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APPLICATION OF WEIGHTED FINITE DIFFERENCE METHOD TO ONE- OR TWO-DIMENSIONAL CONVECTIVE DIFFUSION PROBLEM

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SYNOPSIS

Applications of weighted finite difference method(WFDM) to one- and two-dimensional convective diffusion problems are discussed. There are two methods of solving two-dimensional problems: one is by the use of two-dimensional WFDM and the other one is by the application of one-dimensional WFDM in an ADI scheme(4), which was introduced by Kanoh and Ueda(3), to two-dimensional convective diffusion. In WFDM, a value of the desired point is represented as the sum of the weighted values of the vicinity points, where the weights are obtained from the individual degree polynomial that satisfies the given convective diffusion equation. These methods can obtain higher order accuracy than other finite difference methods or finite element method for a particular transient convective diffusion problem if appropriate mesh size and time step are chosen.

INTRODUCTION

Numerical analysis of transient one- or two-dimensional convective diffusion equations is considered. Finite difference method, finite element method, boundary element method(6) and particle transfer method(2) are used for the analysis. Among these methods, the finite difference method(FDM) is generally used because it is easy to handle. In this method the analytical field is discretised into a mesh(or grid or lattice) and each point value is obtained by difference equations. In applying FDM to heat conduction equation, Watanabe(8) decided the coefficient of FDM using the individual degree polynomial that satisfies the governing equation. In this paper, we present the theoretical background of this method and set up the convergent polynomials that satisfy the transient convective diffusion equation. Furthermore, using these polynomials, the coefficients (weights) of FDM are defined, and FDM is subsequently applied to transient one- or two-dimensional convective diffusion problem. Finally we make comparison among WFDM, exact solution and other methods regarding the accuracy.

DESCRIPTION OF WFDM

To simplify the description of the WFDM, we deal with one-dimensional convective diffusion equation and present the procedure to decide the WFDM for this equation.

Approach to WFDM

If $f_1(X,T), f_2(X,T), \ldots$ denote the polynomials at position X, time T, that satisfy a linear partial differential equation, then Eq. 1 also satisfies this equation because of linearity

$$C(X,T) = a_1 f_1(X,T) + a_2 f_2(X,T) + \ldots \qquad (1)$$

where a_1, a_2, ... are coefficients to be decided from the boundary and initial conditions. Generally in FDM the functional value of the point ⓪ (let its coordinate be X_0, T_0) can be considered as the sum of weighted values of the vicinity points ①∿ⓝ (let their coordinates be $X_1, T_1 \sim X_n, T_n$), as shown in Fig. 1, i.e.,

$$C(X_0,T_0) = P_1 C(X_1,T_1) + P_2 C(X_2,T_2) + \ldots + P_n C(X_n,T_n) \qquad (2)$$

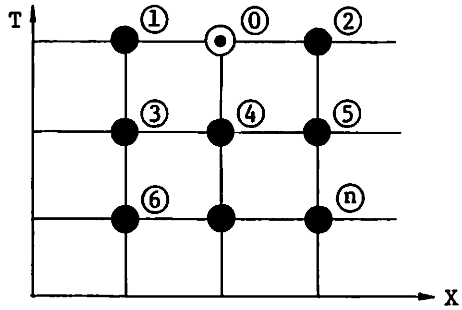

Fig. 1 Lattice points

where P_i is the weight of point i, and $P_1, P_2, \ldots P_n$ are different in various FDM. The method to decide these weights P_i is described below. Substituting Eq. 1 for each term of Eq. 2, we have

$$a_1 f_1(X_0,T_0) + a_2 f_2(X_0,T_0) + \ldots + a_n f_n(X_0,T_0) = P_1[a_1 f_1(X_1,T_1) + a_2 f_2(X_1,T_1) + \ldots + a_n f_n(X_1,T_1)] + P_2[a_1 f_1(X_2,T_2) + a_2 f_2(X_2,T_2) + \ldots + a_n f_n(X_2,T_2)] + \ldots + P_n[a_1 f_1(X_n,T_n) + a_2 f_2(X_n,T_n) + \ldots + a_n f_n(X_n,T_n)] \qquad (3)$$

Here $a_1, a_2, \ldots a_n$ change with the boundary and initial conditions as stated previously. So, in order that Eq. 3(i.e.,Eq. 2) can hold regardless of the values of a_1, a_2, $\ldots a_n$, the following expression must apply. Namely, when the right side of Eq. 3 is arranged with respect to $a_1, a_2, \ldots a_n$, then the corresponding terms of a_1, $a_2, \ldots, a_n$ must be equal, i.e.,

$$\left.\begin{aligned} f_1(X_0,T_0) &= P_1 f_1(X_1,T_1) + P_2 f_1(X_2,T_2) + \ldots + P_n f_1(X_n,T_n) \\ f_2(X_0,T_0) &= P_1 f_2(X_1,T_1) + P_2 f_2(X_2,T_2) + \ldots + P_n f_2(X_n,T_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n(X_0,T_0) &= P_1 f_n(X_1,T_1) + P_2 f_n(X_2,T_2) + \ldots + P_n f_n(X_n,T_n) \end{aligned}\right\} \qquad (4)$$

If the values of $P_1, P_2, \ldots P_n$ obtained by solving this set of linear equations are fed in Eq. 2, we can have the finite difference method which hold regardless of the values of $a_1, a_2, \ldots a_n$. The number n of these superimposed polynomials equals to that of the linear equations(Eq. 4), and also to the number of vicinity points used.

