 F_5(x,y)/\partial x \partial y = 1/\zeta(x) \cdot f_1(x) \cdot f_2\{y/\zeta(x)\} \qquad (5)$$

Similarly, the joint p.d.f. of duration and depth is derived as follows:

$$f_6(x,z) = 1/\xi(x) \cdot f_1(x) \cdot f_2\{z/\xi(x)\} \qquad (6)$$

The derivation and the expression of the joint p.d.f. of peak and depth are more complicated. We shall specialize the model to some extent to facilitate the derivation by assuming the conditions shown in Fig. 4, i.e.,

(1) the curves representing y=const. and z=const. intersect just once on the x-η plane, and

(2) $z \geq y$ for $x \leq x_0$, or $\eta \geq \eta_0$,

---

* Subscripts 1, 2, 3, and 4, respectively designate properties of x, η, y, and z, and subscripts 5, 6, and 7, stand for (x, y), (x, z), and (y, z).

where (x_0, η_0) are the coordinates representing the intersection. Values x_0 and η_0 are calculated from Eqs. 2 and 3, supposing y and z to be constant values. Then,

$$\begin{aligned} F_7(y,z) &= \text{Prob}\{(Y \leq y) \cap (Z \leq z)\} \\ &= \int_0^{x_0} \int_0^{y/\zeta(x)} f_1(x) \cdot f_2(\eta) d\eta dx \\ &\quad + \int_{x_0}^{\infty} \int_0^{z/\xi(x)} f_1(x) \cdot f_2(\eta) d\eta dx \end{aligned} \tag{7}$$

$$f_7(y,z) = \frac{\partial^2 F_7(y,z)}{\partial y \partial z} = (A_1 + A_2) - (B_1 + B_2) \tag{7'}$$

where

$$A_1 = \frac{\partial}{\partial y}\{\frac{\partial x_0}{\partial z} f_1(x_0)\} \cdot \int_0^{y/\zeta(x_0)} f_2(\eta) d\eta$$

$$A_2 = \frac{\partial x_0}{\partial z} f_1(x_0) \cdot \frac{1}{\zeta(x_0)} f_2\{y/\zeta(x_0)\}$$

$$B_1 = \frac{\partial}{\partial z}\{\frac{\partial x_0}{\partial y} f_1(x_0)\} \int_0^{z/\xi(x_0)} f_2(\eta) d\eta$$

$$B_2 = \frac{\partial x_0}{\partial y} f_1(x_0) \cdot \frac{1}{\xi(x_0)} f_2\{z/\xi(x_0)\}$$

The joint c.d.f. of all three variables, x, y, and z, can be derived through a similar procedure based on the definition $\text{Prob}\{(X \leq x) \cap (Y \leq y) \cap (Z \leq z)\}$, i.e.,
(1) for $0 \leq x \leq x_0$, changing the upper bound of integral of the first term of the righthand side of Eq. 7, x_0, to x, and eliminating the second term, and
(2) for $x > x_0$, changing the upper bound of integral of the second term, ∞ to x.

The general expression of the moments is quite simple as follows:

$$\begin{aligned} \nu(r,s,t) &= \int_0^{\infty} \int_0^{\infty} x^r \cdot y^s \cdot z^t f_1(x) \cdot f_2(\eta) dx d\eta \\ &= \int_0^{\infty} x^r \cdot \zeta^s(x) \cdot \xi^t(x) f_1(x) dx \cdot \int_0^{\infty} \eta^{s+t} f_2(\eta) d\eta \end{aligned} \tag{8}$$

where $\nu(r,s,t)$ is the r-th, s-th, and t-th joint moment around the origin with respect to x, y, and z.

SPECIFIC EXPRESSIONS FOR SOME PRACTICAL ASSUMPTIONS

a) The joint and marginal distributions

We shall introduce the following three practical assumptions to advance the derivation:
(1) x and η are Gamma-type distributed;
(2) the expectation of peak intensity is proportional to $(\text{duration})^a$; and
(3) the total depth is proportional to $1/2 \cdot (\text{duration}) \cdot (\text{peak intensity})$.
Assumptions 1 and 3 are empirically reasonable, while 2 stands to reason as long as $0 \leq a \leq 1$, as shown in Fig. 1.

