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LAMINAR DISPERSION IN AN ELLIPTICAL PIPE AND A RECTANGULAR PIPE

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SYNOPSIS

Laminar dispersion coefficients both in an elliptical pipe and a rectangular pipe have been studied theoretically. Change in laminar dispersion coefficient with respect to cross-sectional shape is discussed for a wide range of aspect ratio Ar.

Non-dimensional dispersion coefficient both in an elliptical pipe and a rectangular pipe attains the minimal value at Ar = 1 and is logarithmically symmetrical about it. The behavior of non-dimensional dispersion coefficient for Ar $\neq$ 1, however, is remarkably different between an elliptical pipe and a rectangular pipe: the former monotonously increasing with Ar $\to \infty$ and Ar $\to$ 0. whereas the latter approaching an asymptotic value as Ar $\to \infty$ and Ar $\to$ 0. At Ar = 1, non-dimensional dispersion coefficient in a square pipe is greater than that in a circular pipe. Comparison of dispersion coefficient with different cross-sectional shapes reveals that geometrical shape of a pipe decidedly affects the change in laminar dispersion coefficient with the aspect ratio.

INTRODUCTION

The mixing of soluble matter in the shear flow can be treated as an one-dimensional dispersion, where the convection of the matter by velocity deviation in the cross-section is balanced with the molecular diffusion. The dispersion equation, which is one-dimensional representation of the diffusion equation, is expressed as follows:

$$\frac{\partial \bar{s}}{\partial t} + \bar{u}\frac{\partial \bar{s}}{\partial x} = \frac{\partial}{\partial x} D_x \frac{\partial \bar{s}}{\partial x} \qquad (1)$$

where $\bar{s}$ = the cross-sectional mean concentration of a tracer; $\bar{u}$ = the cross-sectional mean velocity; x = a longitudinal coordinate taken along the direction of mean flow; and t = time. D_x is the longitudinal dispersion coefficient defined as follows:

$$D_x = -\int_A s'u'dA \Big/ A\frac{\partial \bar{s}}{\partial x} \qquad (2)$$

where s' = the deviation of concentration from the cross-sectional mean; u'= the velocity deviation from the cross-sectional mean; and A = the area of a cross-section of a channel.

Analysis of mixing of materials by the dispersion equation (Eq. 1) is widely used in the fields of civil engineering, chemical engineering etc. However, the value of dispersion coefficient for general shape of a cross-section of a channel has not been obtained so far. The definition of Eq. 2 shows that the value of dispersion coefficient may have a wide variation dependent on the velocity distribution in a cross-section. In the case of turbulent flow, which bears a practical importance, the dispersion coefficient is deduced from the results of the field observation or laboratory experiments, because theoretical evaluation of velocity distribution is often different from the real distribution. On the contrary, in the case of laminar flow, velocity distribution is known for pipes with many kinds of cross-sectional shape. Therefore, it is possible to investigate theoretically the dispersion coefficient in some kinds of pipes.

In this paper, analytical solutions of laminar dispersion coefficient will be obtained for the flow in pipes with elliptical and rectangular cross-section. Moerover, the relation between the dispersion coefficient and the cross-sectional shape of the pipe will be discussed for a wide range of aspect ratio of cross-section.

Taylor (6) obtained the longitudinal dispersion coefficient of a laminar flow in a circular pipe under the following assumptions: (i) the longitudinal molecular diffusion term is negligibly small compared with the convection term, (ii) for an observer moving with the mean flow velocity, the temporal variation of the cross-sectional mean concentration is much smaller than the spatial variation of the concentration deviation from the cross-sectional mean, so that the phenomenon is considered to be steady in this system. The obtained dispersion coefficient is expressed by the following non-dimensional form:

$$D_x = \frac{1}{48\,D} R^2 u_m^2 \qquad (3)$$

where D = the molecular diffusion coefficient; R = the hydraulic radius (= r/2, r = the radius of a circular pipe); and u_m = the maximum velocity of the flow.

Analysis of laminar dispersion in an elliptical pipe and a rectangular pipe will be carried out on the basis of Taylor's assumptions in the following sections.

