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THE PROBABILITY DENSITY FUNCTION OF AREAL AVERAGE RAINFALL

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SYNOPSIS

Expression for the probabilistic structure has been derived for the spatial scales of storm rainfall. It is assumed that the rainfall over a certain region is distributed statistically on all small regions as the area elements of the total region. It is then theoretically clarified that probabilistic structure of the areal rainfall over the small averaging area is governed by the exponential type, but gradually transformed into Gamma-distribution and then Normal distribution with increasing averaging area. The structure has been examined for the actual rainfall event in Nagoya City, Japan and the characteristics of the parameters in the probabilistic structure have been also discussed.

INTRODUCTION

The rainfall is one of the fundamental elements in runoff process and its temporal and spatial distributions affect greatly the resultant discharge out of the watershed. The temporal and spatial characteristics of rainfall depend on those of the meteorological disturbance of atmosphere. From the view point of runoff analyses and prediction, therefore, the rainfall characteristics in time and space should be considered in connection with the watershed scales.

Although many efforts have been made to investigate the rainfall characteristics, most of all is dedicated to Depth-Duration Analyses and the spatial characteristics have been rarely discussed except by Ishihara et al (3) and Tomosugi (4). While the empirical research works by Horton (2) and Fletcher (1) have been widely accepted, but there are still remaining problems on the applicability of their works to small and complicated watersheds as in Japan. In the present paper the authors intend to investigate the spatial distributions of rainfall intensity, especially focussing their attention to its probabilistic structures in connection with the spatial scales, in other words, to find out if a specified probabilistic structures exist with respect to the scale of rainfall phenomena, and if so, what kind of relationship should be hold among the parameters involved in the probabilistic structure.

MODEL WATERSHED AND RAINFALL DATA

As the first step of research, to avoid the difficulty due to complicated topological effects, Nagoya City region is here selected as the model area, since

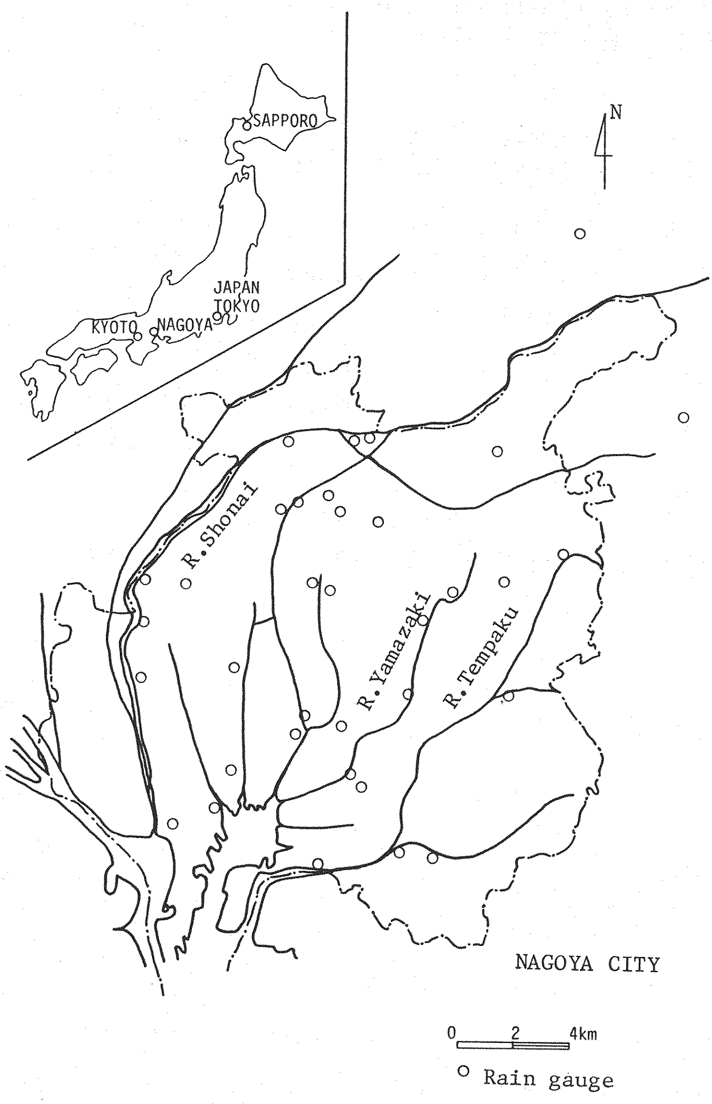

Fig. 1 Site of rain gauge stations

the region is located at the Nohbi-Plain and the effect of local topology on the rainfall characteristics may be ignored.