Then, both the random variables x and η have the following p.d.f.:

$$f(s) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} s^{\alpha-1} e^{-\beta s} \tag{9}$$

where α and β are the shape and the scale parameters ($\alpha > 0$, $\beta > 0$).
Structural Eqs. 1 and 3 are specified by the assumptions 2 and 3, as follows:

$$\zeta(x) = \kappa_1 x^a \quad (\kappa_1 > 0,\ 0 \leq a \leq 1) \tag{10}$$

$$z = \frac{1}{2}\kappa_2 xy \quad (\kappa_2 > 0) \tag{11}$$

By substituting (2) and (10) to (11),

$$z = \frac{1}{2}\kappa_3 x^{1+a}\eta \tag{12}$$

Then,

$$\xi(x) = \frac{1}{2}\kappa_3 x^{1+a} \tag{13}$$

where $\kappa_3 = \kappa_1 \kappa_2$.

By substituting (9) and (13) to (6),

$$f_6(x,z) = c \cdot x^{\alpha_4 - 1} \exp(-\beta_1 x) \cdot z^{\alpha_2 - 1} \exp(-\frac{2\beta_2}{\kappa_3 x^{1+a}} z) \tag{14}$$

where

$$c = (\frac{2}{\kappa_3})^{\alpha_2} \cdot \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \cdot \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}; \quad \alpha_4 = \alpha_1 - \alpha_2 (1+a)$$

It is observed from Eq. 14 that the conditional distribution of z for fixed x is the Gamma-type distribution with the same shape parameter as for η, i.e., α_2, and the scale parameter $2\beta_2/(\kappa_3 x^{1+a})$. Therefore, the shape of the conditional distribution of z is similar to that of η, regardless of x, and the range (and the standard deviation) gets wider for larger x. The conditional distribution of x for fixed z is not a Gamma-type, while the marginal distribution is originally assumed to be so.

The integration of Eq. 14 with respect to x deduces the marginal distribution of z.

$$\begin{aligned} f_4(z) &= \int_0^\infty f_6(x,z) \\ &= c \cdot z^{\alpha_2 - 1} \int_0^\infty x^{\alpha_4 - 1} \exp(-\beta_1 x - \frac{2\beta_2}{\kappa_3 x^{1+a}} \cdot z) dx \end{aligned} \tag{15}$$

We shall specify Eq. 15 by putting a=0 and α_2=1, which designate the case in which x and y are independent and y is exponentially distributed. After some mathematical manipulation,

$$f_4(z) = c' \cdot (\sqrt{\beta z})^{-\nu} K_\nu(\sqrt{\beta z}) \tag{16}$$

where K_ν is the ν-th order modified Bessel function of the second kind, i.e.,

$$K_\nu(s) = \frac{1}{2} \cdot (\frac{s}{2})^\nu \int_0^\infty t^{-\nu-1} \cdot \exp(-t - \frac{s^2}{4t}) dt \tag{17}$$

and

$$c' = 2^{\nu-1}\beta/\Gamma(1-\nu); \quad \beta = 8\beta_1\beta_2/\kappa_3; \quad \nu = -\alpha_4 = 1 - \alpha_1 \tag{18}$$

The assumptions a=0 and α_2=1, are not unrealistic approximations, since the correlation between duration and peak intensity is low, and an exponentially distributed peak intensity is reported in some literature. Then, the p.d.f. of the total amount of rainfall of a single storm may be reasonably approximated by Eq. 16. In view of the type of expression, we shall call the distribution "SQRT-K distribution." It has two parameters, i.e., shape parameter ν and scale parameter β.

Eagleson (2) derived the p.d.f. of the depth of a single rain storm, assuming that both duration and average rainfall intensity distribute exponentially. The p.d.f. is a special expression of Eq. 16 for ν=0 (α_1=1), and, thus, with one parameter. He reported that the fitness of his derived distribution to observed data was much better than that of the exponential distribution,

Table 1 Expressions of moments and correlation coefficients ($\delta=(1+\alpha_2)/\alpha_2,\ \alpha_2=\beta_2$)