LONGITUDINAL DISPERSION COEFFICIENT IN AN ELLIPTICAL PIPE

Fig. 1 illustrates a coordinate system of a pipe whose cross-section is an ellipse with the major axis a and the minor axis b. Relation between the Cartesian coordinate system (y,z) and the elliptical coordinate system (ξ,η) is

$$\left.\begin{aligned}
&y = c\cosh\xi\cos\eta; \qquad z = c\sinh\xi\sin\eta \\
&c^2 = a^2 - b^2 \\
&a = c\cosh\xi_0; \qquad b = c\sinh\xi_0 \\
&\xi_0 = \frac{1}{2}\ln\frac{a+b}{a-b} = \frac{1}{2}\ln\frac{Ar+1}{Ar-1} \\
&h_\xi = h_\eta \\
&\quad = c\sqrt{\sinh^2\xi + \sin^2\eta} = ch
\end{aligned}\right\} \qquad (4)$$

where Ar = the aspect ratio of the cross-section of an elliptical pipe (= a/b).

Provided that the secondary flow does not exist in the cross-section, and that the flow becomes uniform in the direction of x-coodinate, the molecular diffusion equation is written in terms of the general orthogonal curvilinear coordinate system as follows:

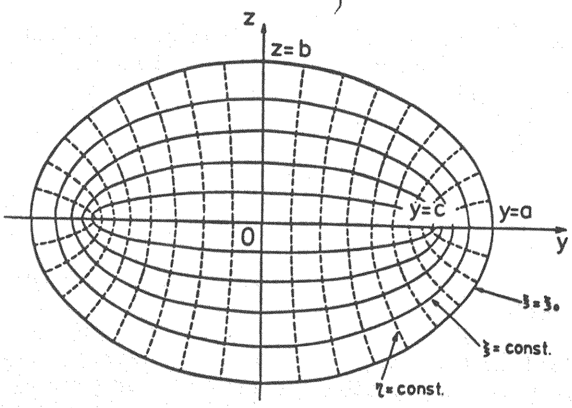

Fig. 1 Elliptical coordinate in an elliptical pipe

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = \frac{D}{h_\xi h_\eta}\left\{ \frac{\partial}{\partial x} h_\xi h_\eta \frac{\partial s}{\partial x} + \frac{\partial}{\partial \xi}\frac{h_\eta}{h_\xi}\frac{\partial s}{\partial \xi} + \frac{\partial}{\partial \eta}\frac{h_\xi}{h_\eta}\frac{\partial s}{\partial \eta} \right\} \tag{5}$$

The molecular diffusion term of x-direction in Eq. 5 is considered to be negligible because it is smaller than the convective term. Moreover, considering that $h_\xi = h_\eta = ch$ in the elliptical coordinate system, we rewrite Eq. 5 as

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = \frac{D}{c^2h^2}\left(\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} \right) \tag{6}$$

Then, the concentration of tracer and the velocity are expressed by the sum of the cross-sectional mean value and the deviation from it, respectively, as follows:

$$\left.\begin{aligned} s &= \overline{s}(x,t) + s'(x,\xi,\eta,t) \\ u &= \overline{u} + u'(\xi,\eta) \end{aligned}\right\} \tag{7}$$

Eq. 6 may be transformed by the use of the moving coordinate system with the cross-sectional mean velocity $\overline{u}$ as follows:

$$\frac{\partial \overline{s}}{\partial t} \quad \frac{\partial s'}{\partial t} + u'\frac{\partial \overline{s}}{\partial x_1} + u'\frac{\partial s'}{\partial x_1} = \frac{D}{c^2h^2}\left(\frac{\partial^2 s'}{\partial \xi^2} + \frac{\partial^2 s'}{\partial \eta^2} \right) \tag{8}$$

where $x_1 = x - \overline{u}t$. Considering the deviation of concentration is much smaller than the cross-sectional mean, i.e. $s' << \overline{s}$ and temporal variation of the mean concentration is negligibly small in comparison with spatial variation, i.e. $|\partial\overline{s}/\partial t| << |u'\partial\overline{s}/\partial x_1|$, we obtain the following equation from Eq. 8

$$u'\frac{\partial s}{\partial x} = \frac{D}{c^2h^2}\left(\frac{\partial^2 s'}{\partial \xi^2} + \frac{\partial^2 s'}{\partial \eta^2} \right) \tag{9}$$

