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INTERACTION BETWEEN A SPHERICAL CONTAINER AND A LIQUID IN IT

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SYNOPSIS

Vibration of a spherical container which is supported by a spring and containing a liquid with a free surface was studied. The governing equations for the case where the free surface keeps the first mode of waves were those of two degrees of freedoms similar to a vibration system.

The concerning system has two basic natural frequencies under the normal free oscillation. When the sphere contains a small amount of liquid, the natural frequency is sharply increasing according to the increase of the liquid amount.

This phenomenon is supported by an experiment. This higher mode, however, disappears suddenly at a certain water level and the lower mode, which is almost same as the simple mass and spring system,takes place. There were some differences between the expected natural frequencies and the experimental figures. This is considered the surface waves of higher modes are predominant in the sphere to a certain extent.

INTRODUCTION

The motion of a hollow sphere containig a liquid with free surface is an interesting problem, especially if it is supported by a spring system, from the point of views of anti-seismological design as well as mass and liquid interaction dynamic systems. Such a system contains a lot of applicability for hydrodynamics as well. If a mass is moving in a fluid with acceleration, the famous virtual mass is of importance from hydrodynamical point of views, while a similar phenomenon takes place when a fluid, especially a liquid with a free surface like the present problem, is stored in the mass.

Relatively small number of studies have been carried out in the past regarding this point. The most important work is the study of Housner (1). He conducted a series of analytical works in order to obtain natural frequencies of structures containing a liquid. Ishikawa and Shi-igai studied a similar problem from a different point of view. They analyzed a two-layered system with a free surface in a cylindrical container in 1973. Though they handled the internal waves in a vessel, the case where the density of the upper layer be zero is considered as the normal one-layer system.

Sogabe conducted a series of comprehensive studies to determine the natural frequencies of the liquid in a cylindrical vessel. Sogabe and his colleagues extended their study for the cases of a sphere with liquid (6),(7),(8),(9),(10). They obtained the natural frequencies of the liquid itself in the spherical vessel which is rigidly supported. Therefore, they have not considered the behavior of the total system. A semi-empirical computation agrees with the experimental data quite well.

Oscillatory behaviors of a sphere supported by a spring will be considered in this paper; especially the interaction between the sphere and the contained water will be investigated related with natural frequencies of the total system.

Regarding this type of problems, Mutoh compiled various experimental results in his comprehensive book (4), and he derived the conception of "free water and attached water", which is now being supported by many engineers. According to Mutoh, a certain part of water in a container, when it is oscillated, does not move, while the rest of water moves with the container. The former is named as "free water" and the latter as "attached water". This is clearly a negative virtual mass effect from the dynamical point of views. Since it reflects an interesting nature of the interaction between a structure system and the liquid, some theoretical examination is tried in this paper regarding the conception of "free water".

Recently, Watabe and others carried out a prototype experiment by using a real LPG storage tank and obtained valuable results (12).

THEORETICAL INVESTIGATION

As explained before, the natural frequency of a liquid contained in a hollow sphere has already been studied, both theoretically and experimentally. However, a dynamical system consisting of a sphere and spring, and of a liquid in the sphere, has not fully been studied, though such a system may be more realistic from practical point of views. Especially, the interaction between the system and the liquid is the present concern.

Fig. 1 shows the schematic diagram of a sphere supported by a leaf spring. This system is the model used in this analysis and the necessary coordinates are specified in this figure.

