

# VELOCITY DISTRIBUTION, INTERFACIAL FRICTION AND ENTRAINMENT IN TWO-LAYERED STRATIFIED FLOWS

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## SYNOPSIS

Saltwedge type of stratified flows has long been providing the interesting themes in hydraulics. Recently, it is a source of the evolving turbulence studies such as organized motion and vortex pairing. In this paper, however, the classical problems of interfacial resistance and entrainment will be discussed.

In order to inquire these problems, we first examine the applicability of the Monin-Obukhov similarity theory on the velocity and temperature distributions in the diabatic atmospheric boundary layer flow to the saltwedge type of stratified flows. Although in such flows the assumption of constant momentum and buoyancy fluxes fails, no rigid boundary exists and the interfacial waves prevail, it is shown that the velocity profiles can be explained by the Monin-Obukhov theory.

A method of estimating the coefficients of interfacial friction and entrainment from the velocity profile measurements is proposed, applying the Monin-Obukhov theory. The coefficients thus determined compare well with the results obtained by the conventional method.

A new theoretical formula on the interfacial friction coefficient is derived by integrating the velocity profiles of Monin-Obukhov's similarity from the interface to the free surface. The present theory does not contain any arbitrary parameters, but it explains the results covering a wide range of flow conditions.

## INTRODUCTION

This paper aims at (i) discussing the applicability of the Monin-Obukhov theory on the velocity profile of diabatic atmospheric boundary layer flow to the saltwedge type stratified flow, (ii) proposing a method to experimentally determine the coefficient of the interfacial friction  $f_i'$  and the entrainment coefficient  $E$  and (iii) presenting a theory on the interfacial friction coefficient formula.

First, a short review of the present state of the arts will be given.

### *Validity of Monin-Obukhov's Theory in Stratified Channel Flows*

The Monin-Obukhov similarity theory on the diabatic atmospheric boundary layer flows is very well known. However, until recently its applications to other types of stratified flows are rather limited except a few. Kishi et al. (16, 25) and Hayashi and Yamada (9) have shown that the Monin-Obukhov "log+linear" law in the form of velocity defects fits the experimental data on the velocity profile of sediment laden flows. On the other hand, Coleman (2) explained the deviation of velocity distribution of suspended sediment flows from the log law in relation to Coles' wake law expressing the wake strength coefficient as a function of the overall Richardson number. As for flows stratified by salt solution, French (5) and McCutcheon (30) examined the validity of Monin-Obukhov's

similarity theory for unidirectional continuously stratified turbulent flows developed on a solid wall. And the latter attempted an extension of the theory to a flow with nonconstant buoyancy flux. As for a two-layered stratified flow of salt wedge type, Hino (10) applied the similarity theory to derive a formula on the interfacial friction coefficient and the writers (Hino et al. (14,15)) presented the experimental verification of the M-O theory. Kantha and Long (20) showed that a mixed layer produced by an oscillating grid is proportional to the Monin-Obukhov length scale. Csanady (4) reexamined the experimental data on the velocity distribution by Lofquist (29) and spoke of the Monin-Obukhov length  $L$  in relation to the entrainment rate but did not test the validity of the theory to the velocity distributions of a two-layered stratified flow.

### *Experiments on Interfacial Friction Coefficient and Entrainment Rate*

The interfacial friction coefficient, which is one of the most important parameters of this problem, has been a theme of numerous experimenters, among those are Keulegan (23), Lofquist (29), Iwasaki (17), Kaneko (18), Suga & Takahashi (41), Partheniades & Dermisis (35) and the authors (Hino, Nguyen and Nakamura (15)). Iwasaki (17) was the first to propose an empirical formula on the interfacial friction coefficient  $f_i'$  as a power law function of a non-dimensional parameter  $\psi$  (named Iwasaki number) which is the Reynolds number divided by the Richardson number and is equal to the inversed cubic square Keulegan number. At present, Kaneko's  $-1/2$  power law ( $f_i' = C\psi^{-1/2}$ ) is supported as the best one.

The interfacial friction coefficient is usually estimated experimentally from the momentum balance equations applied to each layer of the two-layered stratified flow, with the assumptions of negligible rates of fluid entrainment through the interface and of zero average velocity of the lower layer. However, this method frequently requires a troublesome measurement and is apt to bring about error.

The entrainment coefficient is another subject in the problem of two-layered stratified flows. As for the entrainment in a shearing type flows, a variety of experiments have been conducted beginning with the pioneering work by Kato and Phillips (22). Measurement of the entrainment rate is also difficult, because a specially designed apparatus such as an annular endless channel or a channel with moving screen is needed and the results depend seriously on the method of energy supply of mixing (annular tank with rotating screen, channel with moving screen and wind-wave channel) as pointed out by Kit, Berent and Vajda (26).

In this paper a method to experimentally estimate the coefficient  $f_i'$  as well as the entrainment coefficient  $E$  from the measurement of velocity profiles will be developed.

### *Theories on Interfacial Friction Coefficient*

The interfacial friction of the two-layered stratified flows is considered to be caused by (i) the laminar viscous friction, (ii) the internal wave form drag, (iii) the drag caused by the interfacial wave breaking and the entrainment and (iv) the Reynolds stress of turbulent motion. Keulegan (23) presented a model that the interfacial friction is caused by the energy dissipation due to the interfacial waves. Extending this idea, Shi-igai (39) first derived a theoretical formula on the interfacial friction coefficient. He computed the velocity field of internal waves of the non-viscous two-layered flow and then introduced the viscosity to obtain the energy dissipation rate from which the interfacial friction coefficient was derived as a function of the Iwasaki number  $\psi$ . The friction coefficient thus obtained is inversely proportional to the Iwasaki number. This tendency roughly coincides with the experimental data but the absolute value is very small compared to the experimental values. In order to obtain the same order of magnitude it is necessary to introduce a modification of proportionality coefficient forcing the interfacial wave steepness as high as one order of magnitude compared with a real wave steepness. His theory was further refined by Hamada (8) and Tamai (43). The latter also

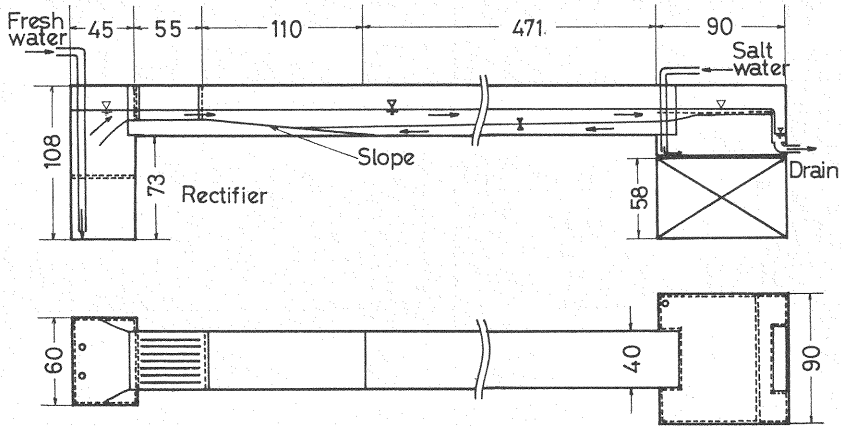


Fig. 1 : Experimental apparatus.

admits the above failure without modification of proportionality constant. From these results, the contribution of the internal wave drag may be concluded to be negligibly small except for a few special cases of low Reynolds number region.

Pedersen (36) and Kashiwamura (21) attempted to relate the entrainment process to the interfacial friction to obtain the formula, relating the friction coefficient to the entrainment coefficient  $E$  as

$$f_{ie}' = 2E \quad (1)$$

As Pedersen pointed out the entrainment process is a part of the interfacial friction and  $f_{ie}'$  given by the above equation can explain a small part of  $f_i'$ .

One of the writers (Hino, 10) has questioned why the interfacial friction coefficient is also a function of the Reynolds number in addition to the Richardson number while many phenomena of stratified flows such as the entrainment phenomenon are only dependent on the Richardson number and he considered that the energy dissipation due to the turbulent fluctuations of flow is more important than that due to the internal waves. Assuming that the Monin-Obukhov similarity theory is applicable in the upper layer of the salt wedge type of stratified flow and integrating it throughout the depth, he (Hino, 10) derived a new interfacial friction coefficient formula without introducing any arbitrary constants. The theory will be developed in an improved form in a subsequent chapter.

#### EXPERIMENTAL APPARATUS AND EQUIPMENT

##### *Experimental Flume*

A schematic diagram of the experimental flume made of transparent acrylic acid resin, together with the related equipments is shown in Fig. 1. The flume having a horizontal bottom is connected with two tanks at both ends: one upstream contains fresh water and another downstream contains salt water. Fresh water was supplied at constant discharge rate from a constant head tank to this upstream tank. The upper fresh water was discharged through the channel into the downstream tank and was drained from a weir installed in it. Discharge rate of drainage could be controlled by vertically sliding the level of the weir edge. In order to suppress the entrainment and mixing of salt water within the

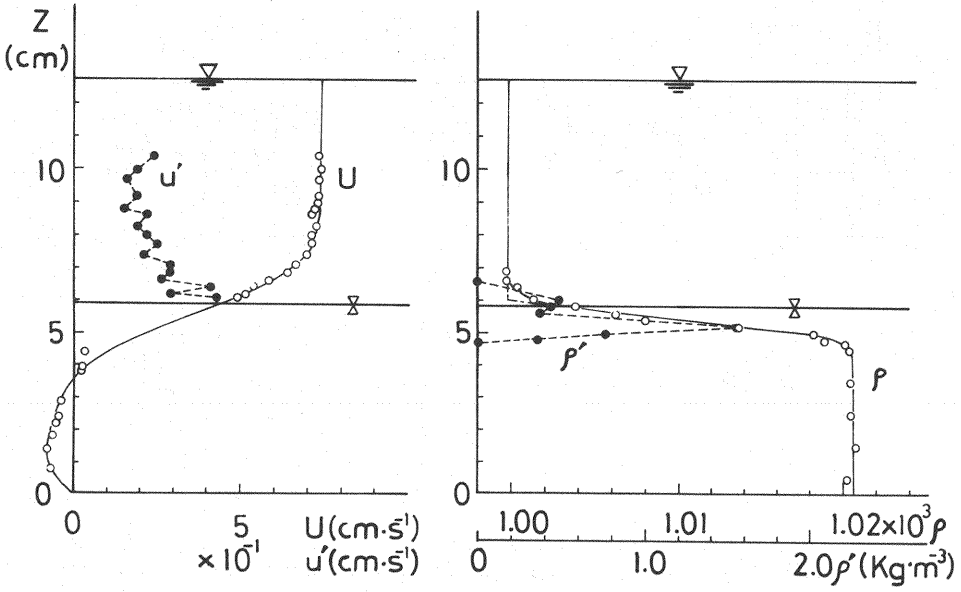


Fig. 2 : Mean velocity and density distribution profiles along with turbulent intensities for a *thin* interface case.