Determination of one-dimensional WFDM

The one-dimensional convective diffusion equation is given as

$$\frac{\partial C}{\partial T} = D\frac{\partial^2 C}{\partial X^2} - V\frac{\partial C}{\partial X} \tag{5}$$

where C is concentration, D is diffusion coefficient, V is velocity, X is position and T is time. Here D and V are constants. For the generalization of the phenomena, following scalar quantities which standardize C_0,V_0 and D_0 are used.

$$c = C/C_0,\quad v = V/V_0,\quad d = D/D_0,$$

$$x = X/X_0 = X/(D_0/V_0),\quad t = T/T_0 = T/(D_0/V_0{}^2) \tag{6}$$

Substituting these quantities into Eq. 5, we have

$$\frac{\partial c}{\partial t} = d\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x} \tag{7}$$

We now consider the numerical analysis of Eq. 7. The polynomials composed of x, t satisfying Eq. 7 are described by Eq. 8, where r in $c^{(r)}$ denotes the maximum degree of x in the polynomials.

$$c^{(0)}(x,t) = 1 \tag{8_0}$$

$$c^{(1)}(x,t) = x - vt \tag{8_1}$$

$$c^{(2)}(x,t) = (x - vt)^2/(2!) + dt \tag{8_2}$$

$$c^{(3)}(x,t) = (x - vt)^3/(3!) + (x - vt)dt \tag{8_3}$$

$$c^{(4)}(x,t) = (x - vt)^4/(4!) + (x - vt)^2/(2!)dt + (dt)^2/(2!) \tag{8_4}$$

. . . .

Generally

$$c^{(r)}(x,t) = \sum_{i=0}^{r/2} \left\{ \frac{(x - vt)^{r-2i}}{(r-2i)!} \cdot \frac{(dt)^i}{i!} \right\} \qquad \text{(r,i: positive integer)} \tag{8_r}$$

Here we assume that t^i equals 1 if t = i = 0. Accordingly the infinite progression(or polynomial progression) composed of superimposed polynomials in Eq. 8 is described as

$$c(x,t) = \sum_{r=1}^{\infty} [\, a_r \sum_{i=0}^{r/2} \{ \frac{(x - vt)^{r-2i}}{(r-2i)!} \cdot \frac{(dt)^i}{i!} \} \,] \tag{9}$$

which correspond to Eq. 1.
Eq. 9 converges if a_r, (x-vt) and dt are finite, because $\lim_{n\to\infty} (b^n/n!)$ equals 0 for an arbitrary b.

Since Eq. 7 is composed merely of differential terms with respect to x,t, it does not change its form even if the origin is moved to an arbitrary position. Therefore if the origin is moved to a desired point, we consider that the FDM is formulated mostly by very close points. Let $\Delta x=h$ and $\Delta t=k$ be the increments of the variables x and t, and x,t be discretised as x=ph, t=qk, where p,q=0,±1, ±2,... Hence p and q do not become large integers, because FDM is constituted

by the points very close to ◎ point as illustrated in Fig. 1.
As Eq. 7 is differentiated once and twice with respect to t and x respectively, we have

$$k = Rh^2 \qquad (R : \text{positive constant}) \tag{10}$$

hence $t = qk = qRh^2$.
Subsequently if we set

$$F = vk/h = vRh \text{ and } \mu = dk/h^2 = dR \tag{11}$$

then Eq. 8_r can be described as in Eq. 12. Here it is also assumed that q^i equals 1 if q=i=0.

$$c^{(r)}(ph,qk) = h^r \sum_{i=0}^{r/2} \left\{ \frac{(p - qF)^{r-2i}}{(r-2i)!} \cdot \frac{(q\mu)^i}{i!} \right\} \tag{12}$$

In case the vicinity points in the model(referred herein as center-scheme model) shown in Fig. 2 are adopted, the weights P_i are decided as given below. Let the concentration of the desired point(ih,jk) be c(i,j) and the weight of of c(i+p,j+q) be P_p^q , and hence the FDM is

$$c(i,j) = P_{-1}^{0}\, c(i-1,j) + P_1^0\, c(i+1,j) + P_{-1}^{-1}\, c(i-1,j-1) + P_0^{-1}\, c(i,j-1) + P_1^{-1}\, c(i+1,j-1) \tag{13}$$

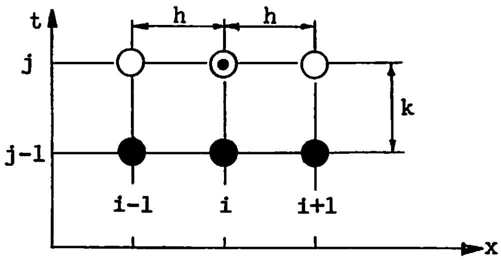

Fig. 2 Center-scheme
●:known point, ○:unknown point,
◉:disired point

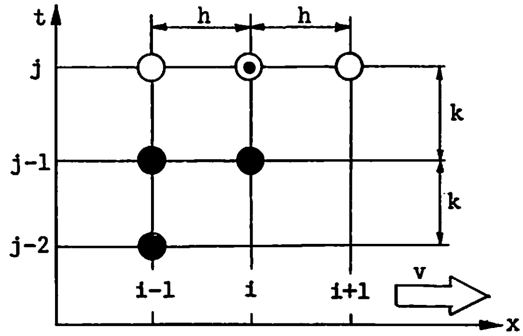

Fig. 3 Upwind-scheme

Next the origin is moved to the desired point as described earlier, i.e., i=j=0 in Eq. 13. And, if we substitute the c values obtained from Eq. 12 (i.e., in five instances of r=0,1,2,3,4) into Eq. 13, we obtain the following linear equations (Eq. 14). These c values correspond to the five vicinity points in Fig. 2. Solving Eq. 14 we obtain the weight of each point.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & F-1 & F & F+1 \\ 1 & 1 & (F-1)^2-2\mu & F^2-2\mu & (F+1)^2-2\mu \\ -1 & 1 & (F-1)^3 -6(F-1)\mu & F^3 -6F\mu & (F+1)^3 -6(F+1)\mu \\ 1 & 1 & (F-1)^4 -12(F-1)^2\mu +12\mu^2 & F^4 -12F^2\mu +12\mu^2 & (F+1)^4 -12(F+1)^2\mu +12\mu^2 \end{bmatrix} \cdot \begin{bmatrix} P_{-1}^{0} \\ P_1^0 \\ P_{-1}^{-1} \\ P_0^{-1} \\ P_1^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