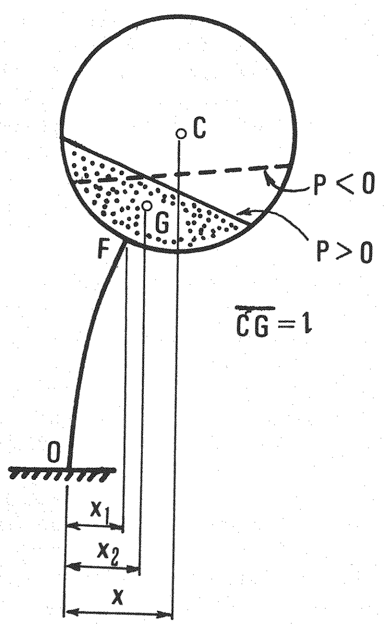

Fig. 1 Ordinate Definitions

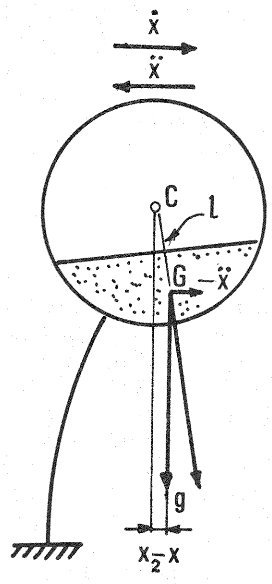

Fig. 2 The Balance of Acceleration and Gravity

Let the mass of the empty sphere be m_1 and the mass of the liquid in the sphere be m_2. The equation of motion is then written as

$$m_1\ddot{x} + m_2\ddot{x}_2 + k_0 x_1 = 0 \qquad (1)$$

where k_0 is the spring constant.

x_1 may be proportional to x which is the horizontal displacement of the center of the gravity of the total system without liquid contained in the sphere. Therefore, we may put

$$x_1 = bx \qquad (2)$$

where b is the proportional coefficient.

x_2 which is the displacement of the center of the gravity of the liquid in the sphere may be expressed as

$$x_2 = x + p\ddot{x} \qquad (p = \pm\ell/g) \qquad (3)$$

Eq. 3 expresses the relative stationary state of liquid motion, when $p=-\ell/g$. If p takes other values rather than eq. 3, it means that the motion of the liquid is out of phase from the combined force of gravitation and the acceleration. This situation is explaind on Fig. 2 of the previous page.

If the sphere is moving to the right hand direction, and the sphere has just passed the equilibrium state, the acceleration to the sphere is working in the left hand direction. Thus the liquid in the sphere receives the counter acceleration from the sphere in the right hand direction, which is expressed as $-\ddot{x}$. Assuming that the free surface of the water is perpendicular to the resultant force F, we obtain the relation that

$$(x_2 - x)/\ell = -\ddot{x}/g \qquad (4)$$

which is by all means the state of relative stationality. Since there may be other possibilities where the surface displacement of the liquid takes some other modes, positive p case cannot be neglected. When the friction term is introduced, positive p case should be considered.

These set of basic equation may be obtained through Lagrange dynamics without any difficulties.

Eq. 1, Eq. 2 and Eq. 3 form a matrix equation expressed by Eq. 5.

$$M\ddot{X} + KX = 0 \qquad (5)$$

where

$$M = \begin{bmatrix} m_1 & m_2 \\ p & 0 \end{bmatrix} \qquad (6)$$

$$K = \begin{bmatrix} bk_0 & 0 \\ 1 & -1 \end{bmatrix} \qquad (7)$$

$$X = \begin{bmatrix} x \\ x_2 \end{bmatrix} \qquad (8)$$

Substituting $X=A\cdot\exp(st)$ for X in Eq. 5, we obtain

$$\begin{vmatrix} m_1 s^2 + k & m_2 s^2 \\ ps^2 + 1 & -1 \end{vmatrix} = 0 \qquad (k = k_0 b) \qquad (9)$$

The solutions of the characteristic equation are

$$s_1^2 \; , \; s_2^2 = -(m_1 + m_2)/2m_2 p \pm \{\sqrt{(m_1 + m_2)^2 - 4kpm_2}\}/2m_2 p \qquad (10)$$

When p is negative, two roots are real, while they are also real for positive p, provided

$$(m_1 + m_2)^2/4km_1 > p \tag{11}$$

Note that s_1 and s_2 are complex in this case.