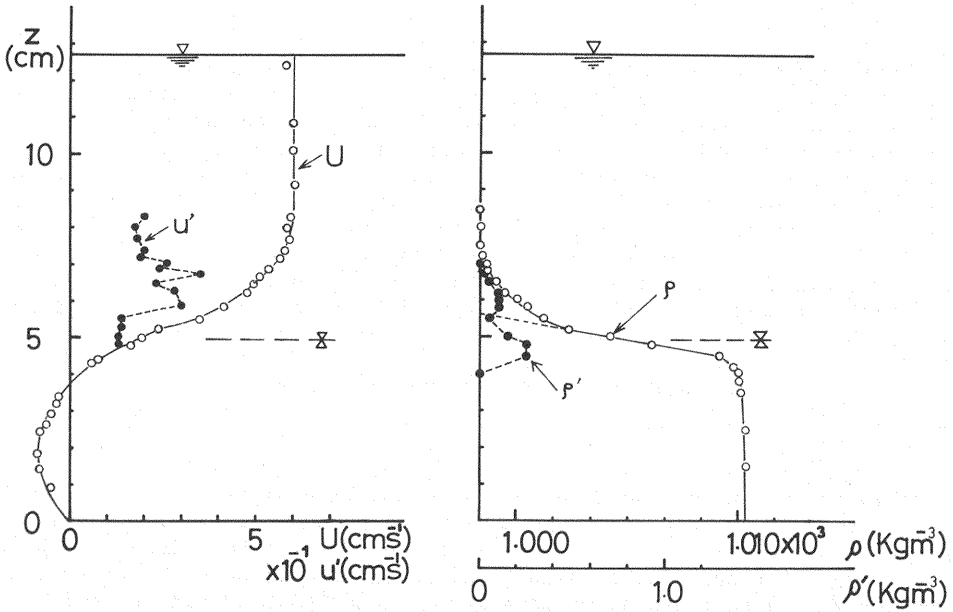


Fig. 3 : Mean velocity and density distribution profiles along with turbulent intensities for a *thick* interface case.

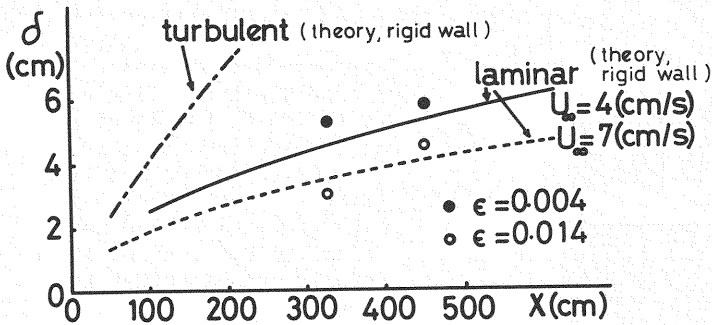


Fig. 4 : Thickness of the boundary or transitional layer,  $\delta$ , compared with the theoretical growth of  $\delta$  on a smooth rigid wall.

downstream tank by discharging the upper layer flow with higher velocity, a plastic net was installed at the exit of the flume just above the interface between salt and fresh water. By this method the amount of salt water withdrawn from the lower tank was remarkably reduced. To keep constant the depth of salt water in the downstream tank and in the flume, salt water was supplied at constant discharge rate into the downstream tank from another salt water supply tank. At the entrance of the flume a tapered bottom and a net spanned horizontally on the salt water interface also helped to keep the mixing of salt and fresh water at much lower level.

#### Measurement Technique

The velocities were measured by the double exposure hydrogen bubble technique. Two stroboscopes installed above and below the flume were flushed simultaneously two or three times at a constant interval while the camera shutter was opened to take 2 or 3 images of time lines of hydrogen bubbles on one frame. The cathode used to generate hydrogen bubbles was thin wires of tungsten with diameter of 0.020mm. In order to obtain clear photos by avoiding violent hydrogen bubble generation from a lower part of wire submerged in a salt water due to the difference of the electric conductivities of salt water and fresh water, the cathode wire was connected below the interface to a fine plastic wire.

The vertical profiles of salinity were measured by traversing a conductivity probe. Density was calculated from salinity.

#### VERTICAL PROFILES OF VELOCITY AND DENSITY

##### Vertical Profiles of Velocity

Vertical profiles of velocity and density were measured at 3 sections  $x = 50\text{cm}$ ,  $x = 80\text{cm}$  and  $x = 175\text{cm}$  from the exit of the main flume. Fig. 2 shows some examples of velocity distributions of the thin interface cases. While, Fig. 3 shows a velocity profile of the thick interface case.

##### (a) Boundary layer thickness growth

From the above figures it is seen that the boundary layer of the upper layer (the transitional region of velocity profile) does not reach to the free water surface. Although the Reynolds number of the upper layer flows are larger than 2000 and the flow may be identified to be turbulent from the velocity fluctuations, the growth rate of the boundary layer is low as if it were a laminar homogeneous flow on a solid wall and the boundary layer thickness seems to be constant over longer distances downstream (Fig.4). This fact has also been reported by Shi-igai (39) and Ashida and Egashira (1). From the field measurements on the velocity profiles of the saline wedges in the natural rivers by Suga (42), the same result can also be recognized. This fact may be considered to be the result of the flexible and movable interface and stratification effects.

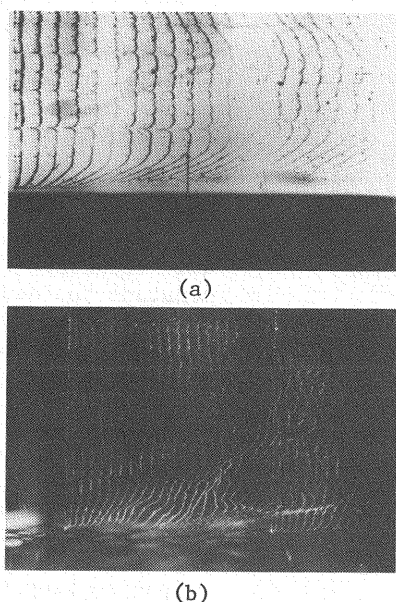


Fig. 5 : Upper layer flow visualized by hydrogen bubble technique. (a) Laminar flow ( $\epsilon = 0.09$ ,  $U_1 = 5.70\text{cm/s}$ ,  $Re = 3762$ ) (b) Turbulent *bursting-like* flow ( $\epsilon = 0.001$ ,  $U_1 =$

number  $Re_c = 450$  is not valid for the cases of large proportionate density effect  $\epsilon$ , although the range of  $\epsilon$  of his experiments overlaps with the authors' one. It is worthy to remember that Keulegan's experiments were performed in the closed flume whereas the authors' ones in the open channel. The upper layer flows along the upper solid wall in Keulegan's experiments are easier to become turbulent.

#### (b) Interfacial velocity

The ratio  $U_i/U_s$  of velocities ( $U_i$  is the velocity at the visual interface and  $U_s$  denotes that at the free water surface) varies from 0.38 to 0.62 for the present experiments. Tsubaki et al. (48) also reported the same result. This ratio in Keulegan's experiments is about 0.61. The phase velocity of interfacial waves closely depends on this interface velocity.

#### Vertical Profiles of Density

Figs. 2 and 3 also illustrate some examples of vertical profiles of density. As these figures show, in the thin interface case, the density distribution in its transitional region can be approximated by linear lines. The density profile of the thick interface is more complex as Fig. 3 shows. In the transitional region, there are two parts : linear part and exponential part. The transitional layer thickness defined by  $\delta_\rho = (\rho_2 - \rho_1)/(\partial\rho/\partial z)_{\max}$  ( $\rho$  = density;  $z$  = vertical coordinate;  $\rho_1$  = density of upper layer fluid; and  $\rho_2$  = density of lower layer fluid) varies in the range of 1.0 to 1.5cm in the present experiments for all three cases. This thickness tends to increase with decreasing of the proportionate density defect  $\epsilon$  and with the distance from the entrance of the flume.

#### (a) Definition of the interface

There are several ways to define the interface between fresh and salt water; e.g. The 'density interface' is the plane where the density gradient is maximum. The 'visual interface' is the one which is determined by eye discrimination without introducing any tracer. It is located, for the thin interface cases,

Lessen, Barcilon and Butler (28) have shown that the boundary layer thickness of turbulent stratified shearing layers in the atmosphere approaches constant due to the marginal stability criterion of the local Richardson number.

On the other hand, in the case of the extremely high proportionate density defect  $\epsilon = 0.09$ , although the Reynolds number  $Re$  of the upper layer is about 4000, the flow still remains likely laminar as Fig. 5 (Photo. 1) shows. The Reynolds number is defined here in terms of the depth, the average velocity and the kinematic viscosity of the upper layer flow. This fact shows that the density of the lower layer strongly affects the flow characteristics of the upper layer and Keulegan's (23) critical Reynolds

at the bottom of the upper fresh water layer, where the density is nearly equal to that of fresh water. In the thick interface case, the visual and density interfaces coincide with each other. Their locations are shown schematically in Fig. 6.

(b) *Transitional layer thickness ratio  $R$  of velocity and density*

In almost present experiments, the ratio  $R = l_u / l_\rho$  of the depth of the upper transitional region of velocity ( $l_u$ ) measured from the density interface, to that of density ( $l_\rho$ ) is greater than 4.0. In such a case, it is called in this paper the *thin density interface* case. For the case where a plastic screen is inserted at the head of saline wedge to augment turbulence generation, the ratio  $R$  is nearly equal to 1.0, this being the case of the *thick (density) interface*.