	x	y	z
mean	α_1/β_1	$\kappa_1 \cdot \frac{\Gamma(a+\alpha_1)}{\beta_1^a\Gamma(\alpha_1)} \cdot \frac{\alpha_2}{\beta_2}$	$(\frac{1}{2}\kappa_3) \cdot \frac{\Gamma(1+a+\alpha_1)}{\beta_1^{1+a}\Gamma(\alpha_1)} \cdot \frac{\alpha_2}{\beta_2}$
the second moment about the origin	$(1+\alpha_1)\alpha_1/\beta_1^2$	$\kappa_1^2 \cdot \frac{\Gamma(2a+\alpha_1)}{\beta_1^{2a}\Gamma(\alpha_1)} \cdot \frac{(1+\alpha_2)\alpha_2}{\beta_2^2}$	$(\frac{1}{2}\kappa_3)^2 \cdot \frac{\Gamma\{2(1+a)+\alpha_1\}}{\beta_1^{2(1+a)}\Gamma(\alpha_1)} \cdot \frac{(1+\alpha_2)\alpha_2}{\beta_2^2}$
variance	α_1/β_1^2	$\kappa_1^2 \cdot \frac{\alpha_2}{\beta_1^{2a}\beta_2^2\Gamma^2(\alpha_1)}$ $\cdot \{\Gamma(\alpha_1)\Gamma(2a+\alpha_1)(1+\alpha_2)-\Gamma^2(a+\alpha_1)\cdot\alpha_2\}$	$(\frac{1}{2}\kappa_3)^2 \cdot \frac{\alpha_2}{\beta_1^{2(1+a)}\beta_2^2\Gamma^2(\alpha_1)}$ $\cdot [\Gamma(\alpha_1)\Gamma\{2(1+a)+\alpha_1\}(1+\alpha_2)-\Gamma^2(1+a+\alpha_1)\cdot\alpha_2]$
	x ∿ y	x ∿ z	y ∿ z
covariance	$\kappa_1 \cdot \frac{\alpha_2}{\beta_1^{1+a}\beta_2\Gamma(\alpha_1)} \cdot a\Gamma(a+\alpha_1)$	$\frac{1}{2}\kappa_3 \cdot \frac{\alpha_2}{\beta_1^{2+a}\beta_2\Gamma(\alpha_1)} \cdot (a+1)\Gamma(a+\alpha_1+1)$	$\kappa_1 \cdot (\frac{1}{2}\kappa_3)\frac{\alpha_2}{\beta_1^{1+2a}\beta_2^2\Gamma^2(\alpha_1)} \cdot \{\Gamma(\alpha_1)\Gamma(1+2a+\alpha_1)$ $\cdot(1+\alpha_2)-\Gamma(a+\alpha_1)\Gamma(1+a+\alpha_1)\alpha_2\}$
correlation coefficient	$\frac{a\Gamma(a+\alpha_1)}{\sqrt{\alpha_1}\sqrt{\Gamma(\alpha_1)\Gamma(2a+\alpha_1)\delta-\Gamma^2(a+\alpha_1)}}$	$\frac{(a+1)\Gamma(a+\alpha_1+1)}{\sqrt{\alpha_1}\sqrt{\Gamma(\alpha_1)\Gamma\{2(1+a)+\alpha_1\}\delta-\Gamma^2(1+a+\alpha_1)}}$	$\frac{\Gamma(\alpha_1)\Gamma(1+2a+\alpha_1)\delta-\Gamma(a+\alpha_1)\Gamma(1+a+\alpha_1)}{\sqrt{\Gamma(\alpha_1)\Gamma(2a+\alpha_1)\delta-\Gamma^2(a+\alpha_1)}}$ $\cdot\frac{1}{\sqrt{\Gamma(\alpha_1)\Gamma\{2(1+a)+\alpha_1\}\delta-\Gamma^2(1+a+\alpha_1)}}$

i.e. conventionally-used one-parameter distribution, and as good as that of the Gamma distribution, which has two parameters. Therefore, the SQRT-K distribution, which is a generalized expression of Eagleson's derived distribution, probably better fits the frequency distribution of the depth of a single storm.

Through a similar procedure we can derive the joint p.d.f. of x and y, and the marginal distribution of y. The joint p.d.f. of (y,z) and (x, y, z) are based on Eq. 7'. It is possible to derive the specific expressions, but somewhat complicated.