Within the region rainfall intensity is observed by Nagoya City, the Meteorological Agency and each Primary and Junior High Schools. Among these gauging stations, 35 sites as shown in Fig. 1 are selected as the useful data sources, at which the storm rainfalls have been measured hourly during a last decade. In relation to the runoff phenomena, much more short sampling time as 10, 15 or 20 minutes is required in the urban region, but since we have lack of the data for such short periods, the hourly records are analyzed. The rainfall occasions treated are those which record the maximum intensity of hourly rainfall in each years at the Nagoya Meteorological Agency.

RAINFALL PROBABILITY IN A CELL REGION

In this section at first we will discuss the rainfall averaged over a small cell region. Consider the volumetric rainfall R_T falling over the total region G in a certain unit time as shown in Fig. 2. Dividing this total region G into N cell regions with equal area a, among these N cell regions write n_i for the number of such cell regions in which we have the volumetric rainfall ${}_1R_i$ in a specified unit time. Then we have the following relationships.

$$R_T = \bar{r}A = \Sigma\, {}_1R_i \cdot n_i \qquad (1)$$
$$N = \Sigma\, n_i$$

in which $\bar{r}$ ($=R_T/A$) is areal average rainfall over the total region. In this Eq. 1 the subscript $_1$ indicates the quantity (rainfall) in one cell region.

The number of way to distribute the total volume of rainfall R_T over the total region to N cell regions in such a manner as ${}_1R_1$ for n_1-cell regions, ${}_1R_2$ for n_2-cell regions and so forth is given by

$$P = \frac{N!}{n_1!\, n_2!\, n_3! \;\ldots.} \qquad (2)$$

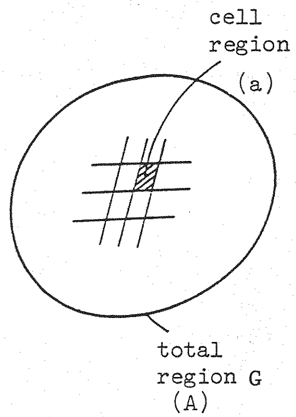

Fig. 2 Conceptual figure of total region G and cell region

Provided each way has equal apriori probability, the probability of such the event that the volumetric rainfall in a cell region ${}_1R_1$, ${}_1R_2$, ... occur in n_1, n_2, ... cell regions, respectively, should be proportional to the value P. Then the distribution maximizing P is most likely to appear in the actual situation. Consequently we obtain the following probability ${}_1p$ which maximizes P under the condition 1, that is, through the so-called maximum entropy principle.

$${}_1p\,({}_1R_i) = \frac{n_i}{N} = \frac{\exp(-\alpha \cdot {}_1R_i)}{f} \qquad (3)$$

$$f = \Sigma \exp(-\alpha \cdot {}_1R_i) \qquad (4)$$

in which α is a constant parameter to be decided by the condition 1. The detailed discussions on the characteristics of this parameter will be made in the later section.

RAINFALL PROBABILITY AND THE SPATIAL SCALES

Based on the probability for a cell region introduced in the previous section, we will discuss the probabilistic structures of rainfall in relation to the spatial scales, that is, the probability of rainfall event over the larger regions with area 2a, 3a, ...

As mentioned in the first section, the spatial scales of rainfall are closely related with those of the atmosphere, but we treat the problems from the probabilistic view point, assuming the independence of rainfall in the individual cell regions. Although this assumption may be seemed as unrealistic, it has been ascertained in advance of the analyses that the rainfall data at the rain-gauge stations concerned here have little of correlation with each other.

In Eq. 3 the rainfall value is treated as discrete value, but in the following discussions in this section we write it as the continuous quantity.

