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A NOTE ON INTERFACIAL STABILITY OF TWO-LAYERED FLOW

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SYNOPSIS

A theoretical study is presented of the effect of viscosity and tangential stress on the hydrodynamic stability of the interface between two horizontal layers of slightly different densities which are in relative motion. The solution is obtained approximately assuming uniform but different velocities in each fluid layer and using the shearing stress formula at the interface found for steady, fully developed laminar flow. Viscosity and tangential stress effects are found to be so small that they do not materially affect the stability criterion for Reynolds numbers greater than 20 if the wave number is taken, as experimentally observed, as one-half of the reciprocal of the thickness of the moving layer.

INTRODUCTION

Study of the stability of an interface between two horizontal layers of fluids of slightly different densities which are in relative motion is important for better understanding of the mechanism of mixing of stratified flows, a phenomenon which is often encountered in water resources engineering.

Non-viscous-flow stability problems of two layered flow with different, but uniform velocity and density profiles saw the famous solution by Helmholtz (5). Viscous effect on the stability of such a flow configuration first received attention in connection with study of air-water interface stability. Thus Jeffreys (3) was the first to study the viscous effect on the stability of such a flow. Tchen (8) derived an approximate solution for the full viscous stability of two-layered flow with different fluid properties moving with uniform but different velocities. Although numerous works have been published on viscous and/or non-viscous stability problems of two-layered flow with realistic velocity and density profiles, this paper is mainly concerned with a viscous effect on the stability of two-layered flow with different, but uniform velocity and density profiles.

Density difference between two layers in this study is chosen to be statically stable and small, representing many problems related to hydraulic engineering. An experimental or semi-empirical study of such a problem is the work of Ippen and Harleman (2), who observed that a uniform flow of salt water under a body of fresh water produced breaking waves when the non-viscous instability criterion for the discontinuous but otherwise uniform density and velocity profiles was satisfied as follows:

$$\Delta U > \sqrt{2(\Delta\rho/\rho)(g/k)} \qquad (1)$$

in which $\Delta U = U' - U$ is the velocity difference between moving layers, the denser fluid being on the bottom; g is the gravitational acceleration; and k is the wave number of interfacial waves. They also found that the observed waves satisfied the relation kh = 2 in which h is the thickness of the moving layer. Eq. 1 could therefore be reduced to

$$F_i = \Delta U/\sqrt{(\Delta\rho/\rho)gh} > 1 \qquad (2)$$

in which F_i is the densimetric Froude number of the lower and moving layer. Criterion (Eq. 1) is strictly valid for an infinite thickness of the fluid layers. However, Ippen and Harleman found that finiteness of the thickness did not affect the solution for $F_i < 1$. The experiments in Ref. (2) covered Reynolds numbers from 100 to 400. [1]The Reynolds number in this case was defined as $R_e = \Delta Uh/\nu$.

Keulegan (4) attempted to take into account the effects of viscosity and rotationality semi-empirically. He arrived at a parameter $\Theta = (\nu g\Delta\rho/\rho)^{1/3}/\Delta U$ which, in combination with the Reynolds number, was used to define a measure of the stability of the interface. The critical values of Θ were found experimentally to be

$$0.127 \text{ for } R_e < 450; \text{ and } 0.178 \text{ for } R_e > 450 \qquad (3)$$

It seems possible then to extend analyses of Jeffreys and Tchen to the case of small density difference and apply it to the experimental results by Ippen and Harleman, and Keulegan. It is true that the velocity profiles assumed in this work, i.e. discontinuous otherwise uniform, may be unrealistic and one can argue that viscous stability of two-layered flow with continuous velocity profile is calculable in this age, e.g. Hayakawa and Unny (1) and Nishida et al. (7). Nevertherless, discontinuous velocity profile is adopted in this work in order to obtain a simple and comprehensive answer to the two-layered flow problem when viscosity effect is small. It is hoped that the assumed velocity profile is a good approximation to the initial growth stage of the boundary layer type flow and the validity of the solution will be tested with reference to the experimental data.