b) Moments and correlation coefficients

By substituting (9), (10), and (13) to (8),

$$\nu(r,s,t) = \int_0^\infty x^r \zeta^s(x) \xi^t(x) f_1(x) dx \cdot \int_0^\infty \eta^{s+t} f_2(\eta) d\eta$$

$$= \kappa_1^s \cdot (\frac{1}{2}\kappa_3)^t \cdot \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \cdot \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}$$

$$\cdot \int_0^\infty x^{r+sa+t(1+a)+\alpha_1-1} e^{-\beta_1 x} dx$$

$$\cdot \int_0^\infty \eta^{s+t+\alpha_2-1} e^{-\beta_2 \eta} d\eta$$

$$= \kappa_1^s \cdot (\frac{1}{2}\kappa_3)^t \cdot \frac{\Gamma\{r+sa+t(1+a)+\alpha_1\}}{\beta_1^{r+sa+t(1+a)}\Gamma(\alpha_1)} \cdot \frac{\Gamma(s+t+\alpha_2)}{\beta_2^{s+t}\Gamma(\alpha_2)} \qquad (19)$$

For example,

$$E(x) = \nu(1,0,0) = \alpha_1/\beta_1 \qquad (20)$$

$$E(x^2) = \nu(2,0,0) = (1+\alpha_1)\alpha_1/\beta_1^2 \qquad (21)$$

$$Var(x) = \nu(2,0,0) - \nu^2(1,0,0) = \alpha_1/\beta_1^2 \qquad (22)$$

$$Cov(x, z) = \nu(1,0,1) - \nu(1,0,0) \cdot \nu(0,0,1) \qquad (23)$$

$$\rho(x, z) = \frac{Cov(x, z)}{\sqrt{Var(x)}\sqrt{Var(z)}} \qquad (24)$$

where Var, Cov, and ρ represent the variance, the covariance and the correlation coefficient of the index variables.

Similarly, the specific expressions of all other joint moments are easily obtained by giving proper integers to r, s, and t in Eq. 19. Some of them are tabulated in Table 1. It should be noted that $\alpha_2=\beta_2$ since $E(\eta)=\alpha_2/\beta_2=1$.

MODEL VERIFICATION WITH OBSERVED DATA

a) Characteristics of observed rainfall

Hashino (3) investigated characteristics of hourly rainfall series at Osaka during the typhoon season, i.e., from June to October, in which severe rainfalls

Table 2 Means and correlation coefficients of observed rain storms

Month	Mean (mm)			Correlation coefficients			Sample size
	$\bar{x}$	$\bar{y}$	$\bar{z}$	x-y	x-z	y-z	
6	21.7	9.4	40.2	0.273	0.581	0.789	121 (1941-1970)
7	21.0	10.1	40.9	0.437	0.660	0.660	163 (1926-1965)
8	19.5	9.3	39.2	0.506	0.829	0.742	35 (1926-1965)
9	20.7	7.2	33.4	0.061	0.630	0.509	91 (1926-1965)
10	20.5	5.1	29.3	0.125	0.237	0.814	111 (1926-1965)

are expected. He defined a single rainfall as a rainy spell with preceding and following dry spells of 4 hours or more, and chose rainy spells of durations equal to or more than twelve hours. Some of the results are shown in Table 2 and Figs. 5 and 6. The averages of duration, peak, and depth, and their cross-correlation coefficients, are tabulated in Table 2. The scatter diagrams are shown in the figures.

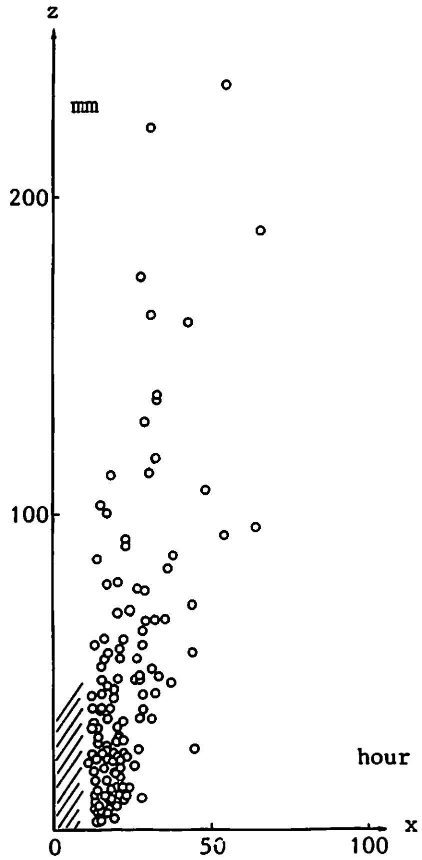

Fig. 5 An example of scatter diagram of duration x and total depth z (Osaka, July, 1900-1965, data for x≧12 hours are plotted. A rainy spell is defined with preceding and following 4-hour dry periods.)