*Vertical Profiles of the Local Richardson Number and Velocity Fluctuations*

(a) *The local Richardson number profiles*

The local Richardson number is defined as follows :

$$R_i(z) = -g \frac{\partial \rho / \partial z}{\rho (\partial U / \partial z)^2} \quad (2)$$

where  $g$  = gravitational acceleration; and  $U$  = mean velocity at height  $z$ .

The local Richardson number distributions calculated from the velocity and density distributions are shown in Fig. 7. For the thin interface case

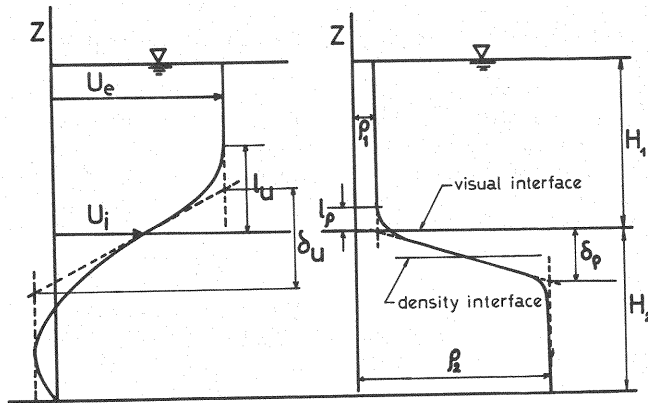


Fig. 6 : Difications of the interface positions, layer thickness and velocities.

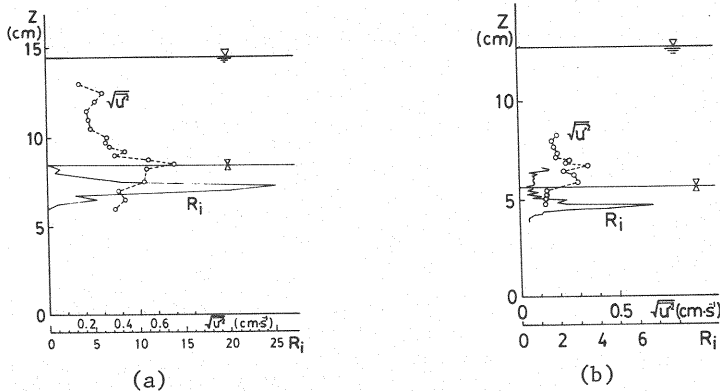


Fig. 7 : Profiles of the Richardson number. (a) *Thin* interface case. (b) *Thick* interface case.

(Fig. 7(a)), the local Richardson number  $R_i(z)$  becomes maximum in the neighborhood of the 'density interface' and decreases to zero at the upper 'visual interface'. On the other hand, for the thick interface case (Fig. 7(b)), the local Richardson number  $R_i(z)$  also takes the maximum value in the neighborhood of the density interface but it decreases to a finite value different from zero in the transitional region of velocity in the upper layer.

*(b) Velocity and density fluctuation profiles*

The vertical profiles of velocity fluctuation as well as those of density fluctuation are included in Figs. 2 and 3. From these figures it is seen that the velocity fluctuation becomes maximum at the visual interface and decreases towards both sides of the interface. The velocity fluctuation in the upper layer in the thick interface case seems to be smaller than that in the thin interface case due to the difference in the local Richardson number. The velocity fluctuation in the neighborhood of the interface is also affected by the interfacial waves, as will be presented elsewhere in relation to the turbulent structure.

Different from the velocity fluctuation, the density fluctuation takes the maximum value in the neighborhood of the density interface at which the density gradient is maximum. The density fluctuation is seen to be closely related with interfacial waves since the period of the salinity variation and that of the interfacial wave agree with each other. The period not only depends on the proportionate density defect  $\epsilon$  but also on the interface velocity  $U_i$ , because the phase velocity of interfacial waves varies with it.

*The Validity of the Monin-Obukhov Similarity Theory*

The Monin-Obukhov similarity theory on the diabatic atmospheric surface layer has been well established in the field of atmospheric turbulence. However, its prevalence to the other types of stratified turbulent flows is rather limited. This may be attributable to the probable failure of the basic assumptions in Monin-Obukhov's theory that the vertical fluxes of momentum and heat (or buoyancy) are constant within the range concerned or to the fact that the thickness of the layer where this assumption may be permissible is in effect very thin. In this section, the validity of the theory will be tested for the two-layered stratified flows such as saltwedges where the lower denser fluid has almost negligible velocity and thus non turbulent.

According to the Monin-Obukhov theory of which a short recapitulation is given in Appendix, the vertical profile of the velocity distribution is described by Eq. (3) (i.e. "log+linear" law),

$$\frac{U(z')}{U_*} = \frac{1}{\kappa} (\ln z' + \beta \frac{z'}{L}) + \text{const.} \quad (|\zeta| \leq 1) \quad (3)$$

where  $z'$  = vertical height measured from the origin;  $\zeta = z'/L$ ;  $U$  = longitudinal velocity;  $U_*$  = friction velocity;  $\beta$  = Obukhov constant; and  $L$  = Monin-Obukhov length defined by

$$L = - \frac{U_*^3}{\kappa B} = \frac{U_*^3}{\kappa g \bar{q} / \rho_0} \quad (4)$$

$$= \frac{U_*^3}{\kappa g \Delta \rho W_e / \rho_0} \quad (4a)$$

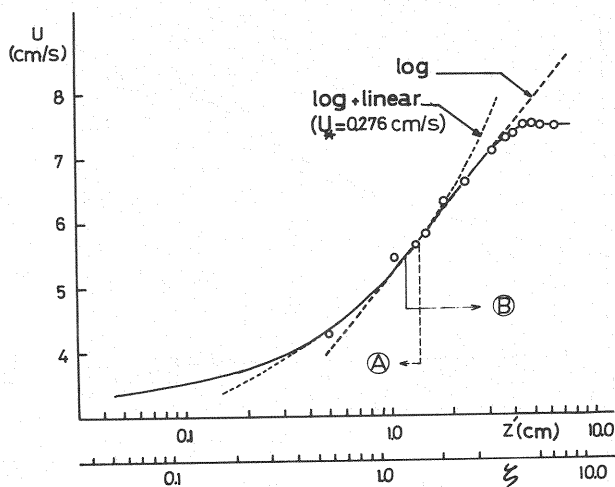
where  $\rho_0$  = reference density;  $B = -g \bar{q} / \rho_0$  = buoyancy flux; and  $\bar{q} = \overline{\rho' w'}$ ;  $\rho'$  and  $w'$  = the fluctuations in density and vertical velocity, respectively.  $\bar{q}$  is rewritten in terms of the density difference across the interface  $\Delta \rho = \rho_2 - \rho_1$  and the entrainment velocity  $W_e$  as  $\bar{q} = \Delta \rho W_e$ . Since  $\bar{q} > 0$  and  $\Delta \rho > 0$  in the stably stratified flow as treated here, the Monin-Obukhov length  $L$  is always positive in this case.

For the extremely high stable stratification, i.e.  $\zeta > 1$ , the recent measurement on the velocity distribution in atmospheric surface layer by Okamoto and Webb (34) and Kondo, Kanechika and Yasuda (27) demonstrate that the (second) "simple log" law holds for the large  $\zeta$ :

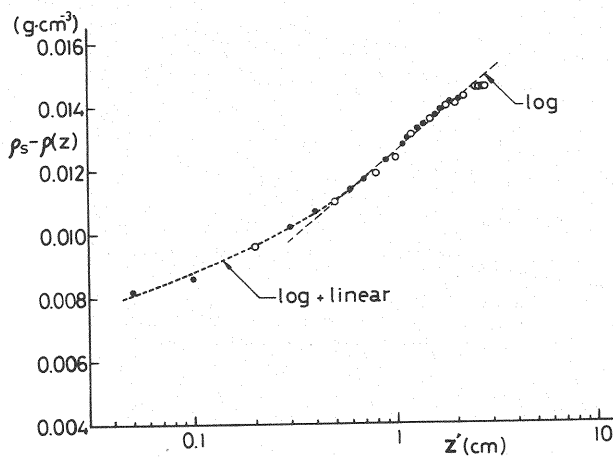
$$\frac{U(z')}{U_*} = \frac{(1+\beta)}{\kappa} \ln z' + \text{const.} \quad (\zeta \geq 1) \quad (5)$$

(a) *The origin of the vertical coordinate*

A problem encountered in applying the Monin-Obukhov theory to the two-layered stratified flow is how to choose the origin of height. In the present investigation, the origin was chosen at the visual interface which is determined by eye discrimination without introducing any tracer to the flow. The reason is as follows: (i) As presented in the previous section, the turbulent velocity fluctuation abruptly decreases below the interface. (ii) The range of the first logarithmic portion of the "log+linear" velocity



(a)



(b)

Fig. 8 : Applicability of the Monin-Obukhov similarity theory  
(a) Velocity profile and (b) density profile.

distribution law becomes widest when the interface is taken as the origin. In what follows the vertical coordinate from the interface is denoted by  $z'$ . According to the similarity theory, the distributions of velocity and density have the same form. Consequently, the origin can be determined more objectively from the distributions of velocity and density, if they are measured with the same good accuracy (Toba and Kawai, (47) and Thom, (45)).

(b) *The validity of the similarity theory*

Some measured distributions of velocity and density difference plotted on the semi-logarithmic paper are shown in Fig.8(a) and (b). Each curve of velocity distribution on the semi-logarithmic graph shows (A) a curved region of log+linear law, Eq. 3 (for  $|\zeta| \leq 1$ ), and (B) a linear region of simple log law, Eq. 5 (for  $|\zeta| \geq 1$ ).