Therefore, we have now three cases where the oscillatory motion takes place. Putting $s=i\omega$, we obtain Eq. 12 for negative p.

$$\omega_1^2 = (m_1 + m_2)[1 - \sqrt{1 - 4kpm_2/(m_1 + m_2)^2}]/2m_2p \tag{12}$$

Similarly, we obtain Eq. 13 and Eq. 14 for positive p.

$$\omega_2^2 = (m_1 + m_2)[1 + \sqrt{1 - 4kpm_2/(m_1 + m_2)^2}]/2m_2p \tag{13}$$

$$\omega_3^2 = (m_1 + m_2)[1 - \sqrt{1 - 4kpm_2/(m_1 + m_2)^2}]/2m_2p \tag{14}$$

The non-dimensional expression for frequency is as follows.

$$(\omega_1/\omega_0)^2 = (1+ m_1/m_2)[-1 + \sqrt{1 + 4k\ell m_2/m_1/gm_1(1 + m_2/m_1)^2}]a/2\ell \tag{15}$$

$$(\omega_2/\omega_0)^2 = (1+ m_1/m_2)[1 + \sqrt{1 + 4k\ell m_2/m_1/gm_1(1 + m_2/m_1)^2}]a/2\ell \tag{16}$$

where $\omega_0^2 = g/a$ and a is the radius of the sphere.

Note that ℓ is the distance between the center of the gravity of the empty sphere and that of the liquid. a being the radius of the sphere, ℓ/a is theoretically calculated as follows.

$$\ell/a = 3\sin^4\alpha/4(2 + \cos^3\alpha - 4\cos\alpha) \tag{17}$$

where α is the half angle of the chord which is the stationary water surface.

It should also be noted that ω_3, which is by all means very low, was not observed at least explicitly throughout the experiment.

EXPERIMENTAL RESULT

A plastic sphere of which diameter is 11.0 cm was used for the container. The sphere is attached to the top of a leaf spring. By measuring the change of the natural frequency of the system, which is caused by the change of the water amount in the sphere, the effect of the water mass with a free surface was examined.

Fig. 3 shows the non-dimensional plot of ω_1^2 and ω_2^2 for various values of a parameter B against the relative amount of water, where $B=4k\ell/m_1g$. This combined theoretical curves shows that ω_1 always takes real values but ω_2 disappears often since it takes imaginary values in a certain range.

Fig. 4 is the dimensional plot of ω_1 and ω_2 according to the change of the total mass together with the experimental data. It should be noted that ω_2 is increasing sharply at the initial stage, in spite of the increase of the total mass. This phenomenon was experimentally supported as was shown on Fig. 4 and Fig. 5 on the next page. Those dimensional plots may be easy to understand this peculiar phenomenon, which may be considered as the negative virtual mass effect.

Another interesting feature of this oscillation is that as ω_2 becomes imaginary, the measured natural frequency jumps to another line of ω_1 . The values of ω_1 are steadily decreasing according to the increase of the total mass, which is by all means a common feature of any oscillation phenomena.

When the amount of water in the container increases, the surface area of the water decreases, and the surface effect disappears. The spring used for the experiment shown on Fig. 4 was perhaps too weak so that it was "bent" when the container was almost filled with water, which caused the off-set of experimental values from the theory. This effect was not observed for the case of Fig. 5, where the stiffer spring was used for the support.