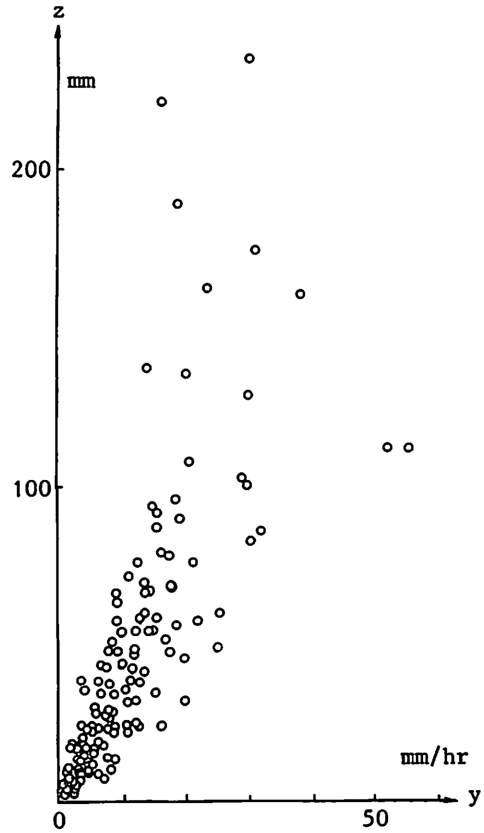

Fig. 6 An example of scatter diagram of peak intensity y and total depth z (for the same conditions as of Fig. 5)

The following characteristics are observed from Table 2.

(1) The sample correlation coefficient between duration and peak r(x, y) is small, i.e., 0 to 0.5, and the average is about 0.3,

(2) the average of r(x, z) is about 0.67 for June through September, while the correlation of October is singularly low, and

(3) r(y, z) ranges from 0.5 to 0.8, and the average is more than 0.7.

In Table 3, climate data of Denver and Boconó (Venezuela) are excerpted from a paper of Córdova and Rodríguez-Iturbe (1). This table uses average intensity i rather than peak intensity y. If we could assume triangular hyetographs, i=1/2y. Then, the probabilistic distribution of i would be approximately the same as that of y.

(1) The correlation coefficient between average intensity i and duration x is 0.043 to 0.12 for Denver, and 0.30 for Boconó.

(2) Shape indices of average intensity, duration, and total amount of Denver

and Boconó are respectively 0.74 to 0.84, 0.63 to 1.21, and 0.32 to 0.38.

Table 3 Storm parameters for climatic data of Denver and Boconó (after Córdova and Rodríguez-Iturbe, 1985)

Climatic variable	Denver						Boconó		
	May 15 to Aug. 15			May 15 to June 15			May 1 to Nov. 15		
	Mean	s.d.	Shape index	Mean	s.d.	Shape index	Mean	s.d.	Shape index
i	1.54	2.08	0.74	1.23	1.34	0.84	0.81	0.89	0.83
x	3.58	4.64	0.66	5.75	7.26	0.63	4.16	3.78	1.21
z	5.93	10.46	0.32	8.28	13.60	0.37	4.39	7.08	0.38
r(x, i)	0.04			0.12			0.30		
No. of storms	790			226			500		

b) Derived characteristics

The characteristic parameters of the proposed model are calculated by substituting proper values for the parameters in the derived expressions, and are compared with those calculated using observed data. Duration x may have an approximately exponential distribution, and the additional random variable η the distribution slightly skewed from the exponential distribution toward the right-hand side. Thus, we shall suppose that $\alpha_1 \doteqdot 1$ and $\alpha_2 \doteqdot 1$ to 2. Another parameter, a, is defined in the range from 0 to 1, and may take a value around 0.5. The parameters calculated based on the expressions in Table 1 are shown in Tables 4 and 5.

Table 4 Correlation coefficients (theoretical estimates)

	ρ(x, y)			ρ(x, z)			ρ(y, z)		
α_2 \ a	0	1/2	1	0	1/2	1	0	1/2	1
1	0	0.402	0.577	0.577	0.623	0.603	0.577	0.800	0.870
2	0	0.524	0.707	0.707	0.741	0.707	0.5	0.801	0.875
∞	0	0.957	1	1	0.969	0.894	0	0.862	0.894

Table 5 Shape indices (theoretical estimates) μ^2/σ^2

	x	y			z		
α_2 \ a		0	1/2	1	0	1/2	1
1	1	1	0.647	1/3	1/3	0.172	1/11
2	1	1	1.099	1/2	1/2	0.245	1/8
∞	1	1	3.663	1	1	0.418	1/5

