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MODELING OF RUNOFF PROCESS IN A FIRST-ORDER BASIN

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SYNOPSIS

This paper deals with runoff analysis in a mountaineous watershed where time of concentration of the river channel is much shorter than that of the slope of the basin. The equation of momentum and continuity are used as basis ones and solved by means of the kinematic wave theory. By introducing the distribution of time of concentration, a response function of runoff system is derived, as a result, the relation between the instantaneous unit hydrograph and the time of concentration is clarified.

The new response function derived here is applied to an experimental basin to obtain the time of concentration from the data of rainfall and runoff. The results show that the distribution of time of concentration is roughly log-normal and similar to that of the slope length of the basin.

INTRODUCTION

There are two different approaches to the problem of rainfall-runoff process in watersheds. One is the stochastic approach, known as black-box analysis and the other is the dynamic approach based on hydraulics. The kinematic wave method belongs to the latter, in which length, gradient, and roughness coefficient of a slope are introduced. In the analysis, however, the average values of these factors are typically taken without considering their variations, even though in actual basins, those values vary according to location.

For instance, a hydrograph of dye concentration in a flow flattens in shape as it flows down, due to diffusion and dispersion caused by variations and fluctuations in flow velocity as shown in Fig. 1. Analogous to this, deviations from the average values of length, gradient, roughness, etc., should play important roles in the rainfall-runoff process.

In this paper, the author researches the runoff analysis in a mountaineous watersheds where the time of concentration of the river channel is negligibly small compared with that of the slopes. The momentum and continuity equations are used as basic equations and are solved by means of the kinematic wave theory. By introducing the distribution function of time of concentration, representing the characteristics of topographical and hydraulic factors, a lumped system model of runoff process is derived.

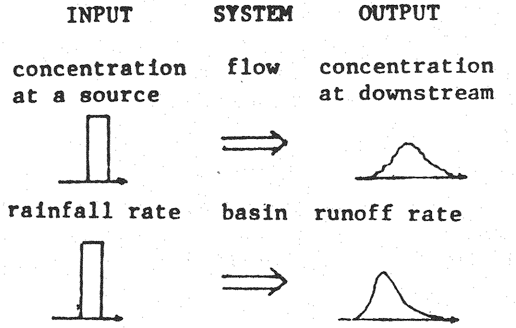

Fig. 1 Analogy of diffusion and runoff

BASIC EQUATIONS

Flow on a slope

The equation of continuity of a flow on a slope is

$$\frac{\partial(\gamma h)}{\partial t}+\frac{\partial q}{\partial x}=(r-f)\cos\theta \qquad (1)$$

where γ= porosity; h= depth; q= discharge unit width; r= rainfall intensity; f= infiltration rate; x= downstream distance; t= time; and θ= angle of a slope as illustrated in Fig. 2

When the slope of the energy gradient is assumed to be equal to the bed slope, the momentum equation can be written as

$$h=Kq^{p} \qquad (2)$$

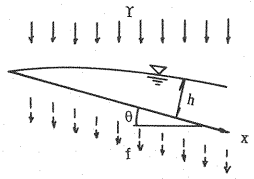

Fig. 2 Definition sketch of slope

where K and p=constants. If Manning's formula is adopted

$$K=(N/\sqrt{\sin\theta})^{p}; \qquad p=0.6 \qquad (3)$$

and if one applies Darcy's law

$$K=1/k_h\sin\theta; \qquad p=1 \qquad (4)$$

where N= Manning's roughness coefficient and k_h= hydraulic conductivity. Equations 3 and 4 may be used for a turbulent surface flow and a subsurface flow, respectively.