EIGENVALUE EQUATION FOR INTERFACIAL STABILITY

Following the usual step of linear stability analysis, one obtains a simplified form of the Orr-Sommerfeld equation with respect to the complex amplitude $\phi(y)$ of the stream function of perturbed flow:

$$(U - c)(d^2\phi/d^2y - k^2\phi) = (\nu/ik)(d^4\phi/dy^4 - 2k^2 d^2\phi/dy^2 + k^4\phi) \qquad (4)$$

where U is the unperturbed flow velocity in x direction, c is the wave celerity of the perturbed motion. The stream function ψ is assumed to take a normal solution described as follows:

$$\psi = \phi(y)\exp ik(x - ct) \qquad (5)$$

Eqs. 4 and 5 both hold for upper and lower fluid motion. In relating upper layer fluid motion hereafter, primed quantities will be used.

Boundary conditions at infinity are given as

$$\phi' \to 0 \text{ as } y \to \infty \text{ ; } \phi \to 0 \text{ as } y \to -\infty \qquad (6)$$

The kinematic condition at the interface, $y = \eta$, is, to the first order

$$\partial\eta/\partial t + U'\partial\eta/\partial x = \partial\psi'/\partial x \text{ at } y = 0 \qquad (7.1)$$

$$\partial\eta/\partial t + U\,\partial\eta/\partial x = \partial\psi/\partial x \text{ at } y = 0 \qquad (7.2)$$

A solution of Eq. 4 satisfying the boundary conditions 6 and 7 is given, for the upper fluid

$$\phi' = B' e^{-ky} + C' e^{-\lambda' y}; \; B' + C' = (U' - c)a \qquad (8.1)$$

where $\nu'\lambda'^2 = \nu'k^2 + ik(U' - c)$ and a is amplitude of interfacial displacement, and for the lower fluid

$$\phi = B e^{ky} + C e^{\lambda y}; \; B + C = (U - c)a \qquad (8.2)$$

where $\nu\lambda^2 = \nu k^2 + ik(U - c)$. Branches of the wave lengths k and λ are taken so that their real parts are positive.

As the dynamic condition at the interface, both normal and tangential stresses at the interface are continuous and are assumed to fluctuate according to the relationships

$$P_{yy} = -P + \rho g\eta + 2\mu(\partial v/\partial y) = \rho a\sigma \exp ik(x - ct) \qquad (9)$$

and

$$\tau_{xy} = \mu\{(\partial u/\partial y) + (\partial v/\partial x)\} = \rho a\, ik\beta \exp ik(x - ct) \qquad (10)$$

where σ is the maximum amplitude of the slope of the shear stress at the interface.

The perturbed piezometric head can be shown to satisfy the Laplace equation and the pressure fluctuation is solved for the upper layer

$$P'/(\rho' g) = -(k/g)(U' - c)e^{-ky} \exp ik(x - ct) \quad \text{for } y > 0 \qquad (11.1)$$

and for the lower layer

$$P/(\rho g) = (k/g)(U - c)e^{ky} \exp ik(x - ct) \quad \text{for } y < 0 \qquad (11.2)$$

For the lower fluid, the substitution of Eqs. 11.1 and 8.2 into Eqs. 9 and 10 leads to

$$a\sigma = ga - k(U - c)B + 2\nu\, ik(B + C) \qquad (12)$$

$$a\, ik\beta = -\nu\{k^2B + \lambda^2C + k^2(B + C)\} \qquad (13)$$

Eliminating B and C from Eqs. 12 and 13, the following expression is obtained for the lower fluid:

$$\sigma = g - 2\nu\lambda k^2[\beta i/\{(U - c)k\} + 2\nu] - k\{\beta + (U - c - 2ik\nu)^2\} + 2i\nu k^2\beta/(U - c) \qquad (14)$$

The same procedure is applied to the upper layer and results in the following expression:

$$\sigma = (\rho'/\rho)g + 2\nu'\lambda'k^2[\beta i/\{(U - c)k\} + 2\nu] + k\{\beta + (\rho'/\rho)(U - c - 2ik\nu')^2\} - 2i\nu'k^2\beta/(U - c) \qquad (15)$$