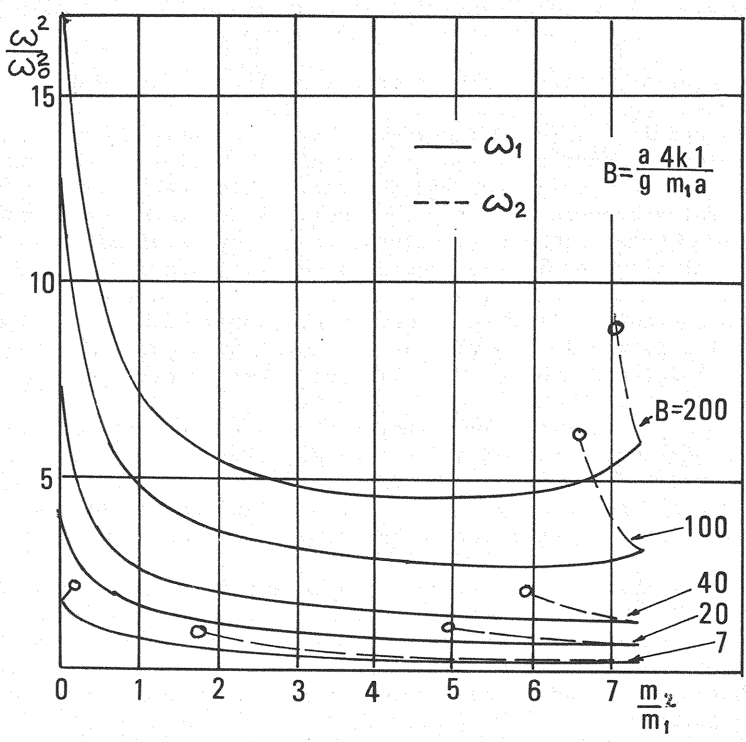

Fig. 3 Non-dimensional Plot of Natural Frequencies

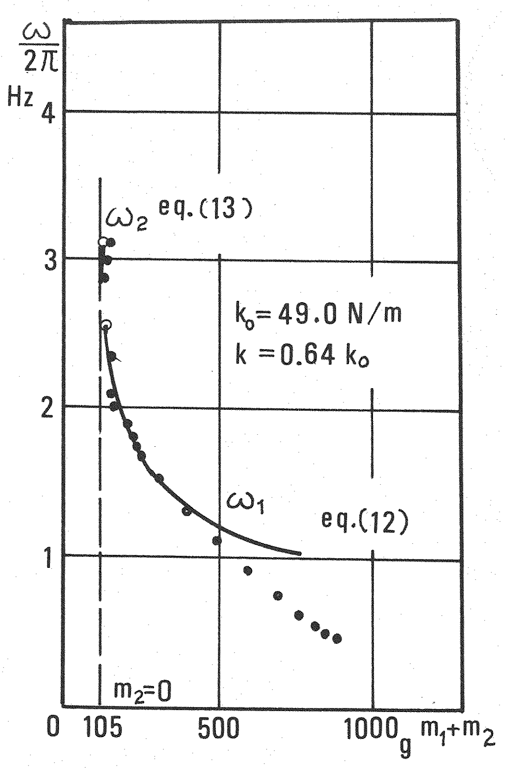

Fig. 5 Dimensional Plot of ω (I)

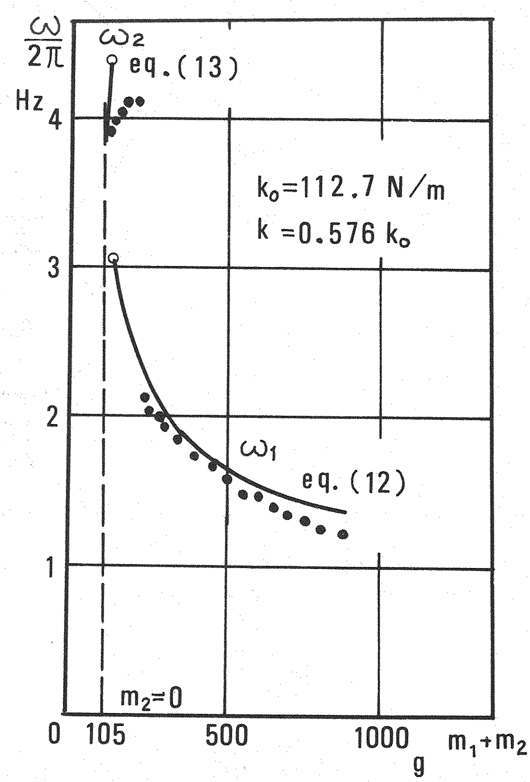

Fig. 6 Dimensional Plot of ω (II)

DISCUSSION AND CONCLUSION

Where the amount of water in the sphere is small, it is found that the natural frequency increases according to the increase of the amount of water in the sphere. This may be explained in the following way. Where the water amount is small, ℓ, the distance between the center of the gravity of the hollow sphere and that of the water in the sphere, becomes shorter and shorter. This means that the natural frequency of the water becomes higher and higher. Since this phenomenon is a coupled oscillation system, the total natural frequency is affected by the change of ℓ when the amount of water in the sphere is small. This phenomenon, however, disappears quickly when the water amount in the sphere is further increased, as this mode becomes unstable. If this higher mode, expressed by ω_2, disappears, then the oscillation is mostly governed by the total mass, and thus the combined frequency becomes lower and lower. This line of total process is supported by experiment quite well.