The velocity distribution of the laminar flow in an elliptical pipe is given in terms of the Cartesian coordinate system as follows (Milne & Thomson (5)):

$$\frac{u}{u_m} = 1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}; \qquad \overline{u} = \frac{1}{2}u_m \tag{10}$$

where u_m = the maximum velocity in a cross-section. The deviation of velocity from the cross-sectional mean is obtained from Eq. 10 as

$$u' = u_m\left(\frac{1}{2} - \frac{y^2}{a^2} - \frac{z^2}{b^2} \right) \tag{11}$$

Expressing Eq. 11 in terms of the elliptical coordinate system and substituting the result into Eq. 9 yields

$$\frac{\partial^2 s'}{\partial \xi^2} + \frac{\partial^2 s'}{\partial \eta^2} = K\, f(\xi,\eta) \tag{12}$$

where

$$\begin{aligned} f(\xi,\eta) = {} & \cosh^2 2\xi_0 \cosh 2\xi - \cosh 2\xi_0 \cosh 4\xi \\ & + \cos 2\eta\, (\cosh 4\xi - \cosh^2 2\xi_0) \\ & + \cos 4\eta\, (\cosh 2\xi_0 - \cosh 2\xi) \end{aligned} \tag{13}$$

$$K = \frac{1}{4\sinh^2 2\xi_0}\frac{u_m c^2}{D}\frac{\partial \overline{s}}{\partial x_1} \tag{14}$$

The boundary conditions for s' are: (i) the flux of tracer becomes zero at the wall of the pipe, and (ii) s' is symmetrical with respect to $\xi = 0$ as follows:

$$\partial s'/\partial\xi = 0 \qquad \text{on} \quad \xi = \xi_0 \quad \text{and} \quad \xi = 0 \tag{15}$$

The solution of the differential equation (Eq. 12), which satisfies the boundary conditions Eq. 15, is obtained as follows:

$$\begin{aligned} s' = K \{ & \frac{1}{4}\cosh^2 2\xi_0 \cosh 2\xi - \frac{1}{16}\cosh 2\xi_0 \cosh 4\xi \\ & + \cos 2\eta \left(\frac{1}{12}\cosh 4\xi - \frac{1}{3}\cosh 2\xi_0 \cosh 2\xi + \frac{1}{4}\cosh 2\xi_0 \right) \\ & + \cos 4\eta \left(\frac{1}{12}\cosh 2\xi - \frac{1}{48\cosh 2\xi_0}\cosh 4\xi - \frac{1}{16}\cosh 2\xi_0 \right) \\ & + s_* \} \end{aligned} \tag{16}$$

in which s_* = an integral constant determined by the condition that the integral of s' in the cross-section is zero.

The longitudinal dispersion coefficient may be calculated from the deviations of concentration and velocity by the definition as Eq. 2. Eq. 2 is rewritten by the elliptical coordinate system as follows:

$$D_x = -4\int_0^{\xi_0}\int_0^{\pi/2} s'u'h^2 d\xi d\eta \Big/ \frac{1}{2}\pi \sinh 2\xi_0 \frac{\partial \bar{s}}{\partial x_1}$$

Substitution of Eqs. 11 and 14 into the above equation and use of Eqs. 4 and 16 lead the longitudinal dispersion coefficient in an elliptical pipe as follows:

$$\frac{D_x D}{u_m^2 c^2} = \frac{3\sinh 14\xi_0 - 11\sinh 10\xi_0 + 13\sinh 6\xi_0 - 5\sinh 2\xi_0}{73728\cosh 2\xi_0 \sinh^5 2\xi_0} = F(\xi_0) \tag{17}$$

By use of the hydraulic radius of elliptical pipe R, Eq. 17 is rewritten as

$$\frac{D_x D}{u_m^2 R^2} = \left(\frac{c}{R}\right)^2 F(\xi_0) \tag{18}$$

where

$$\frac{c}{R} = \frac{8}{\pi \sinh 2\xi_0}\int_0^{\pi/2}\sqrt{\sinh^2\xi_0 + \sin^2\eta}\; d\eta \tag{19}$$

Eq. 19 includes an elliptic integral, therefore, Eq. 18 cannot be expressed by the primary function. The non-dimensional dispersion coefficient is a function only of ξ_0, that is in turn, of Ar, because ξ_0 is a function only of Ar.