Tables 4 and 5 show correlation coefficients and shape indices. The shape index is defined as the square of the inverse of the variation coefficient, μ^2/σ^2,

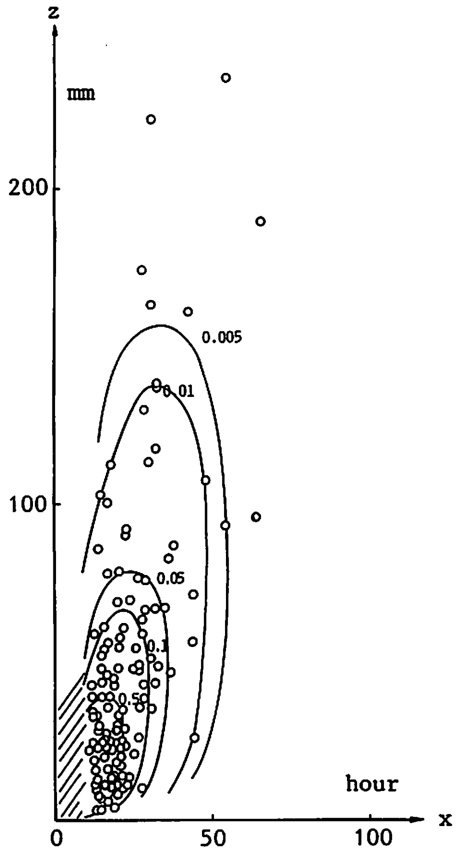

Fig. 7 An example of the comparison of the joint p.d.f. of x and z (Eq. 14) and the scatter diagram (Fig. 5) for $a=1/4$, $\alpha_1=1$, $\beta_1=1/10$, $\alpha_2=\beta_2=2$, $\kappa_3=1.6$, unit: $\times 10^{-3}$

since the shape parameter α of the Gamma distribution is equal to μ^2/σ^2, and, the relation between the value of α and the shape of the distribution is well known, i.e., for $0<\alpha<1$, $\alpha=1$, $\alpha>1$ and $\alpha \gg 1$, the distributions respectively correspond to the reversed-J, the exponential, the skewed bell-shaped, and the normal distributions.

It is observed from Tables 4 and 5 that,

(1) for $a=0$ to $1/2$ and $\alpha_2=1$ to 2, the population cross-correlation coefficients are $\rho(x, y)=0$ to 0.52, $\rho(x, z)=0.58$ to 0.74, and $\rho(y, z)=0.5$ to 0.8, which agree well with the sample statistics summarized in the previous section,
(2) even if x and y are independent ($a=0$), x and z or y and z are highly correlated,
(3) the distribution of y is approximately exponential or a little more skewed than that, since the shape index, μ^2/σ^2, is 0.67 to 1.1, and
(4) the distribution of z is much more skewed than the exponential distribution, since $\mu^2/\sigma^2=0.17$ to $0.5 \ll 1$, and for independently and exponentially distributed x and η ($a=0$, $\alpha_1=\alpha_2=1$), the shape index of z is equal to 1/3.

Theoretical estimates and observed data are compared in Table 6. From this comparison, it is concluded that the proposed model explains quite well the characteristics of observed rainfalls. An example of the comparison between the observed joint frequency distribution and the derived one is shown in Fig. 7.

Table 6 Comparison of theoretically derived and observed characteristics

	Shape indices			Correlation coefficients		
	x	y	z	x-y	x-z	y-z
Theoretically derived	1	0.65-1.10	0.17-0.50	0.0 -0.52	0.57-0.74	0.50-0.80
Observed	0.63-1.21	0.74-0.84	0.32-0.38	0.06-0.51	0.24-0.83	0.51-0.81

Values for Boconó and Denver are calculated with average intensity i, instead of peak intensity y.