Eliminating h from Eqs. 1 and 2 one obtains

$$K\gamma p q^{p-1}\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}=r_e \qquad (5)$$

where $r_e=(r-f)\cos\theta$= effective rainfall intensity. This partial differential equation can be solved by use of the kinematic wave theory as follows:

By combining Eq. 5 with the total diffierential of q, one obtains

$$\frac{dx}{1}=\frac{dt}{K\gamma p q^{p-1}}=\frac{dq}{r_e} \qquad (6)$$

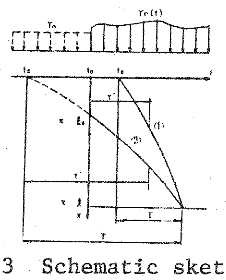

Fig. 3 Schematic sketch of characteristic curve

Hence the well-known equations are obtained as

$$\frac{dx}{dt}=\frac{1}{K\gamma p}q^{1-p} \qquad (7)$$

and

$$q=\left\{\frac{1}{K\gamma}\int_{t_s}^{t}r_e(\tau)d\tau\right\}^{1/p} \qquad (8)$$

where t_s= time when the characteristic starts.

By substituting Eq. 8 into Eq. 7

$$x=x_s+\int_0^{t-t_s}\frac{1}{p(K\gamma)^{1/p}}\left\{\int_{t-t_s-\tau'}^{t-t_s}r_e(t-\tau)d\tau\right\}^{(1-p)/p}d\tau' \qquad (9)$$

where x_s= point where the characteristic originates. Assuming that K and p are constant along a slope and that the effective rainfall began at time t_0, we obtain the following equation by rearranging Eq. 9 with $x_s=0$, $x=l$ and $T=t-t_s$:

$$l=\frac{1}{p(K\gamma)^{1/p}}\int_0^{T}\left\{\int_{T-\tau'}^{T}r_e(t-\tau)d\tau\right\}^{(1-p)/p}d\tau', \qquad (10)$$

where l= length of a slope and T= time of concentration.

In the case that T is larger than $t-t_0$, a characteristic which reaches the end of a slope at time t should depart from a certain point of the slope $(x=x_s)$ at time t_0 as indicated by line ② in Fig. 3. Hence substituting $x_s=l_s$, $x=l$ and $T=t-t_s$ into Eq. 8, we obtain

$$l = l_s + \frac{1}{p(K\tau)^{1/p}} \int_{T-t+t_0}^{T} \left\{ \int_{T-\tau'}^{T} r_e(t-\tau)d\tau \right\}^{(1-p)/p} d\tau' \; ; \qquad T>t-t_0 \tag{11}$$

To find l_s in terms of T, we assume a steady effective rainfall having a constant rate of r_0 before t_0, and a characteristic curve generated by the imaginary rainfall as indicated by a dotted line in Fig. 3. Then l_s is given by substituting $r_e=r_0$, $x_s=0$ and $t=t_0$ into Eq. 9 as

$$l_s = \frac{1}{(K\gamma)^{1/p}} r_0^{(1-p)/p}(T-t+t_0)^{1/p} \tag{12}$$

Combining Eqs. 11 and 12 leads to

$$l = \frac{1}{p(K\gamma)^{1/p}} \int_{\tau_0}^{T} \left\{ \int_{T-\tau'}^{T} r_e(t-\tau)d\tau \right\}^{(1-p)/p} d\tau' + p r_0^{(1-p)/p} \tau_0^{1/p} \tag{13}$$

where

$$\left.\begin{aligned} \tau_0 &= 0 \; ; && T \leqq t-t_0 \\ &= T-t+t_0 \; ; && T>t-t_0 \end{aligned}\right\} \tag{14}$$

From Eq. 7, the discharge per unit width at the end of a slope q_l is given as

$$q_l = \left\{ \frac{1}{K\gamma} \int_0^T r_e(t-\tau)d\tau \right\}^{1/p} \tag{15}$$

Eliminating $K\gamma$ from Eqs. 15 and 13 yields

$$q_l = lR(t,\ T) \tag{16}$$

where

$$R(t,\ T) = p \left\{ \int_0^T r_e(t-\tau)d\tau \right\}^{1/p} \Big/ \left[\int_{\tau_0}^{T} \left\{ \int_{T-\tau'}^{T} r_e(t-\tau)d\tau \right\}^{(1-p)/p} d\tau' + p r_0^{(1-p)/p} \tau_0^{1/p} \right] \tag{17}$$

Stream Flow

The equation of continuity for a stream flow is

$$\frac{\partial A_s}{\partial t} + \frac{\partial Q}{\partial x} = q_* \tag{18}$$

where A_s = cross-sectional area of flow; Q = discharge; and q_* = lateral inflow rate.