The value of the non-dimensional dispersion coefficient at Ar = 1 may be calculated by the use of Eq. 18 in order to make comparison with the value of a circular pipe obtained by Taylor (6) (Eq. 3). In this case, Ar = 1 corresponds to the case of $\xi_0 \to \infty$ in Eq. 18. Limit operation of $\xi_0 \to \infty$ changes Eq. 19 as follows:

$$\frac{c}{R} = 2\sqrt{2}\,\frac{1}{\sinh 2\xi_0}\sqrt{\cosh 2\xi_0}$$

Substitution of the above equation into Eq. 18, and the limit operation give us

$$\lim_{\xi_0 \to 0} \frac{D_x D}{u_m^2 R^2} = \lim_{\xi_0 \to 0} \frac{1}{3702} \left(\frac{\cosh 2\xi_0}{\sinh^2 2\xi_0} \right) \frac{\sinh 14\xi_0}{\cosh 2\xi_0 \sinh^5 2\xi_0} = \frac{1}{48} \qquad (20)$$

Eq. 20 shows that Eq. 18 includes the value of a circular pipe flow as a special case, i.e. Ar = 1.

LONGITUDINAL DISPERSION COEFFICIENT IN A RECTANGULAR PIPE

In this section, the longitudinal dispersion coefficient in a rectangular pipe of width a and height b shown in Fig. 2 will be obtained. Cartesian coodinate system shown in Fig. 2, x being chosen in the flow direction and y, z being in the plane of the cross-section is seleted. The molecular diffusion equation by the use of a coordinate system moving with the cross-sectional mean velocity $\bar{u}$ is as follows:

$$u' \frac{\partial \bar{s}}{\partial x_1} = \frac{D}{a^2} \left(\frac{\partial^2 s'}{\partial \eta^2} + \frac{\partial^2 s'}{\partial \zeta^2} \right) \qquad (21)$$

where $x_1 = x - \bar{u}t$; $\eta = y/a$; $\zeta = z/b$; and Ar = the aspect ratio defined as Ar = a/b.

The velocity dsitribution of laminar flow in a rectangular pipe is obtained in terms of double Fourier series as follows:

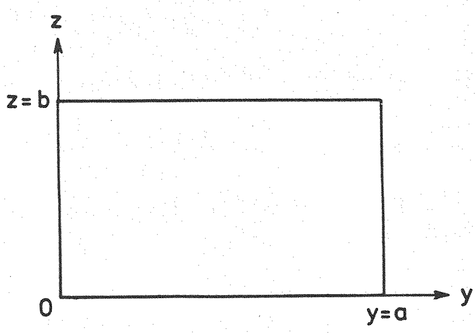

Fig. 2 Cartesian coordinate system for a rectangular pipe

$$u = u_m \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin(2m-1)\pi\eta \sin(2n-1)\pi\zeta \qquad (22)$$

where u_m = the maximum velocity in the cross-section and a_{mn} is Fourier coefficient given as follows:

$$\left. \begin{aligned} a_{mn} &= 1/[\, A_0 (2m-1)(2n-1) \{(2m-1)^2 + Ar^2(2n-1)^2\} \,] \\ A_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n-2} / [(2m-1)(2n-1) \{(2m-1)^2 + Ar^2(2n-1)^2\}] \end{aligned} \right\} \qquad (23)$$

The velocity deviation from the mean is obtained as follows:

$$u' = u_m \left(\sum_{p=1}^{\infty} b_p \cos 2p\pi\eta + \sum_{q=1}^{\infty} c_q \cos 2q\pi\zeta + \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} d_{pq} \cos 2p\pi\eta \cos 2q\pi\zeta \right) \qquad (24)$$

where

$$\left. \begin{aligned} b_p &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{8}{\pi^2} \frac{2m-1}{2n-1} \frac{1}{(2m-1)^2 - 4p^2} \\ c_q &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{8}{\pi^2} \frac{2n-1}{2m-1} \frac{1}{(2n-1)^2 - 4q^2} \\ d_{pq} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{16}{\pi^2} \frac{(2m-1)(2n-1)}{\{(2m-1)^2 - 4p^2\}\{(2n-1)^2 - 4q^2\}} \end{aligned} \right\} \qquad (25)$$