To simplify the equations the following assumption, which was also utilized by Jeffreys(3), is made:

$$\nu k << (U - c);\ \beta << (U - c)^2 \qquad (16)$$

This implies that the wave damping due to viscous dissipation and tangential stress is small. Thus Eqs. 14 and 15 are reduced to

$$\sigma = g - k\{\beta + (U - c - 2i\nu k)^2\} \qquad (17)$$

and

$$\sigma = (\rho'/\rho)g - k\{\beta + (\rho'/\rho)(U' - c - 2i\nu'k)^2\} \qquad (18)$$

Equating 17 and 18, one obtains

$$\rho(U - c - 2ik\nu)^2 + \rho'(U' - c - 2ik\nu')^2 - \Delta\rho g/k + 2\rho\beta = 0 \qquad (19)$$

where $\Delta\rho = \rho - \rho'$. This is the eigenvalue equation of the problem, containing one unknown parameter β which can be found using the semi-empirical steady rectilinear flow resistance law equation at the interface.

INTERFACIAL STABILITY OF A SURFACE CURRENT

If the lower layer is stationary($U = 0$) and the density difference between the two layers is small, it is justifiable to assume that

$$\rho'/\rho \cong 1; \quad \nu' = \nu \tag{20}$$

Eq. 19 will then be reduced to

$$(c + 2ik\nu)^2 + (U' - c - 2ik\nu)^2 - \omega^2 + 2\beta = 0 \tag{21}$$

where $\omega^2 = \Delta\rho g/\rho k$.

The shearing stress at the interface is assumed to take the form

$$\overline{\tau}_{xy} = \frac{1}{2}\rho' c_\tau \overline{U}'^2 \tag{22}$$

where $\overline{U}'$ is the interfacial velocity and c_τ is a shear stress coefficient.

$$\overline{U}' = U' - (\partial\psi'/\partial y) \; ; \qquad \overline{\tau}_{xy} = \frac{1}{2}\rho' c_\tau U'^2 + \tau_{xy} \tag{23}$$

The fluctuating component of shear, τ_{xy}, can be expressed as

$$\tau_{xy} = ik\rho\beta a = -\rho' c_\tau U'(\partial\psi'/\partial y) \tag{24}$$

when second order terms are neglected. Substituting Eq. 7.1 and eliminating B' and C', Eq. 24 can be transformed into

$$\tau_{xy} = \rho c_\tau U'(U' - c)k[(\lambda' - k)/(\lambda' + k) - (\rho/\rho')i\beta/\{\nu'(\lambda' + k)(U' - c)\}]a \tag{25}$$

The assumption previously invoked in Eq. 16 and the fact that the shear stress coefficient c_τ in fully developed flow is proportional to the reciprocal of the Reynolds number (laminar flow) permit replacing the term in brackets by unity, and hence

$$\tau_{xy} = ik\beta a \cong \rho' c_\tau U'(U' - c)ka \tag{26}$$

or

$$\beta = -i(\rho'/\rho)c_\tau U'(U' - c) \tag{27}$$

Substitution of Eq. 27 into Eq. 21 leads to the following expression:

$$2c^2 - 2(U' - 4ik\nu - ic_\tau U')c + U'^2 - \omega^2 - 8k^2\nu^2 - 4ik\nu U' - 2ic_\tau U'^2 = 0 \tag{28}$$

This equation is then solved for c:

$$c = \frac{1}{2}(U' - 4ik\nu - ic_\tau U') \pm [\frac{1}{2}\omega^2 - \frac{1}{4}U'^2 - \frac{1}{4}c_\tau^2 U'^2 - 2k\nu c_\tau U' + \frac{i}{2}c_\tau U'^2]^{1/2} \tag{29}$$

Separation of the imaginary part of c leads to the critical stability condition

$$2(\Delta\rho/\rho)(g/k)/U'^2 = 1 - 16k^2\nu^2/U'^2 + \frac{1}{4} c_\tau^2 U'^2/(2k\nu + c_\tau U'/2)^2 \quad (30)$$

The second and third terms of the right-hand side of this condition are the effects of viscous dissipation and tangential stress respectively. It can be seen that the effect of viscous dissipation is stabilizing, while that of tangential stress is destabilizing. If the viscosity and the interfacial shear are zero, Eq. 30 reduces itself to Eq. 1, as is expected.