Making an assumption of quasi-uniform flow, the momentum equation can be expressed as

$$Q = \frac{1}{n} A_s R^{2/3} I^{1/2} \tag{19}$$

where n = Manning's roughness coefficient; R = hydraulic radius and I = channel gradient. To simplify the following development, we put

$$A_s R^{2/3} = a A_s^b \tag{20}$$

where a and b = constants related to the shape of a cross section. Equation 19 then is written in a convenient form by introducing Eq. 20 as

$$A_s = K_1 Q^{1/b} \tag{21}$$

where

$$K_1 = (n/a\ \sqrt{I})^{1/b}$$

Equations 18 and 21 are of the same form as Eqs. 1 and 2, respectively, making possible to solve by the same method.

The characteristic equation so obtained is

$$\frac{dx}{1}=\frac{dt}{K_1 Q^{1/b-1}}=\frac{dQ}{q_*} \tag{22}$$

Hence, the discharge at the end of a stream segment $Q(L,\ t)$ is

$$Q(L,\ t)=Q(0,\ t-T_L)+\int_0^L q_*(t-\tau)dx \tag{23}$$

where $Q(L,\ t)=$ discharge at the upstream end $(x=0)$, $L=$ length of the segment; $T_L=$ time of concentration of the stream; and $\tau=$ time between 0 to T_L. The relation between these factors are also given as

$$L=\frac{b}{K_1}\int_0^{T_L}\left[\left\{Q(0,\ t-T_L)\right\}^{1/b}+\frac{1}{K_1}\int_{T_L-\tau'}^{T_L} q_*(t-\tau)d\tau\right]^{b-1}d\tau' \tag{24}$$

DERIVATION OF LUMPED SYSTEM FUNCTION OF FIRST-ORDER BASIN

When we consider a watershed of order 1, $Q(0,\ t)$ should be zero, and q_* is to be equal to q_l. Equation 23 then becomes

$$Q(t)=\int_0^L q_l(t-\tau)dx \tag{25}$$

where $Q(t)=$ discharge at the outlet of a basin.

Since flow velocity in a channel is much higher than that of the slopes, the time of concentration of a stream segment is to be negligibly small. Then, Eq. 25 can be approximated by

$$Q(t)=\int_0^L q_l(t)dx=\int_0^L R(t,\ \tau)l dx \tag{26}$$

We now consider a successive effective rainfall of constant rate r_0 over a basin. Then $R(T,\ t)$ is given by rearranging Eq. 17 with $r_e=r_0$ as

$$R(t,\ T)=r_0 \tag{27}$$

Therefore, the discharge caused by this rainfall is

$$Q=r_0\int_0^L l dx \tag{28}$$

On the other hand, the rational formula gives the discharge

$$Q=Ar_0 \tag{29}$$

where $A=$catchment area of the basin. From Eqs. 28 and 29, we obtain

$$A=\int_0^L l dx \quad \text{or} \quad \int_0^L l dx/A=1 \tag{30}$$

This suggests that $l dx/A$ is the probability of a slope of length l. Hence, we can write as

$$\frac{l}{A}dx=f(l)dl=f(l)\frac{dl}{dT}dT \tag{31}$$

where $f(l)=$ probability density function of slope length. Since l is a function of T and t, as seen from Eq. 13, $f(l)dl/dT$ is also a function of both T and t, for a given hyetograph. Then we again write

$$f(l)\frac{dl}{dT}=\varphi(t,\ T) \tag{32}$$

where $\varphi(t, T)$ is considered to be a probability density function of T. By substituting these relations into Eq. 26, the following equation is obtained.

$$Q(t)=A\int_0^{\infty}\varphi(t, T)R(t, T)dT \tag{33}$$

in which, $Q(t)$, $R(t, T)$, and $\varphi(t, T)$ correspond to output, input and response function, respectively, in a runoff system.