Subsequently in case the vicinity points in the model(referred herein as upwind-scheme model) shown in Fig. 3 are adopted, the FDM is

$$c(i,j) = Q_{-1}^{0}\, c(i-1,j) + Q_{1}^{0}\, c(i+1,j) + Q_{-1}^{-1}\, c(i-1,j-1) + Q_{0}^{-1}\, c(i,j-1) + Q_{-1}^{-2}\, c(i-1,j-2) \quad (15)$$

Five weights $Q_{-1}^{0},\ldots,Q_{-1}^{-2}$ are decided from the following linear equations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & F-1 & F & 2F-1 \\ 1 & 1 & (F-1)^2-2\mu & F^2-2\mu & (2F-1)^2-4\mu \\ -1 & 1 & (F-1)^3-6(F-1)\mu & F^3-6F\mu & (2F-1)^3-12(2F-1)\mu \\ 1 & 1 & (F-1)^4-12(F-1)^2\mu+12\mu^2 & F^4-12F^2\mu+12\mu^2 & (2F-1)^4-24(2F-1)^2\mu+48\mu^2 \end{bmatrix} \cdot \begin{bmatrix} Q_{-1}^{0} \\ Q_{1}^{0} \\ Q_{-1}^{-1} \\ Q_{0}^{-1} \\ Q_{-1}^{-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (16)$$

It can be seen from Eqs.14 and 16 that each weight is a non-linear and complicated function of F and μ, and the accuracy of the numerical analyses is also a function of them. If the weights are obtained as mentioned above, the WFDM is therefore formulated. By these methods the concentration at each nodal point is calculated based on the boundary and initial conditions, similar to any other finite difference method.

Determination of two-dimensional WFDM

We formulate the two-dimensional WFDM similar to the one-dimensional WFDM discussed above. The governing equation, which is composed of scalar quantities, is given as

$$\frac{\partial c}{\partial t} = d_1 \frac{\partial^2 c}{\partial x^2} + d_2 \frac{\partial^2 c}{\partial y^2} - v_1 \frac{\partial c}{\partial x} - v_2 \frac{\partial c}{\partial y} \quad (17)$$

where d_1,d_2 and v_1,v_2 are diffusion coefficients and velocities in the x,y coordinate system respectively. The polynomials composed of x,y,t satisfying Eq. 17 are given as

$$c^{(r)}(x,y,t) = \sum_{i=0}^{r/2} \left\{ \frac{(x - v_1 t + y - v_2 t)^{r-2i}}{(r-2i)!} \cdot \frac{(d_1 t + d_2 t)^i}{i!} \right\} \quad (18)$$

The two-dimensional infinite polynomial progression composed of superimposed polynomials in Eq. 18 is described below.

$$c(x,y,t) = \sum_{r=1}^{\infty} \left[a_r \sum_{i=0}^{r/2} \left\{ \frac{(x - v_1 t + y - v_2 t)^{r-2i}}{(r-2i)!} \cdot \frac{(d_1 t + d_2 t)^i}{i!} \right\} \right] \quad (19)$$

Eq. 19 converges if $x-v_1t+y-v_2t$ and d_1t+d_2t are finite for the same reason given for Eq. 9. Here discretisation is introduced in the same manner as in the one-dimensional WFDM. Namely, when $\Delta x=h$, $\Delta y=Gh$ and $\Delta t=k$ are increments of x,y,t, we have

$$x = p_1h,\ y = p_2Gh,\ t = qRh^2 (p_1, p_2, q{:}0, \pm1, \pm2 \ldots; G{:}\text{positive constant}) \tag{20}$$

Subseqently if we set

$$F_x = v_1k/h = v_1Rh\ ,\ F_y = v_2k/Gh = v_2Rh/G$$
$$\mu_x = d_1k/h^2 = d_1R\ ,\ \mu_y = d_2k/(Gh)^2 = d_2R/G^2 \tag{21}$$

then Eq. 18 is described as follows.

$$c^{(r)}(p_1h, p_2Gh, qk) = h^r \sum_{i=0}^{r/2} \left[\frac{\{p_1 + Gp_2 - q(F_x + GF_y)\}^{r-2i}}{(r-2i)!} \cdot \frac{q(\mu_x + G^2\mu_y)^i}{i!} \right] \tag{22}$$

a) Cross center-scheme

In case the vicinity points in the model(referred herein as cross center-scheme model) shown in Fig. 4 are adopted, the FDM is

$$c(i,g,j) = P^{0}_{-1,0}c(i-1,g,j) + P^{0}_{1,0}c(i+1,g,j) + P^{-1}_{0,-1}c(i,g-1,j-1) + P^{-1}_{0,0}c(i,g,j-1) + P^{-1}_{0,1}c(i,g+1,j-1) \tag{23}$$

Where $P^{q}_{p,s}$ is the weight of $c(i+p,g+s,j+q)$.