The boundary conditions for s' is given as follows:

$$\partial s'/\partial\eta = 0 \qquad \text{on } \eta = 0,\ 1$$
$$\partial s'/\partial\zeta = 0 \qquad \text{on } \zeta = 0,\ 1 \tag{26}$$

By substituting Eq. 24 into Eq. 21 and considering the boundary conditions Eq. 26, s' is derived as follows:

$$s' = -\frac{a^2 u_m}{D}\frac{\partial\overline{s}}{\partial x_1}\left(\sum_{p=1}^{\infty} B_p \cos 2p\pi\eta + \sum_{q=1}^{\infty} C_q \cos 2q\pi\zeta + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} D_{pq} \cos 2p\pi\eta \cos 2q\pi\zeta\right) \tag{27}$$

where Fourier coefficients B_p, C_q and D_{pq} are given as follows:

$$B_p = b_p/4p^2\pi^2;\quad C_q = c_q/4Ar^2q^2\pi^2;\quad D_{pq} = d_{pq}/4(p^2+ Ar^2q^2)\pi^2 \tag{28}$$

The longitudinal dispersion coefficient D_x in a rectangular pipe flow is defined as

$$D_x = -\int_0^1\int_0^1 s'u'd\eta d\zeta \Big/ \frac{\partial\overline{s}}{\partial x_1} \tag{29}$$

By substituting Eqs. 24 and 27 into Eq. 29 and carrying out integration of Eq. 29, the longitudinal dispersion coefficient is derived as follows:

$$D_x = \frac{a^2u_m^2}{D}\left(\frac{1}{2}\sum_{p=1}^{\infty} b_pB_p + \frac{1}{2}\sum_{q=1}^{\infty} c_qC_q + \frac{1}{4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} d_{pq}D_{pq}\right) \tag{30}$$

The non-dimensional expression of longitudinal dispersion coefficient is obtained by substitution of Eqs. 23, 25 and 28 into Eq. 30 as follows:

$$\frac{D_x D}{u_m^2R^2} = \frac{32\,(1 + Ar^2)^2}{A_0^{\,2}\,\pi^6}\left[\sum_{p=1}^{\infty}\frac{1}{p^2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2\{(2m-1)^2- 4p^2\}}\cdot\frac{1}{\{(2m-1)^2+ Ar^2(2n-1)^2\}}\right\}^2 + \frac{1}{Ar^2}\sum_{q=1}^{\infty}\frac{1}{q^2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^2\{(2n-1)^2- 4q^2\}}\cdot\frac{1}{\{(2m-1)^2+ Ar^2(2n-1)^2\}}\right\}^2 + 2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{p^2+ Ar^2q^2}\cdot\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\{(2m-1)^2- 4p^2\}\{(2n-1)^2- 4q^2\}\{(2m-1)^2+ Ar^2(2n-1)^2\}}\right\}^2\right] \tag{31}$$

where R = the hydraulic radius defined as $R = ab/2(a+b)$.

Eq. 31 tells us that the non-dimensional form of the longitudinal dispersion coefficient in a rectangular pipe flow is a function of Ar only in a similar mannar to that in an elliptical pipe flow.