In this system, however, the response function is unsteady, making treatment difficult. Some additional arrangements may be needed to obtain a steady response function as follows:

In the case that p equals one

When water flows on a downslope following Darcy's law, p is unity. By substituting $p=1$, Eq. 13 reduces to

$$l=T/K\gamma \tag{34}$$

This means that T is independent of time and rainfall intensity resulting in the stationary response function of the system $\varphi(T)$. For $p=1$, Eq. 17 also reduces to

$$R(t, T)=\int_0^{T}r_e(t-\tau)d\tau/T \tag{35}$$

which shows that the input to the system is the average effective rainfall intensity in the time of concentration. Hence, the relation betweem rainfall intensity and runoff is given by substituting Eq. 35 into Eq. 33 as follows:

$$Q=A\int_0^{\infty}\int_0^{T}r_e(t-\tau)d\tau\varphi(T)/TdT \tag{36}$$

If we define the average effective rainfall intensity as

$$r_m=\int_0^{\infty}\int_0^{T}r_e(t-\tau)d\tau/T\cdot\varphi(T)dT \tag{37}$$

then the rational formula is obtained

$$Q=r_mA \tag{38}$$

Equation 36 is also rewritten in the form of the unit hydrogragh as

$$Q=A\int_0^{\infty}u(\tau)r_e(t-\tau)d\tau \tag{39}$$

where

$$u(\tau)=\int_{\tau}^{\infty}\varphi(T)/TdT \tag{40}$$

This shows that the relation between rainfall anf runoff is linear when p equals one, and that the instantaneous unit hydrograph $u(\tau)$ is the average of reciplocals of time of concentration larger than τ.

In the case that p does not equal one

When a succesive rainfall of a constant rate of r_0 is given on a slope, the relation between the slope length and the time of concentration T_0 for the rainfall is given from Eq. 13 as

$$l=r_0^{(1-p)/p}(T_0/K\gamma)^{1/p} \tag{41}$$

Since T_0 is independent of time, we can define the steady probability density function $\varphi(T_0)$ as follows:

$$f(l)dl=f(l)\frac{dl}{dT_0}\,dT_0=\varphi(T_0)dT_0 \tag{42}$$

Substituting Eq. 42 into Eq. 33, we obtain

$$Q=A\int_0^{\infty}R(t,\ T)\varphi(T_0)dT_0 \tag{43}$$

and the relation between T and T_0 is given by Eqs. 10 and 41

$$T_0^{1/p}=\frac{1}{p}\int_{\tau_0}^{T}\left\{\int_{T-\tau'}^{T}\frac{r_e(t-\tau)}{r_0}\,d\tau\right\}^{(1-p)/p}d\tau'+\tau_0^{1/p} \tag{44}$$

Since T is a function of T_0 and t, $R(t.\ T)$ also is a function of them, i.e. $R(t,\ T)=R_0(t,\ T_0)$. Then Eq. 43 becomes

$$Q(t)=A\int_0^{\infty}R_0(t,\ T_0)\varphi(T_0)dT_0 \tag{45}$$

in which, $\varphi(T_0)$ is considered as a steady response function resulting in the linear relation between the output Q and the input R_0, though the rainfall runoff relation is nonlinear when p does not one.

APPLICATION TO AN EXPERIMENTAL BASIN

There are two kinds of methods to evaluate $\varphi(T)$ or $\varphi(T_0)$ from the data of rainfall and runoff; one is the optimumization technique which has been used for the unit hydrograph method, and the other is the parametric method.

Optimum Response Function

Several optimumization techniques have been used to obtain the unit hydrograph (2),(3). The same method may be used to find the optimum response function in Eq. 36 or Eq. 45. The best fit criterion is minimumization of the sum of the squares between the observed and the computed discharges. Applying the least squares method to evaluate the optimum values of $\varphi(T_0)$, one obtains the following:

$$\rho_{QR}(\tau)=\int_0^{\infty}\varphi_0(t)\,\rho_{RR}(\tau-t)dt \tag{46}$$

where $\varphi_0(t)=$ optimum value of $\varphi(t)$;

$$\left.\begin{aligned}\rho_{QR}(\tau)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{Q(t)}{A}R_0(t,\ \tau)dt\\ \rho_{RR}(\tau_1-\tau_2)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_0(t,\ \tau_1)R_0(t,\ \tau_2)dt\end{aligned}\right\} \tag{47}$$

Equation 46 is well-known as Wiener-Hoph equation and rewritten in the form of discrete time.

$$\rho_{QR}(i)=\sum_{j=0}^{m}\varphi_0(i)\,\rho_{RR}(i-j) \tag{48}$$

This can be easily solved numerically.