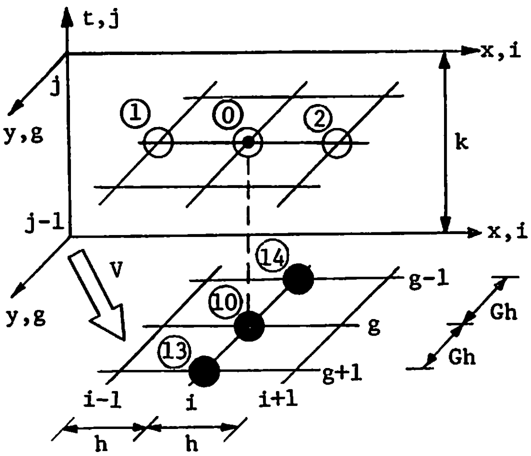

Fig. 4 Cross center-scheme

Next the origin is moved to the desired point, i.e., i=g=j=0 in Eq. 23. And if w substitute the c values obtained from Eq. 22(i.e., in five instances o r=0,1,2,3,4) into Eq. 23, we obtain the following linear equations(Eq. 24). Thes c values correspond to the five vicinity points in Fig. 4. Solving Eq. 24, w obtain the weight of each point.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & F_*-1 & F_* & F_*+1 \\ 1 & 1 & (F_*-1)^2-2\mu_* & F_*^2-2\mu_* & (F_*+1)^2-2\mu_* \\ -1 & 1 & (F_*-1)^3 -6(F_*-1)\mu_* & F_*^3 -6F_*\mu_* & (F_*+1)^3 -6(F_*+1)\mu_* \\ 1 & 1 & (F_*-1)^4 -12(F_*-1)^2\mu_* +12\mu_*^2 & F_*^4 -12F_*^2\mu_* +12\mu_*^2 & (F_*+1)^4 -12(F_*+1)^2\mu_* +12\mu_*^2 \end{bmatrix} \cdot \begin{bmatrix} P^0_{-1,0} \\ P^0_{1,0} \\ P^{-1}_{-1,0} \\ P^{-1}_{0,0} \\ P^{-1}_{1,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (24)$$

Where $F_* = F_x + GF_y$, $\mu_* = \mu_x + G^2\mu_y$.

b) Angle upwind-scheme

In case the vicinity points in the model(referred herein as angle upwind-scheme model(a)) shown in Fig. 5 are adopted, the FDM is

$$c(i,g,j) = Q^0_{-1,0}c(i-1,g,j) + Q^0_{0,-1}c(i,g-1,j) + Q^{-1}_{-1,0}c(i-1,g,j-1) + Q^{-1}_{0,0}c(i,g,j-1) + Q^{-1}_{1,0}c(i+1,g,j-1) \qquad (25)$$

Five weights $Q^0_{-1,0},\ldots,Q^{-1}_{1,0}$ are decided from the following linear equations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -G & F_*-1 & F_* & F_*+1 \\ 1 & G^2 & (F_*-1)^2-2\mu_* & F_*^2-2\mu_* & (F_*+1)^2-2\mu_* \\ -1 & -G^3 & (F_*-1)^3 -6(F_*-1)\mu_* & F_*^3 -6F_*\mu_* & (F_*+1)^3 -6(F_*+1)\mu_* \\ 1 & G^4 & (F_*-1)^4 -12(F_*-1)^2\mu_* +12\mu_*^2 & F_*^4 -12F_*^2\mu_* +12\mu_*^2 & (F_*+1)^4 -12(F_*+1)^2\mu_* +12\mu_*^2 \end{bmatrix} \cdot \begin{bmatrix} Q^0_{-1,0} \\ Q^0_{0,-1} \\ Q^{-1}_{-1,0} \\ Q^{-1}_{0,0} \\ Q^{-1}_{1,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (26)$$

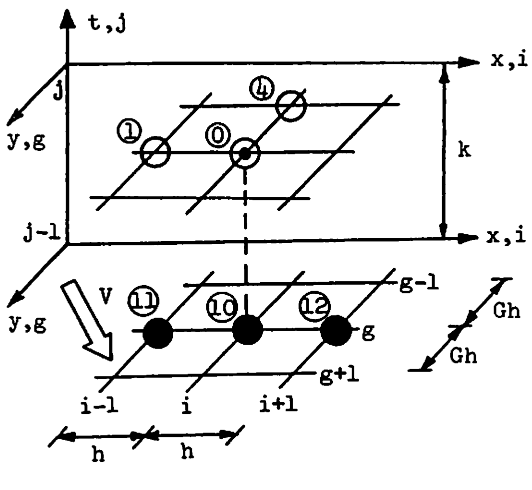

Fig. 5 Angle upwind-scheme(a)

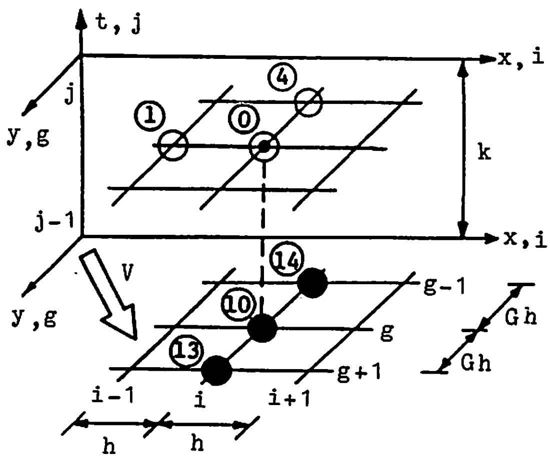

Fig. 6 Angle upwind-scheme(b)