RESULTS AND DISCUSSION

Fig. 3 shows the theoretical value of the non-dimensional longitudinal dispersion coefficient $D_x D/(u_m R)^2$ in a laminar flow which is obtained by Eq. 18 for an elliptical pipe and by Eq. 31 for a rectangular pipe. In Fig. 3, the value of the non-dimensional dispersion coefficient for two-dimensional (plane) Poiseulle flow $D_x D/(u_m R)^2 = 8/945$ is also plotted. In this case, 2R is the distance between two plates and the aspect ratio cannot be defined. Non-dimensional dispersion coefficient is, as is expected, symmetrical about Ar = 1 for both cases. It is already stated that the value of non-dimensional dispersion coefficient becomes minimum at Ar = 1 where it coincides with the value obtained by G.I.Taylor (6). For an elliptical pipe, the value of non-dimensional dispersion coefficient increases as either $Ar \to \infty$ or $Ar \to 0$. On the contrary, in the case of a rectangular pipe, non-dimensional dispersion coefficient has a minimal value at Ar = 1 and increases to a constant value as either $Ar \to \infty$ or $Ar \to 0$. The asymptotic value $D_x D/(u_m R)^2 = 0.064$ is about 7 times the value of plane Poiseulle flow. Comparison of the case for a circular pipe with the case for a square pipe reveals that the latter gives a larger value of non-dimensional dispersion coefficient. This is considered to be rational because the velocity deviation from the mean in a square pipe flow is relatively large compared with that in a circular pipe flow.

Two questions arise from results shown in Fig. 3.

(i) For the limit cases of $Ar \to \infty$ or $Ar \to 0$, the flow in a rectangular pipe is considered to coincide with the plane Poiseulle flow. There is, however, a considerable differenc between those two case regarding the non-dimensional dispersion coefficient.

(ii) For the limit cases of $Ar \to \infty$ or $Ar \to 0$, the value of non-dimensional dispersion coefficient in a rectangular pipe converges to a constant value. On the contrary, the corresponding value for an elliptical pipe diverges.

The following discussion is to answer to the above problems.

Fig. 4 shows the distribution of velocity on the major axis and the deviation disribution of concentration at which the mean concentration decreases with time, both in a rectangular pipe and an elliptical pipe of the same value of aspect ratio Ar. In a rectangular pipe of large value of Ar, velocity becomes small near side wall by viscous effect. However, velocty on the major axis becomes constant for wide region except for the side wall region and velocity distribution in the direction of the minor axis becomes similar to that of the plane Poiseulle flow. At the larger values of Ar, the region approximated by the plane Poiseulle flow becomes expanded, while the side wall regions still

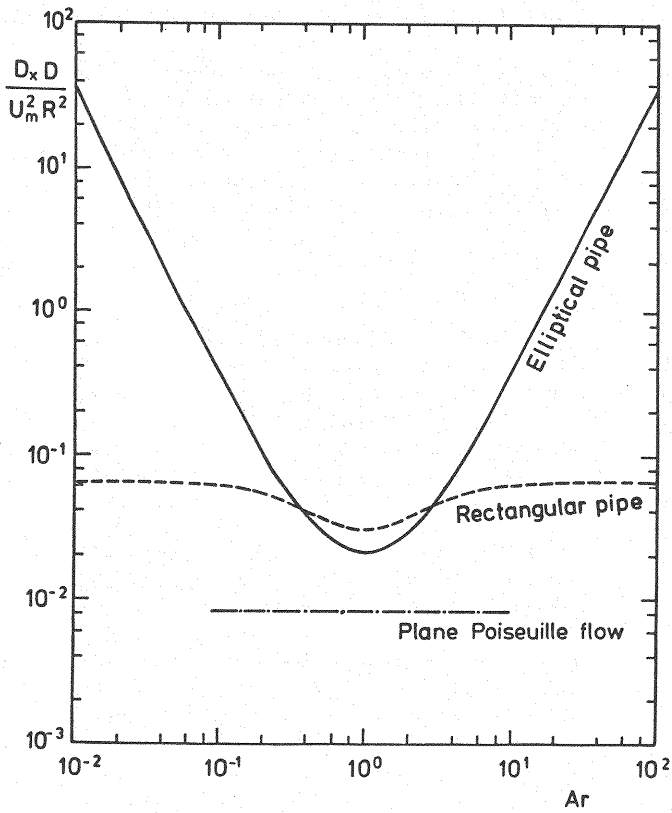

Fig. 3 Relation between the non-dimensional dispersion coefficient and the aspect ratio of cross-section of a pipe