Since for a region of small value of  $z'$ , the "log+linear" law reduced to the simple log law ( $U(z')/U_* \rightarrow 1/\kappa \cdot \ln z' + \text{const.}$ ), the velocity distribution is expressed by the "bi-log" law (Hino and Hagiwara (14)) as Fig.9(a) shows,

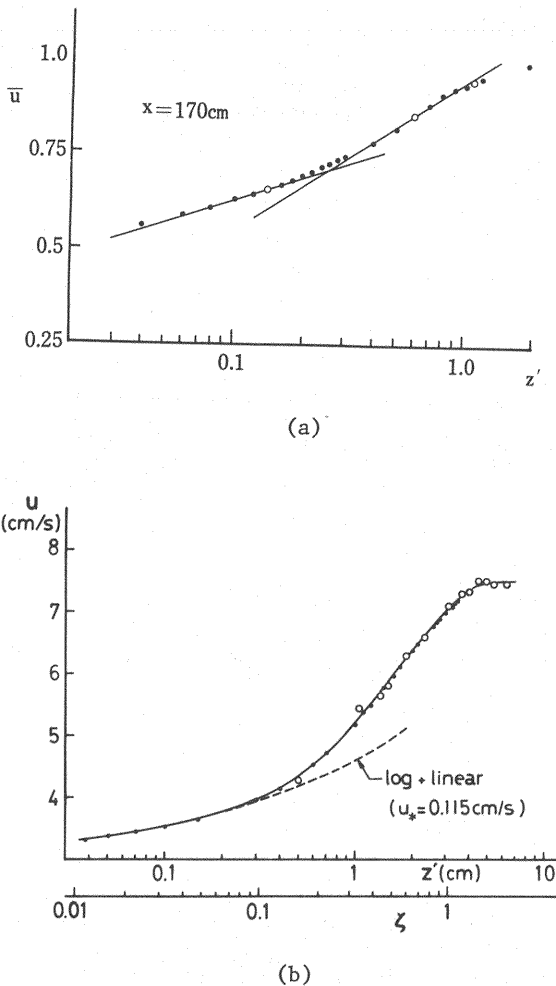


Fig. 9 : (a) "Bi-log" law expression of the velocity distribution of saltwedge type of stratified flow. (b) The log+linear law region becomes narrow if it is fitted in the lowest region.

$$\frac{U(z')}{U_*} = \begin{cases} \frac{1}{\kappa} \ln z' + \text{const.} & (|\zeta| < 1) \\ \frac{(1+\beta)}{\kappa} \ln z' + \text{const.} & (\zeta \geq 1) \end{cases}$$

If the velocity distribution is fitted by the log or log+linear law into the lowest part of the upper layer, the value of  $U_*$  in this region becomes less than that in the second or upper log law region (Hino and Hagiwara, (14)) and the range of the applicability of log+linear law narrows (Fig.9(b)). In this paper, the criterion is adopted that the value of  $U_*$  should be chosen so that the range of the log+linear law becomes widest (Fig.8(a)).

It is not sufficient to conclude from this figure alone that the theory is valid. In order to insist on the validity of the similarity theory of Monin-Obukhov in the present case, it is necessary, strictly speaking, to measure directly the stress  $\tau_z = \rho U_*^2$  and the buoyancy flux  $B$  at the interface. However, since we could not measure these values, its validity is proved indirectly as follows: (i) Eqs. 3 and 5 are assumed to be valid in the regions (A) and (B), respectively, (ii)  $U_*$ ,  $L$  and  $\beta$  are estimated from the measured distributions of velocity by the method to be described subsequently, and then (iii) the interfacial friction coefficient  $f_z'$  and entrainment coefficient  $E$  are calculated from  $U_*$ ,  $L$  and  $\beta$  thus determined and they are compared with the results reported by other researchers.

The values of  $U_*$ ,  $L$  and  $\beta$  are calculated from the measured velocity distributions as follows:

(i) The friction velocity  $U_*$  is determined by the trial and error method, i.e. a value is assumed for  $U_*$  and then  $(\kappa U(z')/U_* - \ln z')$  is computed. This calculation process is repeated until the variation of the value  $(\kappa U(z')/U_* - \ln z')$  determined from measured velocity against the height  $z'$  becomes linear throughout the region (A). The slope of such linear distribution of  $(\kappa U(z')/U_* - \ln z')$  against  $z'$  is just equal to  $S_1 = \beta/L$ .

(ii) Referring to Eq. 5, the coefficient  $\beta$  is calculated by Eq. 6 from the value  $U_*$  found in the first step (i) and the slope  $S_2$  in the region (B) of the linear portion of the velocity distribution plotted on the semi-logarithmic paper:

$$\beta = \kappa \frac{S_2}{U_*} - 1 \quad (6)$$

(iii) Finally the Monin-Obukhov length  $L$  is calculated from the values  $S_1$  found in the first step (i) and the above value of  $\beta$ :

$$L = \beta/S_1 \quad (7)$$

The interfacial friction coefficient  $f_z'$  and the entrainment coefficient  $E$  calculated from the above  $U_*$ ,  $L$  and  $\beta$  agree well with the results of other researchers obtained by the conventional methods, as will be shown in the next section.

The analysis on the measured velocity distributions of the present experiments shows that the "log+linear" and the "second log" laws are valid for the regions of  $0.3 < \zeta < 1$  and  $\zeta \geq 1$ , respectively. The value of  $\beta$  thus obtained ranges;

$$\beta = 0.4 \sim 1.7 \quad (8)$$

In the present experiments, the Reynolds stress was not measured. Goerges (6) measured the velocity and Reynolds stress distributions of the two-layered flows of which the upper fresh water layer almost remains rest. To estimate more directly the value of  $\beta$ , we analysed his experimental results. The value of  $\beta$  is calculated by the following formula,

$$\beta = \kappa \frac{S_2}{U_*} - 1 = \frac{\kappa S_2 (\rho)^{1/2}}{[-(\rho u'w')_i]^{1/2}} - 1 \quad (9)$$

where  $(-\rho u'w')_i$  = Reynolds stress at the interface; and  $S_2$  = slope of the region (B) of the velocity distribution plotted on the semi-logarithmic paper. His data yield the value;  $\beta = 0.83$ .

Therefore the average value of  $\beta$  may be decided to be 1.0, although the value of  $\beta$  in the case of atmospheric surface layer varies from 2 to 10,

$$\beta = 1.0 \quad (\text{average value})$$

From the results obtained above, it can be concluded that the Monin-Obukhov similarity theory is adaptable to the saline wedge type of stratified flows.

#### EXPERIMENTAL DETERMINATION OF INTERFACIAL FRICTION COEFFICIENT FROM VELOCITY MEASUREMENT

##### *Definition of the Interfacial Friction Coefficient*

The interfacial friction stress  $\tau_i$  acting on the interface of two-layered stratified fluid flows of almost zero average lower-layer velocity is expressed in terms of the interfacial friction coefficient  $f_i'$  as

$$\tau_i = f_i' \frac{\rho U_1^2}{2} \quad (10)$$

or

$$f_i' = 2 \left( \frac{U}{U_1} \right)^2 \quad (10a)$$

in which  $U_1$  = average velocity of the flowing layer.

The conventional method experimentally determining  $f_i'$  is based on the modified form of the momentum balance equations applied to each layer of the upper and lower layer flows, with the assumption of negligible rate of fluid entrainment through the interface (Hamada & Horiguchi (7)),

$$f_i' = \frac{(1 - \frac{U_1^2}{\epsilon g H_1}) \frac{dH_1}{dx} + \frac{U_2^2}{\epsilon g H_1} \frac{dH_2}{dx} - f_b \frac{U_2 |U_2|}{\epsilon g H_2}}{\frac{(U_1 - U_2) |(U_1 - U_2)|}{\epsilon g H_1} (1 + \frac{\rho_1}{\rho_2} \frac{H_1}{H_2})} \quad (11)$$

where  $f_b$  = friction coefficient at the bottom of the lower layer;  $U$  = layer averaged velocity;  $H$  = layer thickness;  $\rho$  = layer averaged density;  $\epsilon = (\rho_2 - \rho_1)/\rho$ ; and subscripts 1 and 2 refer to the upper and the lower layer, respectively. It is usually assumed that  $U_2 = 0$  and  $\rho_1 = \rho_2$  but  $\epsilon \neq 0$ .

From numerous experimental results, it is known that the coefficient  $f_i'$  is related to the non-dimensional number  $\psi$  (Iwasaki number named after the first proposer of this relationship) by the power law of the form (Iwasaki (17)),

$$f_i' = c \psi^{-n} \quad (n > 0) \quad (12)$$

where  $\psi$  is defined by

$$\psi = R_e R_i^{-1} \quad (13)$$

in which  $R = U_1 H_1 / \nu$  = Reynolds number; and  $R_i = \epsilon g H_1 / U_1^2$  = Richardson number of the upper layer, respectively. Among various proposals on the exponents  $n$  and the coefficient  $c$  in the expression of Eq. 12, the empirical formula proposed by

Kaneko (18)

$$f_i' = 0.2 \psi^{-0.5} \quad (14)$$

is supported as the best to fit many experimental data.

From their experiments on two-layered stratified flows in a closed duct, Partheniades & Dermisis (35) indicated that the correlation  $f_i'$  with  $Re \cdot R_1'^{-1}$  gives well defined mean lines with quite small scattering in comparison with plotting against  $Re \cdot R_1'^{-1}$  when  $R_1'$  is defined by

$$R_1' = \frac{\varepsilon g R_0}{U_1^2} \quad (15)$$

where  $R_0$  = hydraulic radius of the entire cross-section of the duct.

The scatter of experimental points in the  $f_i' - \psi$  diagram may be predominantly caused by the inaccuracy in measuring the depth especially in cases where the interfacial waves exist, by the effect of the lower layer velocity distribution and by the differences in definition of the position of interface among various researchers. For example, Suga & Takahashi (41) defined it as the plane at which the fluid density is equal to 90% the lower layer density, while for most of other researchers, it is either the visual interface or the density interface.

#### *Determination of the Interfacial Friction Coefficient from the Velocity Measurement*

In the present investigation, the interfacial friction coefficient is calculated from the velocity profile, instead of the conventional method described above. Since the slope of semi-logarithmic plot of velocity distribution in the second logarithmic law region is equal to  $S_2 = (U_* / \kappa) (1 + \beta)$ , the friction velocity is estimated from the following formula, if the value of  $\beta$  is determined,

$$U_* = \frac{\kappa}{1 + \beta} \cdot S_2 \quad (16)$$

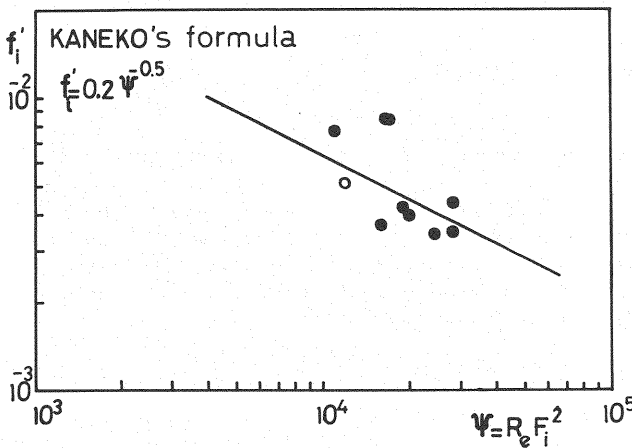


Fig. 10 : Experimental values of the interfacial friction coefficient determined by the present method which are compared with Kaneko's formula.