DISCUSSION OF THE STABILITY CRITERION

For a surface current $\Delta U = U'$. Eq. 30 can be rewritten in terms of the densimetric Froude number, the Reynolds number and the dimensionless wave number $\alpha = kh'$, for the upper layer of thickness h'.

The shearing stress formula for steady, fully developed laminar flow along a solid straight wall or an interface is

$$c_\tau = s/R_e \quad (31)$$

in which s is a constant. The critical stability condition (Eq. 30) then becomes

$$F_i^{-2} = (\alpha/2)[1 - 16\alpha^2/R_e^2 + (s^2/4)/(2\alpha + s/2)^2] \quad (32)$$

It should be noted here that the stability of the interface depends only on the relative motion of the two layers and Eq. 32, though derived for the surface current, is expected to hold for the subsurface current as well.

The value of s depends on the viscosities, the densities, and the thicknesses of the two layers. According to an analysis given by Lock(6), a theoretical value of s = 3 is applied to fully developed flow in two fluid layers of approximately the same density and viscosity occupying the upper and lower half-space respectively. If both layers are of limited depth, however, the s-value for a surface current can range between zero and six depending on the relative thicknesses of the layers and the circulation patterns set up in both layers. The first value refers to an interface not opposing the flow at all, while the higher value represents the resistance found along a solid wall. If the moving upper layer is thin as compared to the lower layer, the interfacial shear will generally be small, and vice versa.

Eq. 32 has been plotted in Fig. 1 with F_i on the abscissa, α on the ordinate, and R_e as a parameter, for s = 5, corresponding to a high shear, and s = 0. The critical condition for nonviscous or frictionless flow (Eq. 1) is a particular solution of Eq. 32 and is obtained by setting s equal to zero and Reynolds number equal to infinity.

The value of the dimensionless wave number α in Eq. 32 cannot be chosen arbitrarily, but must be a specific value found when the waves begin to break. In Ref.(2), a value of $\alpha = 2$ is given as a result of laboratory experiments on bottom density currents. This value also agrees fairly well with

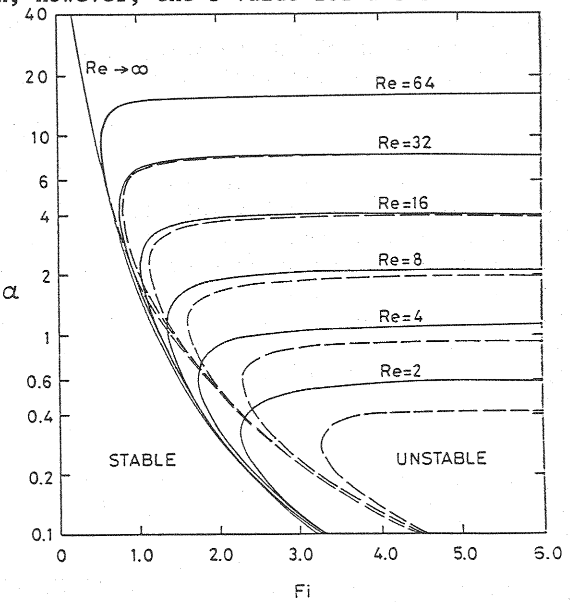


Fig. 1 Critical stability condition in terms of dimensionless wave number α and dimensionless Froude number F_i. Solid lines refer to an interfacial shear coefficient s = 5; broken lines refer to s = 0.