Now let's consider the rainfall ${}_2R$ over the region of area 2a, combining two adjacent cell regions. This event corresponds to the phenomena that rainfall ${}_1R$ occurs over one cell region, ${}_2R-{}_1R$ over the other cell region and totally $[{}_1R+({}_2R-{}_1R)]={}_2R$ falls over the total region 2a. Therefore, the probability ${}_2p$ for this event is given by the following expression with help of the probability $_1p$ introduced in the previous section.

$$_2p(_2R) = \int_0^{_2R} {_1p}(_1R) \cdot {_1p}(_2R - {_1R})\, d_1R$$

$$= \left(\frac{1}{f}\right)^2 \cdot {_2R} \exp(-\alpha \cdot {_2R}) \qquad (5)$$

In case of rainfall $_3R$ over the region with area 3a, combining three adjacent cell regions, the rainfall probability is derived as follows.

$$_3p(_3R) = \int^{_3R} \int^{_2R} {_1p}(_1R) \cdot {_1p}(_2R - {_1R}) \cdot {_1p}(_3R - {_2R}) \cdot d_1R\, d_2R$$

$$= \frac{1}{2}\left(\frac{1}{f}\right)^3 (_3R)^2 \exp(-\alpha \cdot {_3R}) \qquad (6)$$

Similarly, the probability for such the case that the rainfall $_nR$ falls on n-adjacent cell regions with area na is written in the form of

$$_np(_nR) = \frac{1}{(n-1)!} \left(\frac{1}{f}\right)^n (_nR)^{n-1} \exp(-\alpha \cdot {_nR}) \qquad (7)$$

Further, the following relationship should be satisfied from Eq. 3.

$$\int_0^{\infty} \frac{1}{f} \exp(-\alpha \cdot {_1R})\, d_1R = \frac{1}{\alpha f} = 1 \quad ; \quad \alpha = \frac{1}{f} \qquad (8)$$

then Eq. 7 is expressed as

$$_np(_nR) = \frac{1}{(n-1)!} \alpha^n {_nR}^{n-1} \exp(-\alpha \cdot {_nR}) \qquad (9)$$

As well known, this is Gamma-distribution with two parameters n and α so called shape and scale parameters respectively. Equation 9 introduced here with respect to volumetric rainfall $_nR$ in a unit time can be rewritten into the probability density function in terms of areal average rainfall $_nr$ over the region na as follows

$$_np(_nr) = \frac{\beta^n (_nr)^{n-1}}{(n-1)!} \exp(-\alpha \cdot {_nr}) \qquad (10)$$

in which

$$\beta = na\alpha \quad ; \quad {_nr} = {_nR} / na \qquad (11)$$

This is Gamma-distribution which governs the probabilistic structures of areal average rainfall in connection with the spatial scale na.

SPATIAL DISTRIBUTION OF RAINFALL IN NAGOYA CITY

Figure 3 shows some examples of histograms of areal rainfall during one hour with different averaging area for two rainfall occasions in 1973 and 1977. In these figures, areal rainfall is the averaged mean of point hourly rainfall within the circle zone of radius L km with center at each rain-gauge station.