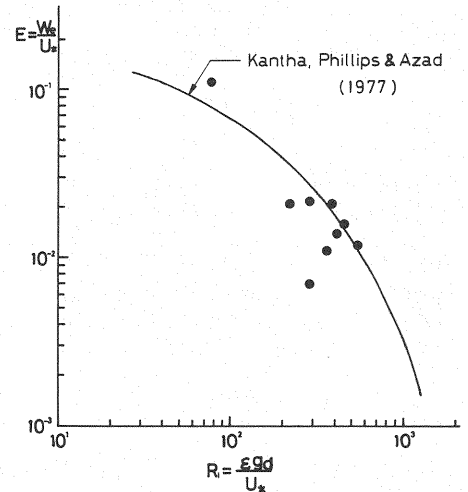


Fig. 11: Experimental values of the entrainment coefficient compared with the Kantha et al. results.

Then from Eqs. 10a and 16, the friction coefficient  $f_z'$  is derived as

$$f_z' = 2 \left\{ \frac{\kappa S_2}{(1 + \beta) U_1} \right\}^2 \quad (17)$$

In the case where the value  $\beta$  is not yet determined, the friction velocity  $U_*$  is calculated from the 'log+linear' region by the method of trial and error described in the previous section. After that, the friction coefficient  $f_z'$  is computed from Eq. 10a.

The present method has the three merits: (i) It is free from the error brought about by the inaccuracy, due to the presence of interfacial waves, of determining the negligibly small slope of interface  $dH_1/dx$ . (ii) There is no need to subtract the effect of sidewall friction, since  $\rho U_*^2$  is the very shear stress exerting at the interface. (iii) The friction coefficient determined from velocity profile gives a local value and not an average value such as that calculated by the conventional method.

In Fig.10 the friction coefficients determined from velocity profile are compared with the empirical formula proposed by Kaneko (18), Eq. 14. It is seen from the figure that the present result agrees well with the previous ones calculated by the conventional methods.

#### EXPERIMENTAL DETERMINATION OF ENTRAINMENT RATE FROM VELOCITY MEASUREMENT

##### *Concept and Definition of Entrainment Velocity*

Entrainment is commonly understood as the process of unilateral transport of fluid mass or salinity across a density interface. Fluid mass is transported, by a highly turbulent fluid motion on one side of the interface, from a low velocity or almost quiescent layer to the turbulent one.

In the usual diffusion process where turbulent fields are almost uniform, mass transport flux  $\bar{q}$  is brought about by turbulent velocity fluctuation  $w'$  of a fluid mass, its average distance of travel or mixing being  $l$ . The density difference between the distance  $l$  is given by  $\Delta\rho' = l d\rho/dz$ . Thus, the mass transport flux  $\bar{q}$  is written as the difference of upward and downward transport:

$$\bar{q} = [\overline{w'\rho(z)} - \overline{w'\rho(z+l)}] \quad (18)$$

$$\approx -\overline{w'l} \frac{d\rho}{dz} - \overline{\rho l} \frac{dw'}{dz}$$

$$\approx -\overline{w'\Delta\rho'} \quad (\because dw'/dz \approx 0)$$

$$= -K \frac{d\rho}{dz} \quad (19)$$

where  $K$  = eddy diffusion coefficient. This is the well-known gradient type of diffusion expression.

However, such a kind of expression is not suited for the case where the two layers of stratified flow are separated by a sharp interface and the density gradient becomes extremely intense and the gradient type expression loses its base of derivation. In such a case, it is very natural to consider that the turbulent motion entrains the denser (by  $\Delta\rho$ ) fluid at a rate of  $W_e$ , i.e.

$$\begin{aligned} \bar{q} &= \overline{w'\Delta\rho'} \\ &= W_e \Delta\rho \end{aligned} \quad (20)$$

where  $\Delta\rho'$  = instantaneous density fluctuation; and  $\Delta\rho$  = its average difference between the two layers. In other words, in the case of stationary stratified flow,  $W_e$  means the mean vertical velocity  $w$  across the density interface.

The concept of the entrainment velocity was initially introduced by Morton, Taylor and Turner (33) in relation to the broadening of buoyancy plumes as  $W_e = \bar{w}$  (the mean transversal velocity to a jet or plume axis).

The entrainment is also discussed in relation to the deepening of the density interface by the shearing action on the surface of water body (for instance, Kato and Phillips, (22)) as  $W_e = dh/dt$  where  $h$  means the depth of the halocline.

These expressions on  $W_e$  taken together indicate that the entrainment velocity may be defined in terms of the movement of the interface and the mean velocity across it as (Hino, (11)):

$$W_e = \frac{dh}{dt} + \bar{w} \quad (21)$$

Tamai and Kan (44) has given an extended expression as

$$W_e = \frac{Dh}{Dt} + \bar{w} = \frac{\partial h}{\partial t} + \bar{u} \frac{\partial h}{\partial x} + \bar{w} \quad (22)$$

#### *Empirical Formula on the Entrainment Velocity*

(i) The entrainment velocity  $W_e$  may be easily estimated to be proportional to  $w'$  and thus to  $U_*$ , defining the entrainment coefficient  $E_*$ ,

$$W_e = E_* U_* \quad \text{or} \quad E_* = W_e / U_* \quad (23)$$

Since in stratified flows turbulence is effected by the overall Richardson number  $R_i$  or its rooted inverse, the internal Froude number  $F_i$ , the entrainment coefficient may be a function of  $R_i$  or  $F_i$ . From experiments, it is known that  $E_*$  is expressed as

$$E_* \propto R_{i*}^{-n} \quad (24)$$

As for the value of the exponent  $n$ , it is known that  $n = 3/2$  for stirring tank experiments (Rouse and Dodu (1955), Cromwell (1960), Ellison and Turner (1959), Linden (1973), Wolanski and Brush (1975), Thompson and Turner (1975)).

For (recirculating) stratified shear flows with constant density gradient,  $n = 1$  (Kato and Phillips (22)). But for two-layered stratified flows in the same apparatus as Kato and Phillips' (22), Kantha, Phillips and Azad (19) reported that  $n = 1$  for the lower Richardson number range but gave the more general relation between the entrainment coefficient and the Richardson number. Price (38) and Thompson (46) argued that  $n = 1/2$ .

(ii) On the other hand, Suga and Takahashi (42) defined the entrainment coefficient  $E$  as the ratio of entrainment velocity  $W_e$  to the upper average velocity  $U_1$ ,

$$E = W_e / U_1 \quad (25)$$

and derived the following formula, from the large scale laboratory experiments and the field observations of saline wedges in natural rivers:

$$\begin{aligned} E &= 2 \times 10^{-3} R_i^{-3/2} \\ &= 2 \times 10^{-3} F_i^3 \end{aligned} \quad (26)$$

It is to be noted that the definition of the entrainment coefficient  $E$  in the above formula is different from  $E_*$ .

(iii) Keulegan (23) suggested an experimental formula of the mean mixing velocity  $U_m$  from the lower layer to the upper layer of arrested saline wedges,

$$U_m / U_1 = k (1 - 1.15 U_c / U_1) \quad (27)$$

where  $U_c$  is the critical velocity of mixing and the coefficient is  $k=3.5 \times 10^{-4}$  (Keulegan, (23) or  $2.14 \times 10^{-4}$  (Keulegan, (24))

(iv) Hino et al. (15) determined separately the upward and downward entrainment velocities of a saline wedge based on the conservation laws of mass and salinity applied between the two neighbouring sections. It is shown that the downward entrainment rate is smaller, but not negligible and of the same order with the upward entrainment rate. (The ordinate of the original figure of reference (15) was erroneously written and should be reduced one order of magnitude).

#### *Determination of the Entrainment Coefficient from the Velocity Measurement*

The conventional method determining the entrainment coefficient at the interface is based on the experiments on the interface deepening or the measurement of the difference of mass fluxes at two neighbouring control sections. It is much easier, compared to the conventional methods, to apply the Monin-Obukhov similarity theory to derive buoyancy flux from vertical profile of velocity. From the definition of the Monin-Obukhov length, Eq. 4a, the entrainment coefficient can be written as follows, applying Eq. 7,

$$E_* = \frac{W_e}{U_*} = \frac{U_*^2}{\kappa \epsilon g L} = \frac{S_1 U_*^2}{\kappa \epsilon g \beta} \quad (28)$$

Substitution of Eq. 16 into the above equation yields

$$E_* = \frac{\kappa S_1}{\epsilon g \beta} \left( \frac{S_2}{1 + \beta} \right)^2 \quad (29)$$

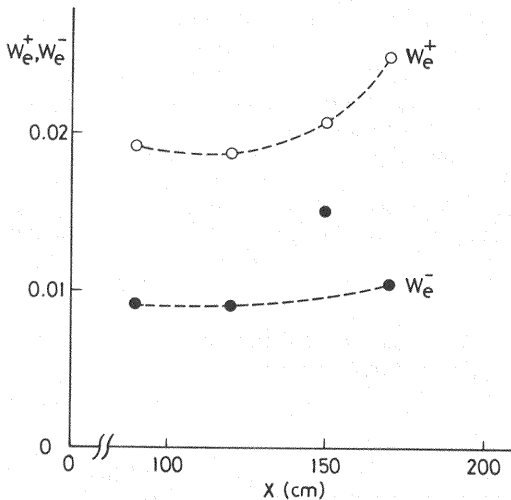


Fig.12: Upward and downward entrainment velocities. The ordinate of (15) is corrected.