Subsequently in case the vicinity points in the model(referred herein as angle upwind-scheme model(b)) shown in Fig. 6 are adopted, the FDM is

$$c(i,g,j) = R^{0}_{-1,0}\ c(i-1,g,j) + R^{0}_{0,-1}\ c(i,g-1,j) + R^{-1}_{0,-1}\ c(i,g-1,j-1) + R^{-1}_{0,0}\ c(i,g,j-1) + R^{-1}_{0,1}\ c(i,g+1,j-1) \quad (27)$$

Five weights $R^{0}_{-1,0},\dots,R^{-1}_{0,1}$ are decided from the following linear equations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -G & F_*-G & F_* & F_*+G \\ 1 & G^2 & (F_*-G)^2-2\mu_* & F_*^2-2\mu_* & (F_*+G)^2-2\mu_* \\ -1 & -G^3 & \begin{matrix}(F_*-G)^3\\ -6(F_*-G)\mu_*\end{matrix} & \begin{matrix}F_*^3\\ -6F_*\mu_*\end{matrix} & \begin{matrix}(F_*+G)^3\\ -6(F_*+G)\mu_*\end{matrix} \\ 1 & G^4 & \begin{matrix}(F_*-G)^4\\ -12(F_*-G)^2\mu_*\\ +12\mu_*^2\end{matrix} & \begin{matrix}F_*^4\\ -12F_*^2\mu_*\\ +12\mu_*^2\end{matrix} & \begin{matrix}(F_*+G)^4\\ -12(F_*+G)^2\mu_*\\ +12\mu_*^2\end{matrix} \end{bmatrix} \cdot \begin{bmatrix} R^{0}_{-1,0} \\ R^{0}_{0,-1} \\ R^{-1}_{0,-1} \\ R^{-1}_{0,0} \\ R^{-1}_{0,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (28)$$

In a two-dimensional angle upwind-scheme, the c values at time($t+\Delta t$) are obtained by the model given by Eq. 25(described in Fig. 5), at time($t+2\Delta t$) by the model given by Eq. 27(depicted in Fig. 6), at time($t+3\Delta t$) again by Eq. 25, at time($t+4\Delta t$) again by Eq. 27, and so on. Hence these two models are used alternately so that the two-dimensional upwind WFDM can satisfy all the boundary conditions.

From Eqs. 24 , 26 and 28, each weight is a non-linear function of F_* and μ_*, i. e., a function of F_x, F_y,μ_x,μ_y and $G(=\Delta y/\Delta x)$. And the accuracy of two-dimensional numerical analysis is a non-linear and complicated function of these terms.

If the weights are obtained from Eqs. 24, 26 and 28, the two-dimensional WFDM is therefore formulated.

The problem where the velocities change in space(or further in time) especially arouse interest in two-dimensional convective diffusion problem.

To apply WFDM to this problem, it is necessary that linear equations be solved and weights be defined on each mesh point. This additional work is easily performed by electronic computers, since the number of the linear equations is only five.

COMPARISON AMONG EXACT SOLUTION, WFDM AND OTHER METHODS

One-dimensional analysis

(1) Exact solution

When the following initial and boundary conditions are considered,

$$c(x,t=0) = \exp(-x/\sqrt{d})\ ,\ c(x=0,t) = \exp[(1+v/\sqrt{d})t]\ ,\ c(x=\infty,t) = 0 \quad (29)$$

an exact solution of Eq. 7 is given as

$$c(x,t) = \exp[-x/\sqrt{d} + (1 + v/\sqrt{d})t] \quad (30)$$

(2) Numerical solutions

a) WFDM solution

The one-dimensional domain is discretised by x=ih,t=jk. We set the various quantities as : h=0.2, R=1/4, $k=Rh^2=0.01$, i,j= 0 ∿ 50. We obtain the initial condition by setting t=0 and the boundary conditions by setting x=0(at one end of the domain) and x=50h(at the other end of the domain) in Eq. 30. The accuracy is estimated by the relative error E(= (|exact solution - solution)|/ exact solution). Two cases are investigated. First in the case of F=vRh=0.31 and μ=dR=0.05 with v=6.2 and d=0.2, the exact and WFDM solutions are shown in Table 1. And second in the case of F=0 ∿ 1 and μ =0.01 ∿ 1, the WFDM solutions are shown in Figs. 7 and 8. The shaded portions on these figures are the regions where E_{max} is less than h^4(=0.0016). Here E_{max} is the maximum among the relative errors obtained from one combination of F and μ values. In this method the maximum degree of polynomials is four, so the truncation error can be regarded as almost in the order of h^5. Then we may consider that the relative error becomes approximately less than h^4.

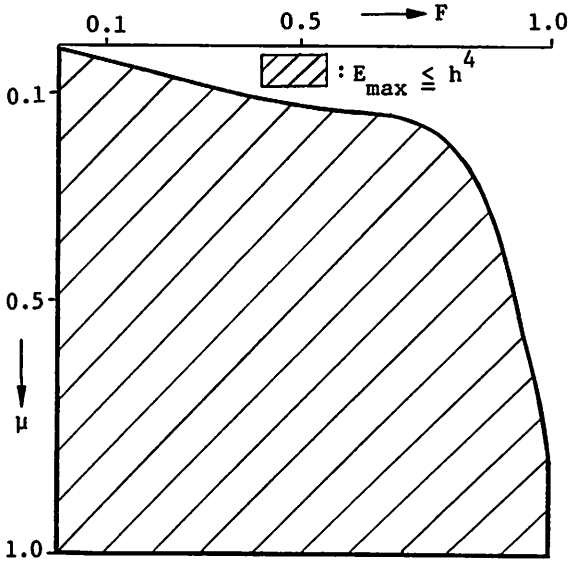

Fig. 7 1-dim. WFDM(Center-scheme)