The curves drawn in the figure correspond to the Gamma-distribution fitted

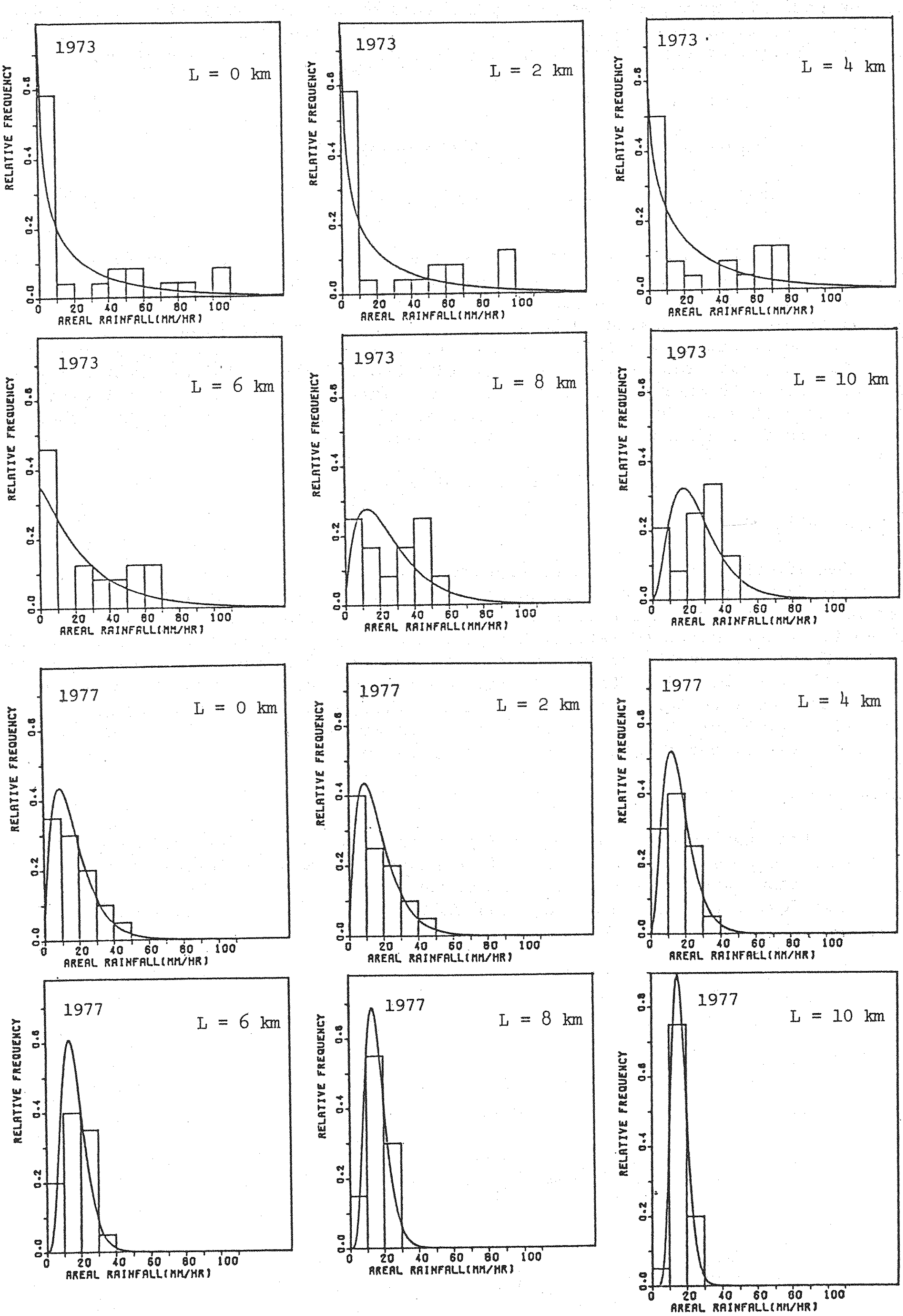

Fig. 3 Areal rainfall distribution

to the each histogram, that is, the parameters of Gamma-distribution n, α(or β) are estimated by means of the moment method. As well known, the expectation E(r) and variance Var(r) of Gamma-distribution (10) becomes as follows respectively

$$E(r) = n/\beta \quad ; \quad Var(r) = n/\beta^2 \tag{12}$$

On the other hand, those μ and σ^2 for the sampled data are

$$\mu = \frac{1}{m} \Sigma r \quad ; \quad \sigma^2 = \frac{1}{m-1} \Sigma (r - \bar{r})^2 \tag{13}$$

in which m is the number of data. Hence, the parameters n and α (or β) are estimated by the given μ and σ^2 for each averaging scale L.

In Fig. 3 we can see how the distribution of areal rainfall changes with averaging area, i.e. spatial scale. For the small averaging area (small L), in the figure, the probability density function is governed by the exponential type of distribution, but gradually transformed into Gamma-distribution and then Normal-distribution with increasing averaging area. This process observed in the probability according to the spatial scale may be explained very well by the probabilistic structure discussed in the previous sections.