Though the Wiener-Hoph equation has been widely used to obtain unit hydrograph, resulting unit hydrograph usually exhibits oscillation. From Eq. 36, the relation between the instantaneous unit hydrograph and $\varphi(T)$ is expressed as

$$\varphi(T)/T=-du(T)/dT \tag{49}$$

Since $\varphi(T)$ is a nonnegative value, $du(T)/dT$ must be negative, i.e., $u(t)$ should decrease monotonically with time, the oscillation in the unit hydrograph results in unrealistic negative values of resulting response function as illustrated in Fig. 4.

To evaluate the physically reliable values of $\varphi(T)$, we now need to solve Eq. 39 under constraint $\varphi(t)\geqq 0$ as follows:

$$\text{minimize} \quad \sum_{i=0}^{m} e(i) \tag{50}$$

subjecting to the constraints $\varphi(j) \geq 0$ and $e(i) \geq 0$
where

$$e(i) = \frac{Q(i)}{A} - \sum_{j=0}^{m} R_0(i,\ j)\varphi_0(j) \tag{51}$$

Equation 46 is readily solved by use of the simplex method (1).

The analysis method mentioned above was applied to an experimental basin of area 0.226km². The topographical map of the basin is shown in Fig. 5. The results obtained are shown in Figs. 6 - 10.

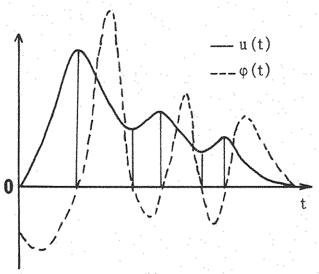

Fig. 4 Relation between u(t) and φ(t)

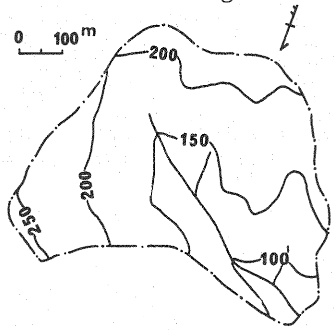

Fig..5 Topographical map of experimental basin

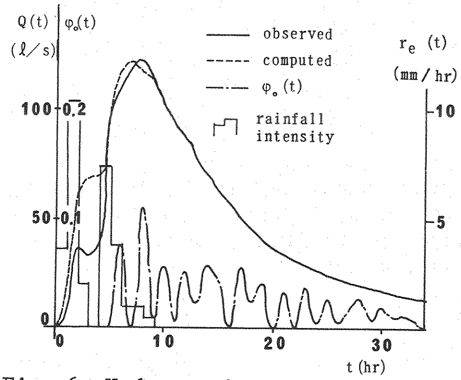

Fig. 6 Hydrographs and time of concentration (p=1)

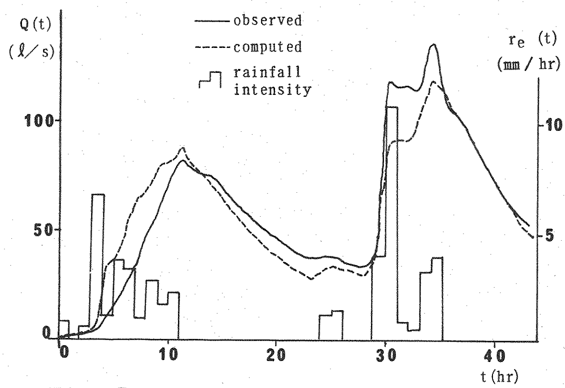

Fig. 7 Comparison between predicted and observed discharge

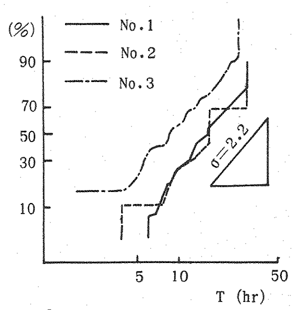

Fig. 8 Distribution of time of concentration (p=1)