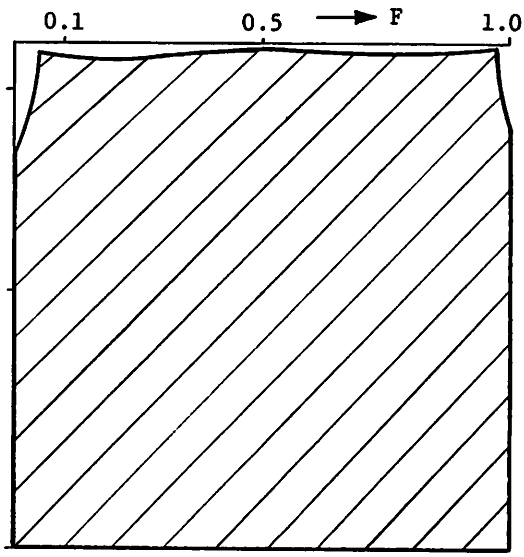

Fig. 8 1-dim. WFDM(Upwind-scheme)

b) Solution by Crank-Nicolson method(5)

This method, which is shown by the same scheme presented in Fig. 2, is described below.

$$\begin{aligned} c(i,j) = {} & [(2\mu+F)/4(1+\mu)]c(i-1,j) + [(2\mu-F)/4(1+\mu)]c(i+1,j) \\ & + [(2\mu+F)/4(1+\mu)]c(i-1,j-1) + [(1-\mu)/(1+\mu)]c(i,j-1) \\ & + [(2\mu-F)/4(1+\mu)]c(i+1,j-1) \end{aligned} \qquad (31)$$

The calculated result by this method is shown in Table. 1; the shaded portion in Fig. 9 is the region where E_{max} is less than h^4.

c) Solution by D.A.Bella method(1)

In this method the convection term is expressed by the explicit backward space difference of first order and the diffusion term by implicit center space difference of second order. The pseudo-dispersion is corrected by using the numerical dispersion coefficient D_n, where D_n is converted to a dimensionless quantity. Hence c(i,j) can be described as,

$$\begin{aligned} c(i,j) = {} & [1/(1+2\mu-2D_n)][(\mu-D_n)c(i-1,j) + (\mu-D_n)c(i+1,j) \\ & + F\cdot c(i-1,j-1) + (1-F)c(i,j-1)] \end{aligned} \qquad (32)$$

The calculated result by this method is shown in Table. 1, and there is no region where E_{max} is less than h^4.

d) FEM solution by implicit scheme

The one-dimensional domain is divided into 50 linear elements. Then the application of the Galerkin method(7) to Eq. 7 and the replacement of the time differencial term with backward finite difference result in

$$\mathbf{A}\,\mathbf{C}^j = \mathbf{B}\,\mathbf{C}^{j-1} + \mathbf{D} \qquad (33)$$

where $\mathbf{C}^j$ is the vector of c^j(=c of time j) and **A**, **B** and **D** are the coefficient matrices composed of F, μ and the boundary conditions. The calculated result by this method is shown in Table. 1, and the shaded portion in Fig. 10 is the region where E_{max} is less than h^4.

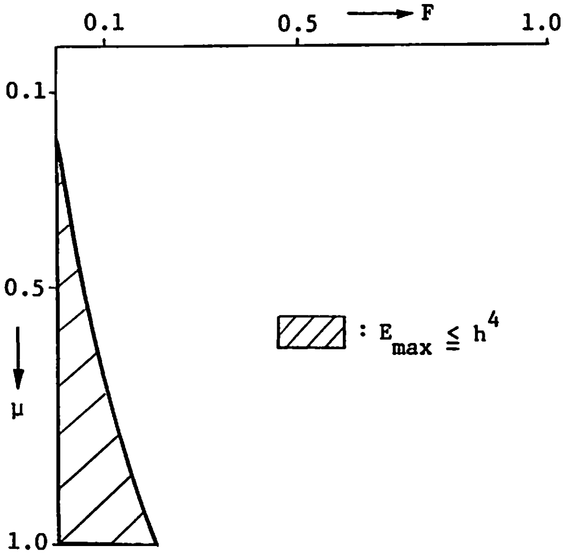

Fig. 9 Crank-Nicolson method

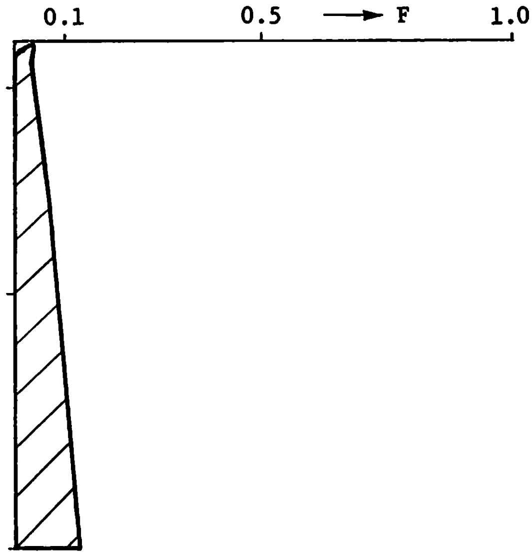

Fig. 10 FEM

Table.1 Solutions and Errors of Exact,WFDM,FDM and FEM solution
(F=0.31,μ=0.05,h=0.2,k=0.01) ,E:relative error. (for ex. 5.2···E-4= $5.2\cdots\times10^{-4}$)