CHARACTERISTICS OF PARAMETERS

Parameter β

The effect of spatial scales appears in the distribution of rainfall Eq. 10 through the parameters n and β. According to Eq. 11, β is proportional to n. The parameter α involved in the coefficient of the proportional relationship should be essentially determined to satisfy the condition 1, depending on the total rainfall volume R_T and the number of cell regions N, and is a constant value for a certain rainfall occasion. Since the cell region a is constant, the value $a\alpha$ should be also a constant for a certain rainfall occasion. This relationship between β and n are shown in Fig. 4 for the each rainfall occasion analyzed. From the figure, we can see the clear linear relationship between β and n.

Parameter $a\alpha$

The inclination of each line in Fig. 4 is $a\alpha$ and has different value in each rainfall. In this sub-section, we will discuss the characteristics of $a\alpha$, especially in relation to average rainfall intensity $\bar{r}$.

From Eqs. 1 and 3, the average rainfall intensity over a cell region is given by

$$a\bar{r} = \frac{\Sigma\, ar_i \exp(-a\alpha r_i)}{\Sigma \exp(-a\alpha r_i)} \tag{14}$$

$$f(\alpha) = \Sigma \exp(-a\alpha r_i) \tag{15}$$

Differentiating f(α) with respect to α, we obtain

$$\frac{\partial f(\alpha)}{\partial \alpha} = -\Sigma\, ar_i \exp(-a\alpha r_i) \tag{16}$$

and then

$$a\bar{r} = -\frac{\partial f(\alpha)}{\partial \alpha} \Big/ f(\alpha) = -\frac{\partial}{\partial \alpha} [\ \ln f(\alpha)\ ] \tag{17}$$

In the observation, we measure the rainfall with some unit measure. Write the unit as r_o, then the rainfall intensity is to have only the discrete values