FITTING PROCEDURE

The authors proposed a practical procedure to fit the derived joint distribution to observed data, and to estimate the parameters. It is desirable to estimate all parameters in the model at once by direct application of the maximum likehood method to the joint probability distribution of x, y, and z. However, since the expression of the joint distribution is complicated and the model has six independent parameters, α_1, β_1, $\alpha_2(=\beta_2)$, κ_1, κ_3, and a, it is quite difficult to stably estimate them by the direct maximum likelihood method. The alternative procedure is presented as follows:

(1) estimate α_1 and β_1, to maximize the logarithmic likelihood function ℓ_1 of the marginal distribution of x, i.e.,

$$\ell_1 = \log_e \prod_{i=1}^{n} f_1(x_i) = \sum_{i=1}^{n} \log_e f_1(x_i) \rightarrow \max \tag{25}$$

(2) then, estimate α_2, κ_1, κ_3 and a to maximize the following quasi-logarithmic likelihood function ℓ_2:

$$\ell_2 = \log_e \prod_{i=1}^{n} \{f_5(x_i, y_i)\cdot f_6(x_i, z_i)\}$$
$$= \sum_{i=1}^{n} \log_e \{f_5(x_i, y_i)\cdot f_6(x_i, z_i)\} \rightarrow \max \tag{26}$$

Several effective maximizing techniques and their standard computer programs have been published, such as Powell's method (7) which is utilized by the authors.

It is recommended to apply this technique to the data which are treated beforehand as follows:

(1) subtract a small threshold value, e.g., 1mm/hr, from the original data and set the negative data at zero, to eliminate the effect of negligibly small rainfall data, and,

(2) if hourly rainfall data is analyzed, subtract x_* from the duration of each rainfall, where $x_*=0.5$, since the rainy spells whose duration is seemingly n hours actually have a duration of (n-1) to n hours.

Rainfall of a single storm is practically defined as a rainy spell with preceding and following n dry periods, where n is somewhat arbitrary. The rainy spells are sampled from data treated in step (1).

The application of the above estimating procedure to data observed in Osaka so far have suggested that $a \fallingdotseq 0.5$, $\alpha_1 \fallingdotseq 1$, and $\alpha_2=1$ to 2.

ADDITIONAL COMMENTS

In the present paper, the expressions (10) and (11) are assumed for the structural Eqs. 1 and 3. Eq. 1 is a fundamental equation to relate the micro characteristics of the rainfall time sequence, observed in a short time interval, e.g., hourly rainfall sequence within a single storm, to the macro characteristics of rainfall of a single storm. Therefore, the specific expression of Eq. 1 should be determined based on the probabilistic model of rainfall within a single storm. This important process may adjust the relation between the probabilistic models of rainfall of and within a single storm. Taking account of this role of Eq. 1, the authors call Eq. 1 or Eqs. 1 and 3 the structural equations.

Employing expressions other than Eqs. 10 and 11 in 1 and 3 will change the subsequent equations. Even in this case, analysis can proceed along the analytical process presented here, and Eqs. 1 to 8 can be used for the fundamental equations.

CONCLUSIONS

A probabilistic model of rainfall of a single storm is presented. Each rainfall event is represented in terms of three characteristic variables, i.e., duration, maximum intensity, and total amount. A joint probability density function of the variables and analytical expressions of the moments are derived, including correlation coefficients among the variables. The validity of the proposed model is verified through its application to observed rainfall events. For example, the model gives correlation coefficients between (a) duration and maximum intensity, (b) duration and total amount, and (c) maximum intensity and total amount, are 0 to 0.5, 0.6 to 0.75, and 0.5 to 0.8, respectively.
The derived shape indices are 1, 0.65 to 1.1, and 0.17 to 0.5 for duration,

maximum intensity, and total amount, respectively. As summarized in Table 6, these theoretical estimates using the proposed model agree well with historical observations. Further, the distribution function of total rainfall of a single storm is theoretically derived, and is named "SQRT-K distribution," which is a generalized form of the Eagleson's derived distribution.

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APPENDIX - NOTATION

The following symbols are used in this paper:

- a = exponent of duration x to express the nonlinear regression between duration x and peak intensity y
- Cov = covariance of index variables
- E = expectation of index variable
- f = probability density function
- F = cumulative distribution function
- i = average intensity of a single storm
- ℓ = logarithmic likelihood
- r = sample correlation coefficient
- Var = variance of index variable
- x = duration of a single storm
- y = peak intensity of a single storm
- z = total depth of a single storm
- α = shape parameter of Gamma distribution
- β = scale parameter
- ζ = function to express regression of peak intensity y to duration x
- η = random component of peak intensity y for fixed duration x
- κ = constant
- μ = mean
- ν = shape parameter of SQRT-K distribution, joint moment of duration, peak intensity and total depth
- ξ = function to express regression of total depth z to duration x
- ρ = population correlation coefficient; and
- σ = standard deviation.