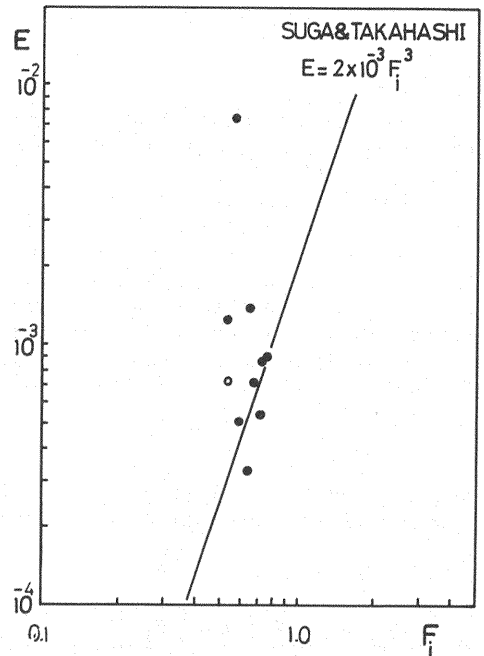


Fig.13: Experimental results on the entrainment coefficient derived from vertical velocity profiles compared with Suga and Takahashi's empirical formula.

For practical purposes, the entrainment coefficient is also defined to be the ratio  $E$  of the entrainment velocity  $W_e$  to the upper layer average velocity  $U_1$  as Eq. 25 . Therefore,

$$E = \frac{S_1}{\kappa \epsilon g \beta} \frac{U_*^3}{U_1} \quad (30)$$

Plotting of the entrainment  $E_*$  against the overall Richardson number  $R_{1*} = \epsilon g l_u / U_*^2$  is compared with Kantha, Phillips and Azad's result in Fig.11. The length  $l_u$  is the depth of the transitional region of velocity distribution. The entrainment coefficient  $E$  against the internal Froude number of the upper layer flow  $F_i = R_{1*}^{-1/2} = U_1 / \sqrt{\epsilon g H_1}$  is compared with Suga and Takahashi's empirical formula, Eq. 26, in Fig.13. These figures show that the entrainment coefficient determined from the velocity distribution almost agrees with that obtained by the conventional method.

### A NEW THEORY ON INTERFACIAL FRICTION COEFFICIENT

#### General Derivation

In this section, a new theory on the interfacial friction coefficient is derived from the similarity theory on the velocity distribution. Recently, one of the writers (Hino (10), also Hino et al. (15)) has derived an interfacial friction coefficient formula by integrating the vertical profile of velocity distribution from the interface to the free surface. This idea is parallel to the theoretical derivation of the Karman-Prandtl formula on the wall friction coefficient in turbulent pipe and channel flows.

In what follows, the previous theory will be further developed symbolically and then given in an improved form.

#### (a) Friction coefficient $f_i'$

The upper and lower layers of a two-layered stratified flow may be either turbulent or laminar. In our case of saltwedge type two-layered stratified flows, the upper layer is divided into three ranges; i.e., (i) outer layer, (ii) stably stratified layer and (iii) viscous interfacial layer (Fig.14). The velocity distributions in these three regions may be written symbolically as

$$\begin{aligned} \text{(i)} \quad \frac{U(z)}{U_*} &= \phi_1(z') \equiv \phi_2(h') = \text{const.} \\ &\quad (h' < z' \leq H_1 : \text{outer layer}) \\ \text{(ii)} \quad \frac{U(z)}{U_*} &= \phi_2(z') \quad (\delta_*' < z' \leq h' : \text{stratified layer}) \\ \text{(iii)} \quad \frac{U(z)}{U_*} &= \phi_3(z') \quad (0 < z' < \delta_*' : \text{viscous interfacial layer}) \end{aligned} \quad (31)$$

in which  $z'$  = vertical distance from the interface;  $H_1$  = depth of the upper layer;  $h'$  = height from the interface to the boundary of the stratified layer; and  $\delta_*'$  = depth of the transitional layer. In the first layer the velocity becomes constant and is equal to that at the outer edge ( $z' = h'$ ) of the second layer;  $\phi_1(z') \equiv \phi_2(h')$ .

The scaling parameters for the second and third layers are, respectively, the Monin-Obukhov length  $L$  and the length factor  $\nu/U_*$  formed from the governing factors in the viscous interfacial layer. Although there exists the interfacial waves, its contribution as a roughness parameter is assumed to be negligible due to a small difference between the velocity of the upper flow and internal wave celerity and to a lesser density difference compared to the case of the wind-water waves. Consequently, we can rewrite the velocity distribution functions  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  as:

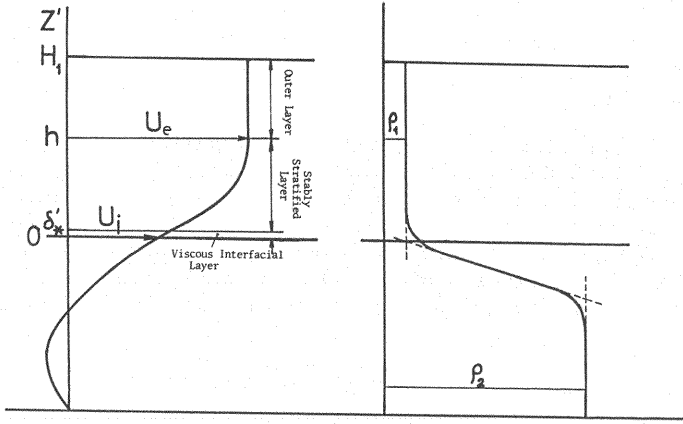


Fig. 14 : Definition of the three regions of velocity distribution used in the derivation of the interfacial friction formula.

$$\phi_1(z') = \phi_2 \left( \frac{h'}{L} \right) = \text{const.}$$

$$\phi_2(z') = \phi_2 \left( \frac{z'}{L} \right) \quad (32)$$

$$\phi_3(z') = \phi_3 \left( \frac{U_* z'}{\nu} \right)$$

It is to be noted that the interface velocity  $U_i$  is also included in the velocity distribution function  $\phi_3$  in form of  $U_i/U_*$ .

By integrating Eq. 31 from the interface ( $z' = 0$ ) to the free surface ( $z' = H_1$ ), the mean velocity is given as

$$\begin{aligned} U_1 &= \frac{U_*}{H_1} \left[ \int_0^{\delta_*'} \phi_3(z') dz' + \int_{\delta_*'}^{h'} \phi_2(z') dz' + \int_{h'}^{H_1} \phi_1(z') dz' \right] \\ &= \frac{U_*}{H_1} \left[ \int_0^{H_1} \phi_u(z') dz' \right] \end{aligned} \quad (33)$$

in which  $\phi_u$  represents the velocity distribution functions;  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  of the upper layer. From Eq. 33 the interfacial friction coefficient  $f_i'$  can be derived as follows:

$$\begin{aligned} f_i' &= 2 \left( \frac{U_*}{U_1} \right)^2 \\ &= 2H_1^2 \left[ \int_0^{H_1} \phi_u(z') dz' \right]^{-2} \end{aligned} \quad (34)$$

$$f_i' = 2 \left[ \left(1 - \frac{h'}{H_1}\right) \phi_2 \left(\frac{h'}{L}\right) + \frac{L}{H_1} \left\{ f_2 \left(\frac{h'}{L}\right) - f_2 \left(\frac{\delta_*'}{L}\right) \right\} + \frac{\nu}{U_* H_1} f_3 (U_* \delta_*' / \nu) \right]^{-2} \quad (35)$$

where  $f_i (\cdot)$ 's mean

$$\begin{aligned} f_2 (\cdot) &= \int \phi_2 (\zeta) d\zeta \\ f_3 (\cdot) &= \int \phi_3 (\zeta) d\zeta \end{aligned} \quad (36)$$

Now, the Monin-Obukhov length is a measure of the range within which the Monin-Obukhov theory is valid (Monin & Yaglom (32), Hino, Honma and Senshu (12)) and the parameter  $h'/L$  is a universal constant of the order of unity, then we write

$$J = h'/L \quad (= \text{a constant}) \quad (37)$$

The concept that  $J$  is a universal constant is equivalent to the marginal stability criterion proposed by Lessen, Barcilon and Butler (28).

On the one hand,  $\delta_*'$  may be considered to be equivalent to the depth of the viscous sublayer and determined by the critical Reynolds number  $Re_{\delta_*'}$ ,

$$Re_{\delta_*'} = \frac{U_* \delta_*'}{\nu} \quad (= \text{a universal const.}) \quad (38)$$

Consequently, the terms  $\delta_*'/L$ ,  $H_1/L$ ,  $\nu/U_* H_1$  and  $h'/H_1$  in Eq. 35 can be transformed in terms of these parameters and the universal constants as follows;

$$\begin{aligned} \zeta_{\delta_*'} &= \frac{\delta_*'}{L} \\ &= \kappa \cdot \frac{U_* \delta_*'}{\nu} \cdot \frac{We}{U_1} \cdot \frac{\epsilon g H_1}{U_1^2} \cdot \frac{\nu}{U_1 H_1} \cdot \left(\frac{U_1}{U_*}\right)^4 \\ &= \kappa \cdot Re_{\delta_*'} \cdot E(R_1) \cdot R_1 \cdot Re_e^{-1} \cdot (f_i'/2)^{-2} \\ &= \kappa \cdot Re_{\delta_*'} \cdot E(R_1) \cdot \psi^{-1} \cdot (f_i'/2)^{-2} \end{aligned} \quad (39a)$$

$$\begin{aligned} \zeta_{H_1} &= \frac{H_1}{L} \\ &= \kappa \cdot \frac{\epsilon g H_1}{U_1^2} \cdot \frac{We}{U_1} \cdot \left(\frac{U_1}{U_*}\right)^3 \\ &= \kappa \cdot R_1 \cdot E(R_1) \cdot (f_i'/2)^{-3/2} \end{aligned} \quad (39b)$$

$$\begin{aligned} \frac{\nu}{U_* H_1} &= \frac{\nu}{U_1 H_1} \cdot \frac{U_1}{U_*} \\ &= Re_e^{-1} \cdot (f_i'/2)^{-1/2} \end{aligned} \quad (39c)$$

$$\begin{aligned} \frac{h'}{H_1} &= \frac{h'}{L} \cdot \frac{L}{H_1} \\ &= \frac{J}{\kappa} \cdot R_1^{-1} \cdot E^{-1}(R_1) \cdot (f_i'/2)^{3/2} \end{aligned} \quad (39d)$$

in which  $R_e = U_1 H_1 / \nu =$  the Reynolds number;  $R_i = \varepsilon g H_1 / U_1^2 =$  the Richardson number of the upper layer flow; and  $E = W_e / U_1 =$  the entrainment coefficient given as a function of  $R_i$ . Therefore Eq. 23 becomes an implicit expression on the formula of the interfacial friction coefficient  $f_i'$  as a function of the Reynolds number  $R_e$  and the Richardson number  $R_i$  along with universal constants  $\kappa, J, R_{e\delta_*}'$ :

$$f_i' = F_n (R_e, R_i, \frac{U_i}{U_1}; \kappa, J, R_{e\delta_*}') \quad (40)$$

It must be emphasized that there is left no freedom to choose arbitrarily the values of constants contained in Eq. 40. The parameters  $\kappa, J$  and  $R_{e\delta_*}'$  are all the universal constants which have been determined already from the other phenomena.