time \ space	solution	Exact	WFDM analysis	WFDM E	Bella's FDM analysis	Bella's FDM E	Crank-Nicolson analysis	Crank-Nicolson E	FEM analysis	FEM E
10k	20h	5.2226E-4	5.2207E-4	3.7E-4	5.6824E-4	0.09	5.3025E-4	0.01	5.2801E-4	0.01
	40h	6.8145E-8	6.8120E-8	3.7E-4	7.4145E-8	0.09	6.9189E-8	0.02	6.8896E-8	0.01
20k	20h	9.0268E-4	9.0225E-4	4.7E-4	1.0787E-3	0.20	9.3211E-4	0.04	9.2520E-4	0.03
	40h	1.1778E-7	1.1773E-7	4.7E-4	1.4075E-7	0.20	1.2162E-7	0.03	1.2072E-7	0.03
40k	20h	2.6967E-3	2.6958E-3	3.4E-4	3.8873E-3	0.44	2.8803E-3	0.07	2.8407E-3	0.05
	40h	3.5188E-7	3.5176E-7	3.4E-4	5.0723E-7	0.44	3.7583E-7	0.07	3.7066E-7	0.05

Two-dimensional analysis

(1) Exact solution

When the following initial and boundary conditions are considered

$$c(x,y,t=0) = \exp(-x/\sqrt{d_1} - y/\sqrt{d_2}) ,$$

$$c(x=0,y,t) = \exp[-y/\sqrt{d_2} + (2 + v_1/\sqrt{d_1} + v_2/\sqrt{d_2})t] \qquad (34)$$

$$c(x,y=0,t) = \exp[-x/\sqrt{d_1} + (2 + v_1/\sqrt{d_1} + v_2/\sqrt{d_2})t]$$

$$c(x=\infty,y,t) = c(x,y=\infty,t) = 0$$

an exact solution of Eq. 17 is given as

$$c(x,y,t) = \exp[-x/\sqrt{d_1} - y/\sqrt{d_2} + (2 + v_1/\sqrt{d_1} + v_2/\sqrt{d_2})t] \qquad (35)$$

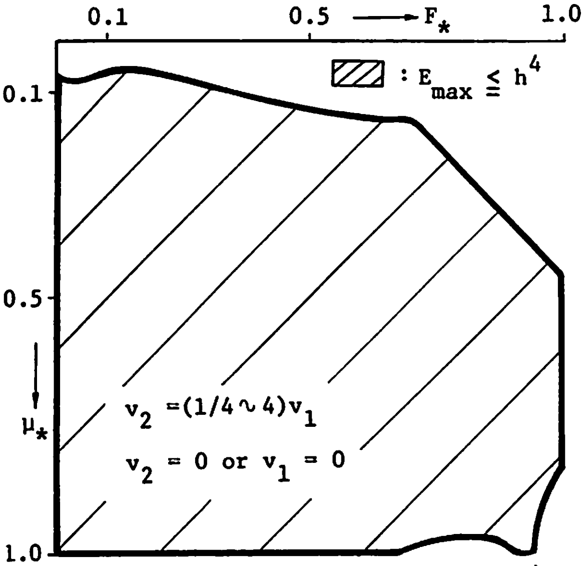

Fig. 11 2-dim.WFDM(Cross center-scheme)

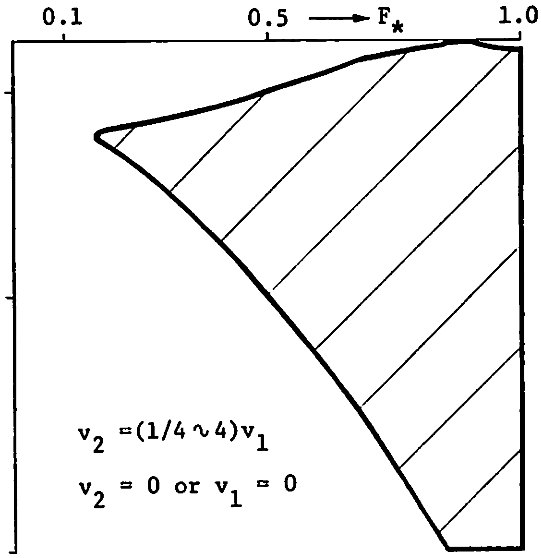

Fig. 12 2-dim.WFDM(Angle upwind-scheme)

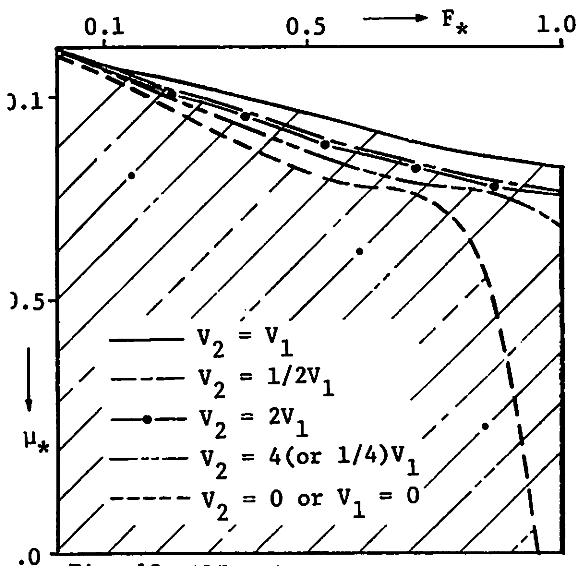

Fig. 13 ADI solution (by 1-dim. Center-scheme)