While this theory is derived by assuming a flat water surface, the observed water surface was very wavy and not quite flat at all. Waves of the first and higer modes were observed throughout the experiment. This fact will be of importance when the system faces with the external oscillatory forces. The waves in the sphere have already been recorded by the authors, which will be reported in near future.

Perhaps the fact that the position of the center of gravity does not change very much for the higher waves gives a good result for this simplified analysis.

The theory also explains the meaning of "free water", proposed by Mutoh (4). In order to express the conception of the free water and the attached water theoretically, Mutoh suggested a model shown on Fig. 6, where two masses are connected by two springs.

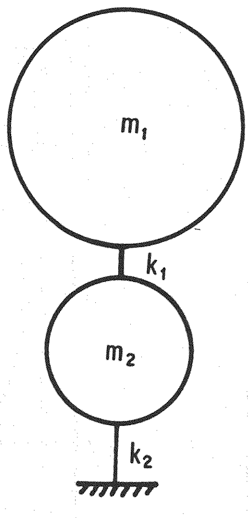

Fig. 7 Model of Free Water by Mutoh

The equation which governs this model is, neglecting any damping terms, also Eq. (5), where M and K are given by

$$M = \begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix} \qquad (18)$$

$$K = \begin{bmatrix} k_1 + k_2 & 0 \\ 1 & -1 \end{bmatrix} \qquad (19)$$

They are somewhat different from Eq. 6 and Eq. 7. The main difference is the effect of ℓ/g. If we use the model of Fig. 6, we have to estimate m_1 always by using empirical or experimental data. On the other hand, the model by authors derives the following equation.

$$m_2 p \cdot d^4x_1/dt^4 + (m_1 + m_2)d^2x_1/dt^2 + bk_0x_1 = 0 \qquad (20)$$

Then, assuming $d^4x_1/dt^4 = -\omega^2 d^2x_1/dt^2$, Eq.20 is modified as

$$[m_1 + m_2(1 - p\omega^2)]d^2x_1/dt^2 + bk_0x_1 = 0 \qquad (21)$$

The above expression is correct as far as x_1 has a harmonic oscillation shape. Thus, it is clear that the term $m_1 + m_2(1 - p\omega^2)$ represents a "mass" of the system, where $m_2p\omega^2$ is the negative "virtual mass coefficient", which is a function of frequency. This term never appears from a series connection of two masses like Fig. 6.

As the conclusion, we obtained the following.

1. The natural frequency of a sphere, supported by a spring, and containing a liquid, is reasonably described by Eq. 5.
2. Experimental data verified the tendency of natural frequencies of the system. The frequency increases initially for a short period as the total mass increases. It becomes soon decreasing afterwards according to the mass increase.
3. The mathematical model proposed by the authors presents a new conception of negative virtual mass. This conception explains the idea of "free water" clearly.

ACKNOWLEDGEMENT

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APPENDIX - NOTAION

The following symbols are used in this paper:

b = proportional coefficient;

B = non-dimensional spring coefficient ($4k\ell/m_1 g$);

g = gravitaional acceleration;

k = virtual spring coefficient;

k_0 = real spring coefficient;

K = vectorial spring coefficient;

ℓ = distance between the center of the hollow sphere and the center of gravity of the water in the sphere;

m_1 = mass of the empty sphere;

m_2 = mass of water in the sphere;

M = vectorial mass;

p = $\pm\ell/g$;

s = complex frequency;

t = time;

x = displacement of the center of the sphere;

x_1 = displacement of the end of spring;

x_2 = displacement of the center of gravity of water in the sphere;

X = complex displacement;

α = half angle against the chord consisting of the flat water surface;

ω = frequncy in general;

ω_1 = general mode of frequency;

ω_2 = higher mode of frequency; and

ω_3 = imaginary or lower frequency which was not observed.