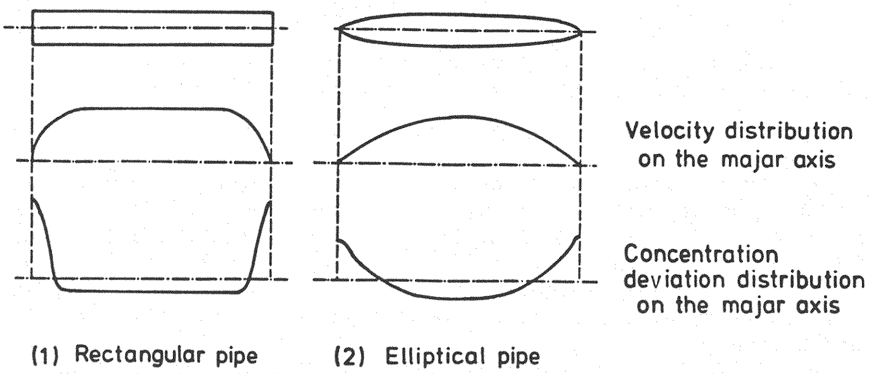

Fig. 4 Velocity and concentration deviation distributions on the major axis

exist. Therefore, tracer is retarded in the side wall regions and this mechanism makes non-dimensional dispersion coefficient become larger than that of the plane Poiseulle flow. As Ar becomes large, the area of the side wall regions becomes relatively small, and then the value of the non-dimensional dispersion coefficient converges to a constant. On the contrary, for the case of an elliptical pipe, the thickness to the direction of the minor axis varies continuously in the direction of the major axis and velocity also varies continuously corresponding to the thickness. Therefore, as Ar becomes large, the velocity deviation from the mean becomes large. This is the reason that non-dimensional dispersion coefficient becomes large as Ar becomes large.

Fischer (3) pointed out that the dispersion coefficients in natural rivers become about 10 times larger than those obtained by Elder (1), in which the log-law for the velocity distribution was assumed. He considered that in natural rivers, depths vary in the lateral direction and so the velocity deviation becomes large in that direction. Although large differences between laminar flow and turbulent flow exist, it is an interesting fact that at large (or small) value of Ar, the dispersion coefficient in an elliptical pipe, where the velocity deviation in the lateral direction is relatively large, is remarkably larger than that in a rectangular pipe.

CONCLUSION

The longitudinal dispersion coefficient of laminar flow both in an elliptical pipe and a rectangular pipe is theoretically studied and the relation between the dispersion coefficient and the aspect ratio of a pipe is discussed.

The conclusion obtained in this study is as follows:

1. The non-dimensional longitudinal dispersion coefficient in an elliptical pipe is analytically obtained. Its value is symmetrical about Ar = 1 and increases noticeably as both $Ar \to \infty$ and $Ar \to 0$. At Ar = 1, its value coincides with the value of a circular pipe obtained by Taylor (6).
2. The non-dimensional longitudinal dispersion coefficient in a rectangular pipe is also symmetrical about Ar = 1 and gradually increases to a constant value of about 0.064 as both $Ar \to \infty$ and $Ar \to 0$. At Ar = 1, the value of non-dimensional dispersion coefficient in a square pipe is greater than that in a circular pipe.
3. Comparison of the dispersion coefficients in two kinds of pipes which have different shapes of a cross-section shows that the longitudinal dispersion of laminar flow is noticeably affected by the geometrical shape of a pipe.

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APPENDIX - NOTATION

The following symbols are used in this paper:

a = major axis of an ellipse or width of a rectangular pipe;
A = area of a cross-section;
Ar = aspect ratio of a cross-section;
b = minor axis of an ellipse or height of a rectangular pipe;
D = molecular diffusion coefficient;
D_x = longitudinal dispersion coefficient;
h_ξ, h_η = scale factors of elliptical coordinate system;
R = hydraulic radius;
$\overline{s}$ = cross-sectional mean concentration of a tracer;
s' = deviation of concentration from the cross-sectional mean;
t = time;
$\overline{u}$ = cross-sectional mean velocity;
u' = velocity deviation from cross-sectional mean;
u_m = maximum velocity in the cross-section;
x = longitudinal coordinate taken along the mean flow direction;
y, z = Cartesian coordinates;
ξ, η = elliptical coordinates; and
η, ζ = non-dimensional Cartesian coordinates.