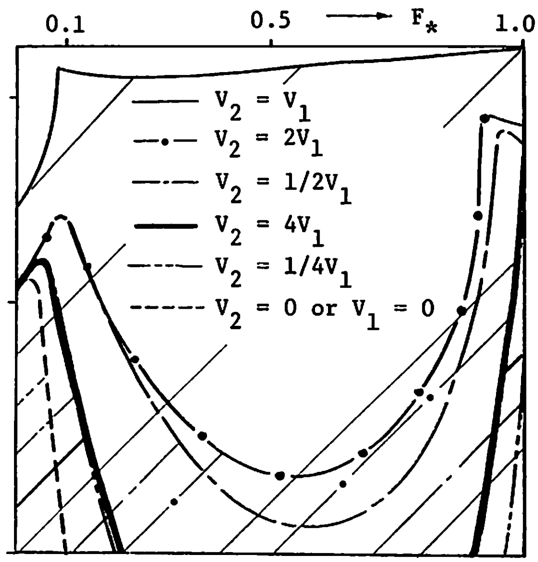

Fig. 14 ADI solution (by 1-dim.Upwind-scheme)

(2) Numerical solution

We discretise the two-dimensional domain by $x=ih$, $y=gGh$ and $t=jk$ and set the various quantities as: $h=0.2$, $G=1$ or $3/4$, $R=1/4$, $k=Rh^2=0.01$, $d_1=d_2$, $v_2=(1/4\sim4)\cdot$

v_1, i=g=0 ∿ 8,j=0 ∿ 50. We obtain the initial condition by setting t=0 and the boundary conditions by setting x=0 and x=8h and y=0 and y=8Gh at the horizontal and vertical ends of the domain respectively in Eq. 35.

a) Two-dimensional WFDM solution

In the case of F_*=0 ∿ 1,μ_*=0.01 ∿ 1, the two-dimensional WFDM solutions are shown in Figs. 11 and 12, where the shaded portion is the region in which E_{max} is less than h^4. We also tried other models using two-dimensional WFDM. However the resulting shaded portions are smaller than the ones shown in Figs. 11 and 12.

b) ADI solution(8) applying one-dimensional WFDM

In this method, the two-dimensional convective diffusion equation(Eq. 17) is analyzed by applying twice the one-dimensional WFDM(Eq. 13 or 15), first on the x-direction and then on the y-direction.
The ADI solutions are shown in Figs. 13 and 14 for the case of F_*=0∿1, μ_*=0.01 ∿ 1. The shaded portions on these figures are the regions where E_{max} is less than h^4.

CONCLUSION

Numerical analysis of a particular transient one- and two-dimensional convective diffusion problems is discussed. In this WFDM, the analytical field is discretised into a mesh and a value of the desired point is represented as the sum of the weighted values of the vicinity points. The weights are obtained from the individual degree polynomial that satisfies the governing equation.
And its application to two-dimensional problem through the use of the ADI method is discussed in this paper. The two-dimensional WFDM is also presented.

The following conclusions are obtained:

1. This method can obtain accuracy approximately two order higher than other methods can get, particularly on one-dimensional convective diffusion problem, if appropriate mesh size and time step are chosen.
2. WFDM has advantage over D.A.Bella FDM method, Crank-Nicolson method and FEM in the range of accuracy. Especially the center-scheme model and upwind-scheme model of the one-dimensional WFDM and the cross center-scheme model and the angle upwind-scheme model of the two-dimensional WFDM are performed well.
3. Each weight of the one-dimensional WFDM is a non-linear function of F and μ and that of the two-dimensional WFDM is a non-linear function of F_x , F_y ,μ_x ,μ_y (Eq. 21) and G(=$\Delta y/\Delta x$). In other words, each weight of WFDM can correspond to changes in velocity, diffusion coefficient, mesh size and time increment which are related to F, μ and G. This considers the reason why WFDM can give high accuracy.

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APPENDIX - NOTATION

The following symbols are used in this paper:

A,B,D	= coefficient matrices composed of F, μ and the boundary conditions in FEM;
c	= dimensionless concentration;
C	= concentration;
C^j	= vector of c^j(= c at time j);
d	= dimensionless diffusion coefficient;
d_1, d_2	= dimensionless diffusion coefficients along x and y;
D	= diffusion coefficient;
D_n	= dimensionless form of numerical dispersion coefficient;
E	= relative error
E_{max}	= the maximum among the relative errors obtained from one combination of F and μ values;
$F=v\Delta t/\Delta x$	= dimensionless form of velocity(or Courant number)
$F_x=v_1\Delta t/\Delta x$	= dimensionless form of velocity along x;
$F_y=v_2\Delta t/\Delta y$	= dimensionless form of velocity along y;
$F_*=F_x+GF_y$	= two-dimensional Courant number;
$G=\Delta y/\Delta x$	= ratio of Δy to Δx;
$h=\Delta x$	= mesh size along x;
$k=\Delta t$	= time increment;
P_p^q	= weight of the point at position $(i+p)\Delta x$, time $(j+q)\Delta t$;
$Q_{p,s}^q$	= weights of the point at position $(i+p)\Delta x$ and $(g+s)\Delta y$, time $(j+q)\cdot\Delta t$;
R	= positive constant;
t	= time;
v	= dimensionless velocity;
v_1, v_2	= dimensionless velocities along x and y;
V	= velocity;
$\Delta x, \Delta y, \Delta t$	= increments of x,y and t;
$\mu=d\Delta t/\Delta x^2$	= diffusion number;
$\mu_x=d_1\Delta t/\Delta x^2$	= diffusion number along x;
$\mu_y=d_2\Delta t/\Delta y^2$	= diffusion number along y; and
$\mu_*=\mu_x+G^2\mu_y$	= two-dimensional diffusion number.