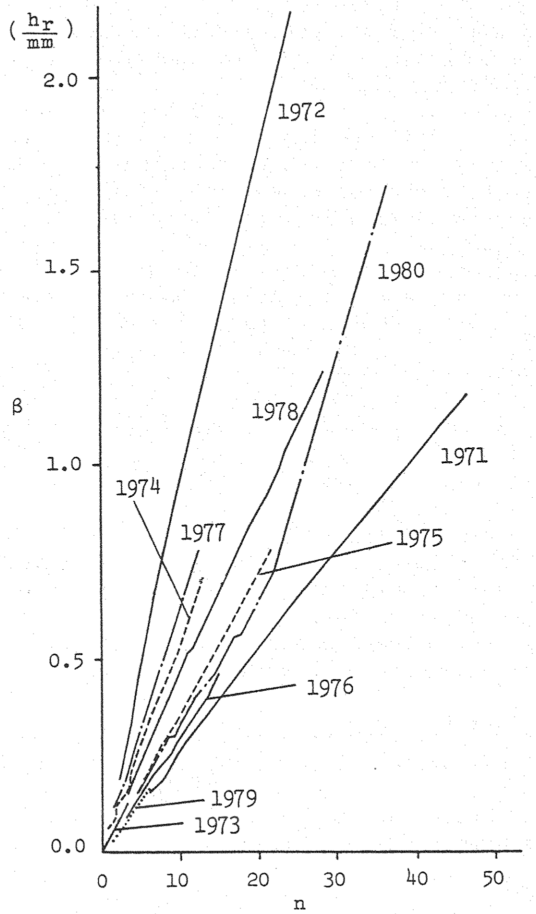

Fig. 4 Relationships between β and n

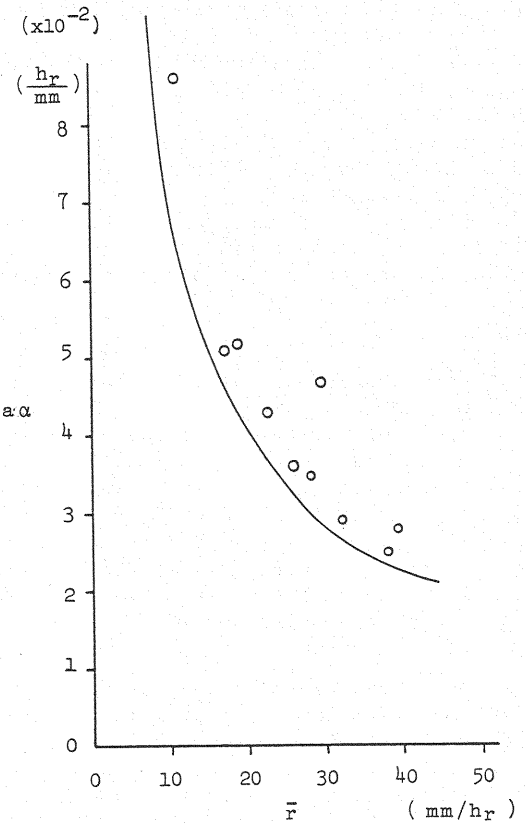

Fig. 5 Relationship between $\bar{r}$ and aα

$$0,\ r_o,\ 2r_o,\ 3r_o,\ \ldots\ldots\ldots \tag{18}$$

Using the unit r_o, the function $f(\alpha)$ is expressed as

$$f(\alpha) = 1 + e^{-a\alpha r_o} + e^{-2a\alpha r_o} + e^{-3a\alpha r_o} + \ldots\ldots\ldots$$

$$= \frac{1}{1 - e^{-a\alpha r_o}} \tag{19}$$

Applying this to Eq. 17, the following relations are derived.

$$a\bar{r} = \frac{ar_o \exp(-a\alpha r_o)}{1 - \exp(-a\alpha r_o)} \tag{20}$$

$$a\alpha = \frac{1}{r_o} \ln\left(1 + \frac{r_o}{\bar{r}}\right) \tag{21}$$

This is the expression which defines the parameter aα in terms of the unit measure

r_o and the areal average rainfall $\bar{r}$ over the total region G.

The theoretical relationship 21 is drawn in Fig. 5 together with the actual data analyzed for the value r_o=10 mm/hr. It may be concluded from the figure that the parameter $a\alpha$ observed is accurately expressed by Eq. 21.

The characteristics of the specified region are involved in the distribution implicitly as the form of a, which relates the spatial scales with the actual area. The cell region a itself should be identified by the observed data and may vary due to the meteorological situations such as front, typhoon and so forth. But it is interesting that the value $a\alpha$ can be obtained theoretically for all the rainfall types in Nagoya City.

What we would especially take to emphasize now is the fact that the probabilistic structures of spatial scale of rainfall are according to the Gamma-distribution 10 and the actual parameters $a\alpha$, β can be estimated by Eqs. 11 and 21.

CONCLUSION

The conclusions obtained through the present paper may be summarized as follows.

i) The probability of rainfall over equally divided small cell region is given by the exponential distribution.

ii) Probability density function of areal average rainfall over combined cell regions is governed by Gamma-distribution (10) affected by the spatial scale.

iii) The parameters involved in Eq. 10 are evaluated in terms of the unit measure r_o and the averaging rainfall $\bar{r}$ over the total region.

iv) The probabilistic structures derived in the present paper explain very well the spatial distributions of the actual phenomena (storm rainfall) in Nagoya City in Japan.

As is described in the introduction, the spatial distribution of rainfall is very important for the evaluation of runoff discharge out of watersheds. Although the problems should be further discussed on the smallest scale of the cell region etc, the theoretical approach proposed in this paper may become a clue to clarify the rainfall characteristics in relation to the spatial scales of watersheds.

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APPENDIX - NOTATION

The following symbols are used in this paper:

A	=	area of total region G;
a	=	area of cell region (=A/N);
f	=	$\Sigma \exp(-\alpha \cdot {}_1R_i)$;
G	=	total region;
L	=	averaging scale of rainfall;
m	=	number of sampled data;
N	=	number of cell regions in total region;
n	=	parameter indicating the spatial scale;
n_i	=	number of cell regions in which volumetric rainfall ${}_1R_i$ occurs;
R_T	=	total rainfall in volume;
${}_nR$	=	volumetric rainfall falling on the subregion with area na;
${}_nR_i$	=	i-th level of volumetric rainfall falling on the subregion with area na;
$\bar{r}$	=	average rainfall intensity (R_T/A);
r_i	=	i-th level of rainfall intensity;
${}_nr$	=	${}_nR/na$;
α	=	parameter of Gamma-distribution;
β	=	$na\alpha$;
μ, $E(r)$	=	spatial average of rainfall; and
σ^2, $Var(r)$	=	spatial variance of rainfall.