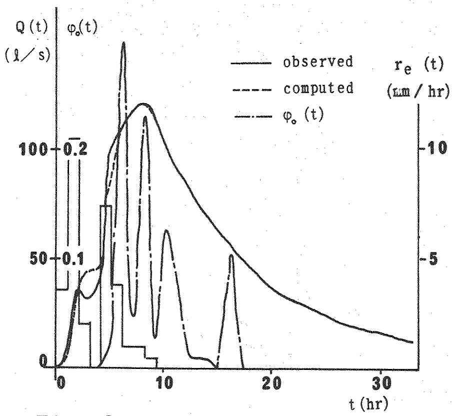

Fig. 9 Hydrographs and time of concentration (p=0.6)

Figure 6 shows the resulting optimum time of concentration for $p=1$ and comparison of the observed and computed hydrographs of event 1. To test the validity of the results the predicted hydrograph of event 2 using $\varphi(T)$ of even 1 is shown and compared with the obsreved hydrograph in Fig. 7. The optimum values of time of concentration for events 1,2 and 3 are given in Fig. 8 which shows that they are log-normally distributed with the same standard deviation 2.2.

Figures 9 and 10 are the results obtained using $p=0$ and $r_0=10\text{mm/hr}$. In this case, the distribution of the optimum time of concentration are also log-normal but with less standard deviations. If K and l are independent of each other, the standard deviation of time of concentration σ_T is given from Eq. 41 as

$$\log\sigma_T=\sqrt{(p\log\sigma_l)^2+(\log\sigma_K)^2} \qquad (52)$$

where σ_l, σ_K = standard deviations of l and $K\gamma$, respectively. Equation 52 indicates that the value of σ_T decrease with decreasing p.

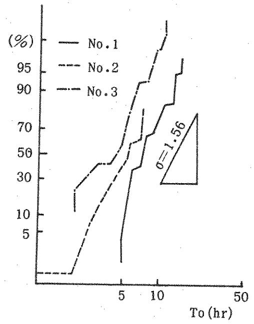

Fig. 10 Distribution of time of concentration (p=0.6)

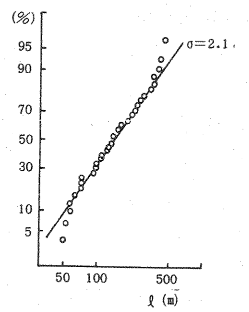

Fig. 11 Distribution of slope length

A slope length can be measured on the topographical map as the length of a curve running orthogonally to the contours from the diving ridge to the nearest stream. As shown in Fig. 11, the measured slope length also shows a log-normal distribution with the standard deviation of 2.1 which is very close to the value of σ_T for $p=1$. This suggests that $\log\sigma_K$ is very small compared with $\log\sigma_T$ or that there is a certain correlation between K and l . If the former is true,

$$\log\sigma_T\fallingdotseq p\log\sigma_l \quad \text{or} \quad \sigma_T\fallingdotseq\sigma_l^{\,p} \qquad (53)$$

Substituting $p=0.6$ and $\sigma_l=2.1$ into Eq. 53 yields $\sigma_T=1.56$ which is considered to be a reasonable value of the standard deviation for $p=0.6$, as seen in Fig. 9.