a limited number of observations on surface density currents. It may therefore be used to reduce Eq. 32 to

$$F_i^{-2} = 1 - 64/R_e^2 + (s^2/4)/(4 + s/2)^2 \qquad (33)$$

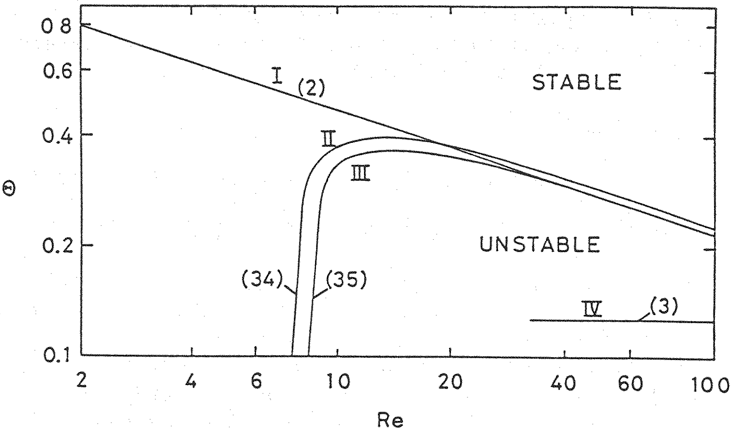
Expressed in terms of Keulegan's Θ values, Eq. 33 for high interfacial shear, or s = 5, is

$$\Theta = F_i^{-2}R_e^{-1/3} = R_e^{-1/3}(1.15 - 64/R_e^2) \qquad (34)$$

For zero interfacial shear, or s = 0, it is

$$\Theta = R_e^{-1/3}(1.0 - 64/R_e^2) \qquad (35)$$

Eqs. 34 and 35 have been plotted in Fig. 2 as curves II and III respectively. The non-viscous flow condition (Eq. 2), which becomes $\Theta = 1/R_e$, and Keulegan's empirical conditions have been added as curve I and IV, respectively.



Eq. 35 indicates an absolute stable region for Reynolds numbers less than about seven which is due primarily to viscous dissipation. A slightly destabilizing effect of viscosity can be noted for Reynolds numbers larger than 20 for high interfacial shear (curve II). The difference between curves II and III is fairly small, indicating that the interfacial shear is of little importance. At Reynolds numbers above 20, viscosity and interfacial stress effects are also small and affect the stability criterion very little.

CONCLUSION

The effects of viscosity and tangential stress on the interfacial stability of superposed fluids in laminar flow with discontinuous density and viscosity profiles have been demonstrated in an approximate but comprehensive way. Viscosity and tangential stress effects are so small that they do not materially affect the stability criterion for Reynolds numbers greater than 20. There exists an absolutely stable region, where any disturbance is damped out, for Reynolds number less than about seven. At first approximation the tangential stress seems to have only a slightly destabilizing effect, if any.

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APPENDIX - NOTATION

The following symbols are used in this paper:

a = internal wave height;

c = complex wave velocity;

c_τ = shear stress coefficient;

F_i = densimetric Froude number;

g = gravitational acceleration;

h = thickness of lower layer;

i = unit of imaginary number;

k = wave number ($= 2\pi/\lambda$);

p = perturbation pressure;

$\underline{P}_{yy}$, τ_{xy} = perturbation normal and tangential stress, respectively;

τ_{xy} = resultant tangential stress;

R_e = Reynolds number;

s = coefficient in laminar shear stress law;

t = time;

u, v = components of perturbed velocity in x- and y-direction, respectively;

U = basic flow velocity;

$\overline{U}$ = resultant flow velocity = $U + u$;

x,y = space coordinates;

α = dimensionless wave number;

η = interfacial elevation;

Θ = Keulegan number = $1/(F_i^2 R_e)^{1/3}$;

λ = wave length;

ν,μ = kinematic viscosity and dynamic viscosity, respectively;

ρ = density of lower fluid;

$\Delta\rho$ = density difference = $\rho - \rho'$;

σ,β = normal and shear stress coefficients, respectively;

ϕ = amplitude of stream function;

ψ = stream function;

Ω = $(g/k)^{1/2}$;

ω = $(\Delta\rho g/\rho k)^{1/2}$;

' = added to variable if referring to upper layer(unless defined otherwise.