(b) *Other expression of interfacial friction coefficient,  $\lambda_i$*

From the theoretical view point, it might be better to define the friction coefficient in terms of  $U_e$  (the velocity at the outer edge of the second layer) instead of  $U_1$  (the mean velocity in the upper layer), because the velocity in the first layer is constant, and to define the Richardson number as well as the entrainment coefficient in terms of the friction velocity  $U_*$  and the depth  $h'$ , i.e.,

$$\lambda_i = 2 \left( \frac{U_*}{U_e} \right)^2 \quad (41)$$

$$R_{i_*}' = \frac{\varepsilon g h'}{U_*^2} \quad \text{or} \quad R_{i_*}' = \frac{\varepsilon g h'}{U_e^2} \quad (42)$$

$$E_* = \frac{W_e}{U_*} \quad (43)$$

The constant velocity in the first layer can be rewritten as follows;

$$\frac{U_e}{U_*} = \phi_2 \left( \frac{h'}{L} \right) = f_n \left( \frac{h'}{L}, \frac{\delta_*'}{L}, \frac{U_i}{U_*} \right) \quad (44)$$

In this case, the term  $\delta_*'/L$  can be rewritten:

$$\begin{aligned} \frac{\delta_*'}{L} &= \kappa \frac{U_* \delta_*'}{\nu} \cdot \frac{\nu}{U_e h'} \cdot \frac{\varepsilon g h'}{U_*^2} \cdot \frac{W_e}{U_*} \cdot \frac{U_e}{U_*} \\ &= \kappa \cdot R_{e\delta_*}' \cdot R_e'^{-1} \cdot R_{i_*}' \cdot E(R_{i_*}') \cdot \left( \frac{\lambda_i}{2} \right)^{-1/2} \end{aligned} \quad (45)$$

$$\frac{U_i}{U_*} = \frac{U_i}{U_e} \cdot \frac{U_e}{U_*} = \frac{U_i}{U_e} \cdot (2/\lambda_i)^{1/2} \quad (46)$$

in which the Reynolds number  $R_e'$  is defined by  $R_e' = U_e h' / \nu$ . Therefore, the interfacial friction coefficient  $\lambda_i$  has generally the following form,

$$\lambda_i = G_n (R_e', R_{i_*}', \frac{U_i}{U_e}; \kappa, J, R_{e\delta_*}') \quad (47)$$

### Formula on the Interfacial Friction Coefficient

In what follows, the velocity distribution functions  $\phi_i$  in the above mentioned formula will be given the explicit forms.

#### (a) Application of the "log + linear" law

In the upper part of the second layer (ii), the "second log" law may be a better expression than the mere extension of the "log+linear" as shown in the previous chapter. But for the sake of simplicity, the velocity distribution in this layer is first approximated only by the "log+linear" law; i.e.

$$\phi_{2a}(\zeta) = \frac{1}{\kappa} [\ln(\zeta/\zeta_{\delta_*}') + \beta(\zeta - \zeta_{\delta_*}')] + \left( \frac{U_i}{U_*} + \text{Re}_{\delta_*}' \right) \quad (48)$$

$$(\zeta_{\delta_*}', < \zeta \leq J')$$

in which  $\zeta_{\delta_*}' = \delta_*'/L$ ;  $J' = h''/L$ ; and  $h'' =$  height at which the velocity described by the "log+linear" law reaches a constant velocity  $U_e$ . Therefore, the constant velocity  $U_e$  in the first layer (i) becomes:

$$\frac{U_e}{U_*} = \phi_{2a} \left( \frac{h''}{L} \right) = \frac{1}{\kappa} [\ln(J'/\zeta_{\delta_*}') + \beta(J' - \zeta_{\delta_*}')] + \left( \frac{U_i}{U_*} + \text{Re}_{\delta_*}' \right) \quad (49)$$

$$(J' < \zeta < \zeta_{H1})$$

Finally since the flow in the third layer is strongly affected by viscosity, the velocity distribution can be considered to be linear one:

$$\frac{U}{U_*} (U_* z' / \nu) = \phi_3 \left( \frac{U_* z'}{\nu} \right) = \frac{U_* z'}{\nu} + \frac{U_i}{U_*} \quad (0 \leq \frac{U_* z'}{\nu} \leq \text{Re}_{\delta_*}') \quad (50)$$

Substituting Eqs. 48, 49 and 50 into Eq. 35, we obtain the explicit expression on the interfacial friction coefficient  $f_i'$

$$\left( 1 - \frac{U_i}{U_1} \right) / \sqrt{f_i'/2} = - \frac{1}{\kappa} \cdot \frac{J'}{\zeta_{H1}} \cdot \left( 1 + \frac{\beta J'}{2} \right) + \frac{1}{\kappa} (\ln J' + \beta J') - \frac{1}{\kappa} (\ln \zeta_{\delta_*}' + \beta \zeta_{\delta_*}') \\ + \text{Re}_{\delta_*}' + \frac{\zeta_{\delta_*}'}{\zeta_{H1}} \left( \frac{1}{\kappa} + \frac{\beta}{2\kappa} \zeta_{\delta_*}' - \frac{1}{2} \text{Re}_{\delta_*}' \right) \quad (51)$$

Now substituting the value of  $\zeta_{\delta_*}'$ ,  $\zeta_{H1}$  described by Eqs. 39a and 39b into Eq. 51 and neglecting the last term in the right side (since  $\zeta_{\delta_*}' \ll \zeta_{H1}$ ), we obtain

$$\frac{1}{\kappa} (\ln \psi f_i'^2) - \left( 1 - \frac{U_i}{U_1} \right) f_i'^{-1/2} - \frac{\ln E}{\kappa} - \beta \text{Re}_{\delta_*}' E (\psi f_i'^2)^{-1} \\ - \left( 1 + \frac{\beta J'}{2} \right) \frac{J'}{\kappa^2} (E R_1)^{-1} f_i'^{3/2} = -D \quad (52)$$

in which  $f_i' = f_i'/2$  and

$$D = \text{Re}_{\delta_*}' + \frac{\beta J'}{\kappa} + \frac{1}{\kappa} \ln \frac{J'}{\kappa \text{Re}_{\delta_*}'} \quad (= \text{const.}) \quad (53)$$

(b) *Application of the "log + linear" and the "second log" laws*

The more exact way of deriving the formula on  $f_{ia}'$  is to connect the "log + linear" law with the second log law until the constant velocity  $U_e$  is reached; i.e.

$$\frac{U(\zeta)}{U_*} = \phi_{2a}(\zeta) = \frac{1}{\kappa} [\ln(\zeta/\zeta_{\delta_*}) + \beta(\zeta - \zeta_{\delta_*})] + \left( \frac{U_i}{U_*} + Re_{\delta_*} \right) \quad (\zeta_{\delta_*} < \zeta \leq \zeta_*) \quad (54a)$$

$$\frac{U(\zeta)}{U_*} = \phi_{2b}(\zeta) = \frac{1}{\kappa} [(1+\beta)\ln\zeta - \ln\zeta_{\delta_*} + \beta(\zeta_* - \zeta_{\delta_*} - \ln\zeta_*)] + \left( \frac{U_i}{U_*} + Re_{\delta_*} \right) \quad (\zeta_* < \zeta \leq J) \quad (54b)$$

$$\frac{U_e}{U_*} = \phi_{2b}\left(\frac{h'}{L}\right) = \frac{1}{\kappa} [(1+\beta)\ln J - \ln\zeta_{\delta_*} + \beta(\zeta_* - \zeta_{\delta_*} - \ln\zeta_*)] + \left( \frac{U_i}{U_*} + Re_{\delta_*} \right) \quad (J < \zeta \leq \zeta_{H1}) \quad (55)$$

where  $\zeta_* = z_*/L$  is a constant of about unity. At the heights  $h''$  and  $h'$ , the velocities given by Eqs. 49 and 55 must coincide; i.e.

$$\phi_{2a}(J') = \phi_{2b}(J) \quad (56)$$

Substitution of Eqs. 50, 54a, 54b and 55 into Eq. 35 yields similarly the more exact formula on the interfacial friction coefficient

$$\begin{aligned} \left(1 - \frac{U_i}{U_*}\right) / \sqrt{f_i'/2} = & - \frac{1}{\kappa \zeta_{H1}} [J(1+\beta) - \beta \zeta_* (1 - \frac{\zeta_*}{2})] + \frac{1}{\kappa} [\ln J + \beta(\ln \frac{J}{\zeta_*} + \zeta_*)] \\ & - \frac{1}{\kappa} (\ln \zeta_{\delta_*} + \beta \zeta_{\delta_*}) + Re_{\delta_*} \\ & + \frac{\zeta_{\delta_*}'}{\zeta_{H1}} \left( \frac{1}{\kappa} + \frac{\beta}{2\kappa} \zeta_{\delta_*}' - \frac{1}{2} Re_{\delta_*} \right) \end{aligned} \quad (57)$$

It is to be noted that Eq. 51 is different from Eq. 57 only in two first terms in the right-hand sides and that  $J$  is larger than  $J'$ , because the "log+linear" law reaches the constant velocity  $U_e$  at a shorter distance than the "second log" law. In usual cases, the last term in the right-hand side of Eqs. 51 and 57 may be negligible since  $\zeta_{\delta_*}'/\zeta_{H1} = \delta_*'/H_1 \ll 1$  and the values of the terms in the last parentheses are not so large.

(c) *Extreme cases*

It is useful to examine some extreme cases of Eq. 52. In the cases that  $h'/H_1 \ll 1$  and  $\delta_*'/L \ll 1$ , Eq. 52 reduces to,

$$\frac{1}{\kappa} (\ln \psi f_i'^2) - \left(1 - \frac{U_i}{U_1}\right) f_i'^{-1/2} - \frac{1}{\kappa} \ln E = -D \quad (58)$$

If the sum of last two terms in the left hand side of Eq. 58 is negligibly small, Eq. 58 yields an approximate formula

$$f_i' = 2e^{-\kappa D/2} \psi^{-1/2} \quad (59)$$

The relationship ( $f_i' \propto \psi^{-1/2}$ ) is exactly identical with the mostly supported empirical formula proposed by Kaneko (18) which fits considerably well for wide ranges of experimental data. The same result is also obtained in the case of Eq. 57.