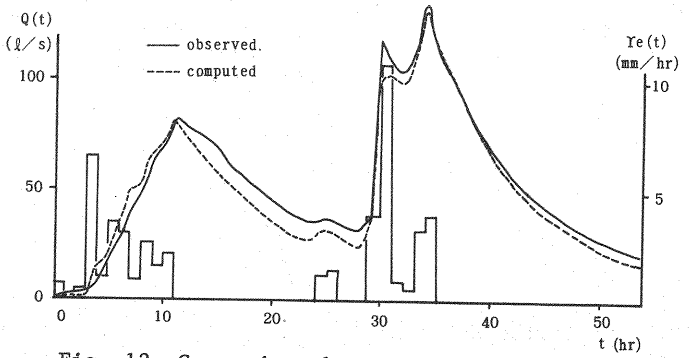

Fig. 12 Comparison between computed and observed discharge (p=0.6 and r_0=10mm/hr)

Parametric Approach

Since the respone function derived here is the probability density function, it is reasonable to fit a certain probability density formula to treat it as a parametric system model. Doing so, the mean and the standard deviation becomes the unknown parameters to be determined. If one uses Eq. 53 to estimate the standard deviation, the mean is the only parameter.

In Fig. 12, the computed hydrograph is compared with the observed one, in which, the distribution of time of concentration is assumed to be log-normal with standard deviation of 1.56 and the mean was determined so as to obtain the best fitting. Good agreement was obtained.

The instantaneous unit hydrograph is expressed by the probability density function of time of concentration as seen in Eq. 40. Applying the log-normal function to $\varphi(T)$, Eq. 40 becomes

$$u(\tau)=\frac{1}{2}\exp\left[-c(\overline{X}-\frac{1}{2}cS_T{}^2)\right]\left(1-\mathrm{erf}(X)\right) \quad (54)$$

where

$$X=\frac{\log\tau-\overline{X}+cS}{2S_T}\ ;\ c=\ln 10=2.303\ ;\ S_T=\log\sigma_T\ ;$$

and $\overline{X}$ is the mean of $\log T$.

CONCLUSION

The new response function of runoff system has been derived by use of the kinematic wave theory. It is clarified from the equation that the instantaneous unit hydrograph is expressed by the probability density function of time of concentration.

The time of concentration obtained from the data of the experimental basin is distributed log-normally as is the slope length of the basin.

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APPENDIX - NOTATION

The following symbols are used in this paper:

a	= a constant;
A	= chatchment area;
As	= cross-sectional area;
b	= an exponent;
c	= $Ln\ 10=2.303$
$e(i)$	= prediction error given by Eq. 51;

f	= infiltration rate;
$f(l)$	= probability density function of slope length;
g	= acceleration of gravity;
h	= water depth;
g	= gradient of river bed;
k_h	= hydraulic conductivity;
K	= a constant;
K_1	= a constant;
l	= slope length;
l_s	= length from the top to the origin of a characteristics;
L	= channel length;
N, n	= Manning's roughness coefficients of river channel;
p	= an exponent;
q	= discharge per unit width;
q_l	= discharge per unit width at the end of slope;
q_*	= lateral inflow rate;
Q	= discharge of river;
r	= rainfall intensity;
r_e	= $(r-f)\cos\theta$, effective rainfall intensity;
r_0	= rate of a steady effective rainfall intensity;
r_m	= average effective rainfall intensity defined by Eq. 37;
R	= hydraulic radius;
$R(t, T)$	= function of time and T given by Eq. 17;
$R(t, T_0)$	= function of time and T_0 given by Eqs 17 and 44;
S_T	= $\log\sigma_T$
t	= time;
t_0	= time when effective rainfall begins;
t_s	= time when a characteristic starts;
T, T_L	= time of concentration of slope and river channel;
T_0	= time of concentration of slope due to a steady effective rainfall having a rate of r_0 ;
$u(\tau)$	= instantaneous unit hydrograph;
x	= downstream distance;
x_s	= position where a characteristic originates;
X	= $\log T$
$\overline{X}$	= mean value of $\log T$;
γ	= porosity
θ	= angle of slope;
ρ_{QR}	= cross correlation function;
ρ_{RR}	= auto correlation function;

σ_l = standard deviation of slope length;

σ_K = standard deviation of $K\gamma$;

σ_T = standard deviation of time of concentration;

τ = time valuable;

τ_0 = time given by Eq. 14;

τ' = time defined in Fig. 3;

φ = probability density function of time of concentration; and

φ_0 = optimum probability density function of time of concentration.