In the limiting case that  $E$  approaches zero ( $E \rightarrow 0$ ), the Monin-Obukhov length  $L$  reaches to infinity ( $L \rightarrow \infty$ ) and the scaling law which derives this formula loses its foundation and the roughness length  $k_s$  which is caused by the wave drag of internal waves but discarded in this theory must be taken into account.

### Numerical Results

In order to calculate-numerically the theoretical formula on  $f_i'$ , (Eq. 52 or Eq. 57), it is necessary to know the universal constants  $\kappa$ ,  $\beta$ ,  $J$ ,  $Re_{\delta_*}$ , the values of  $U_i/U_1$  and the entrainment coefficient  $E$ . From the present experiments we obtain

$$\beta = 1.0$$

$$J = 3.0 \quad (\text{or equivalently } J' = 5.0)$$

And it is well known that

$$\kappa = 0.4$$

$$Re_{\delta_*} = 10.$$

As for the entrainment coefficient, the formula by Suga and Takahashi (Eq. 26) was used here. Since the value of the relative interfacial velocity  $U_i/U_1$  varies depending on the flow conditions of the lower layer, the experimental result presented in a previous chapter was used as a parameter,

$$U_i/U_1 = 0.3 \sim 0.6$$

The theoretical interfacial resistance formula derived above (Eq. 52) is compared with the experimental values obtained in laboratories and in natural rivers in Fig. 15. As the figure shows, it should be emphasized that no arbitrary constants are used in this theory, but the theory can explain the results in the wide range of the flow characteristics.

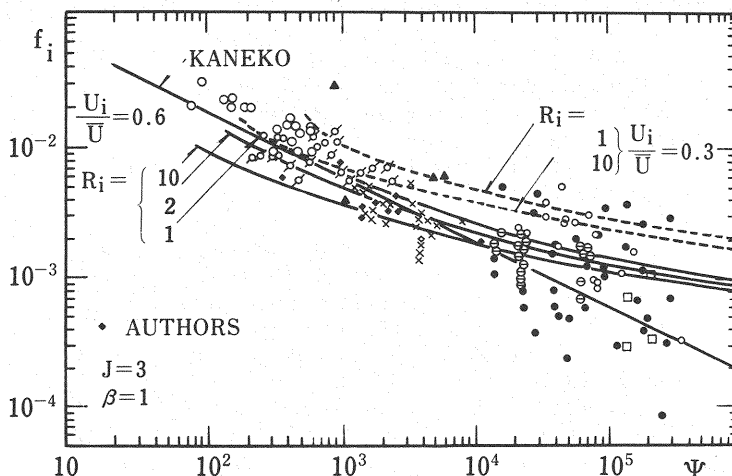


Fig. 15 : Comparison of the theoretical interfacial friction coefficient with experimental values in laboratories and natural saline wedges.

## CONCLUSIONS

The results obtained are summerized as follows.

1. The Monin-Obukhov similarity theory on the diabatic atmospheric boundary layer flow is applicable to the velocity profiles in the upper layer of two layered saltwedge type of stratified flows where no rigid boundary exists and interfacial waves prevail and in which a constant momentum and buoyancy flux assumption fails.

2. The coefficients of interfacial friction and entrainment can be determined from the vertical profile of velocity by the application of the Monin-Obukhov similarity theory. The results thus obtained agree well with those determined by the conventional methods.

3. A theory on the interfacial friction coefficient is derived from the similarity theory of velocity distribution. The theory does not include any arbitrary parameter, but can explain the results in laboratories and natural rivers in the wide range of the flow characteristics.

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#### APPENDIX (I)

##### *The Monin-Obukhov similarity theory*

Assuming the horizontal homogeneity of turbulent field and the constant fluxes of momentum and heat, the Monin-Obukhov theory predicts that velocity and density profiles are described as (Monin & Yaglom (32))

$$\frac{dU}{dz} = \frac{U_*}{\kappa z} \phi_M(\zeta) \quad (A1)$$

$$\frac{d\bar{\rho}}{dz} = \frac{\rho_*}{\kappa z} \phi_\rho(\zeta) \quad (A2)$$

where  $\rho_* = -\overline{\rho'w'}/U_*$ ,  $\phi_M$  and  $\phi_\rho$  are the universal functions and

$$\zeta = \frac{z}{L} \quad (A3)$$

where

$$L = -\frac{U_*^3}{\kappa B} = -\frac{U_*^3}{\kappa \bar{q}g/\rho_0} \quad (A4)$$

is called the Monin-Obukhov length; and  $B = g\bar{q}/\rho_0$  = the buoyancy flux;  $\bar{q} = -\overline{\rho'w'}$ . The Monin-Obukhov length can be rewritten

$$L = \frac{U_*^3}{\kappa g \frac{\Delta\rho}{\rho_0} W_e} \quad (A5)$$

where  $L$  takes a positive value for stable stratification because  $\Delta\rho > 0$ ,  $W_e > 0$  and  $\bar{q} < 0$ .  $\phi(\zeta)$  is to be determined from the so-called KEYPS equation,

$$\phi^4 - \gamma\zeta\phi^3 = 1 \quad (A6)$$

Integrating the above equations (A1) and (A2), one obtains

$$\frac{U(z)}{U_*} = \frac{1}{\kappa} \left[ f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right) \right] \quad (\text{A7})$$

$$\frac{\bar{\rho}(z)}{\rho_*} = \frac{1}{\kappa} \left[ f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right) \right] + \rho_0(z) \quad (\text{A8})$$

in which  $f$  = the non-dimensional universal function and  $z_0$  = the roughness length. For near-natural conditions (for small  $|z/L|$ ), one obtains, from the Taylor expansion of  $\phi(\zeta) \approx 1 + \beta\zeta$ , the famous "log+linear" law

$$\frac{U(z)}{U_*} = \frac{1}{\kappa} \left( \ln \frac{z}{L} + \beta \frac{z}{L} \right) + \text{const.} \quad (|\zeta| < 1) \quad (\text{A9})$$

in which  $\beta$  = Obukhov's constant.

For the extremely high stable stratification, i.e.  $\zeta > 1$ , the recent measurements of the velocity distribution on the atmospheric surface layer by Okamoto & Webb (34) and Kondo, Kanechika & Yasuda (27) demonstrate that the second "simple log" law holds for large  $z/L$ ,

$$\frac{U(z)}{U_*} = \frac{(1 + \beta)}{\kappa} \ln z + \text{const.} \quad (\zeta > 1) \quad (\text{A10})$$

#### APPENDIX (II) - NOTATION

The following symbols are used in this paper:

|                 |                                                                      |
|-----------------|----------------------------------------------------------------------|
| $B$             | = $-g\bar{q}/\rho_0$ , buoyancy flux;                                |
| $D$             | = constant, Eq. 53;                                                  |
| $E$             | = $W_e/U_1$ , entrainment coefficient;                               |
| $E_*$           | = $W_e/U_*$ , entrainment coefficient;                               |
| $f_i'$          | = interfacial friction coefficient;                                  |
| $f_{ia}'$       | = theoretical value of $f_i'$ given by Eq. 51 or Eq. 52;             |
| $f_{ie}'$       | = interfacial friction coefficient contributed by entrainment;       |
| $g$             | = gravitational acceleration;                                        |
| $h'$            | = thickness of stratified layer, Eq. 31;                             |
| $h''$           | = layer thickness given by Eq. 49;                                   |
| $H_1$           | = upper layer thickness, Eq. 31;                                     |
| $J$             | = universal constant, Eq. 37;                                        |
| $J'$            | = $h''/L$ ;                                                          |
| $l_u$           | = transitional layer thickness of velocity profile defined in Fig.6; |
| $l_\rho$        | = transitional layer thickness of density profile defined in Fig.6;  |
| $L$             | = Monin-Obukhov length, Eq. 4;                                       |
| $\bar{q}$       | = $\rho'w'$ ;                                                        |
| $R$             | = $l_u/l_\rho$ ;                                                     |
| $Re$            | = $U_1 H_1 / \nu$ , Reynolds number;                                 |
| $Re_{\delta_*}$ | = $U_* \delta_*' / \nu$ ;                                            |
| $R_i$           | = $\epsilon g H_1 / U_1^2$ , Richardson number;                      |

|              |                                                                         |
|--------------|-------------------------------------------------------------------------|
| $R_{i*}$     | = $\epsilon gh' / U_*^2$ ;                                              |
| $S_1$        | = slope of $[(\kappa U(z') / U_* - \ln z') \text{ v.s. } z']$ ;         |
| $S_2$        | = slope of the second simple log law plotted on semi-logarithmic graph; |
| $\bar{U}(z)$ | = velocity at height $z$ ;                                              |
| $U_1$        | = upper layer mean velocity;                                            |
| $U_e$        | = outer edge velocity of stratified layer;                              |
| $U_i$        | = interface velocity;                                                   |
| $U_*$        | = friction velocity;                                                    |
| $w'$         | = vertical velocity fluctuation;                                        |
| $W_e$        | = entrainment velocity;                                                 |
| $x$          | = coordinate, downstream direction;                                     |
| $z$          | = coordinate, vertical direction;                                       |
| $z'$         | = distance from the interface;                                          |
| $\beta$      | = Obukhov constant;                                                     |
| $\delta_*'$  | = thickness of the viscous interfacial layer;                           |
| $\epsilon$   | = $(\rho_2 - \rho_1) / \rho_1$ ;                                        |
| $\zeta$      | = $z / L$ ;                                                             |
| $\kappa$     | = Karman constant;                                                      |
| $\lambda$    | = $2(U_* / U_e)^2$ ;                                                    |
| $\nu$        | = kinematic viscosity;                                                  |
| $\rho$       | = density;                                                              |
| $\Delta\rho$ | = density difference of the upper and lower layer;                      |
| $\tau_i$     | = interfacial friction;                                                 |
| $\phi_i$     | = velocity distribution functions, Eq. 32; and                          |
| $\psi$       | = $R_e R_i^{-1}$ , Iwasaki number, Eq. 13.                              |