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CHARACTERISTICS OF WAVES IN THE WATER WITH CONTINUOUS DENSITY GRADIENT

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SYNOPSIS

The wave motion in the finite depth of water with a vertical density gradient is theoretically discussed.

The fundamental equation of motion which has been proposed by one of the authors is slightly simplified and then used as the basic equation of motion.

A theoretical solution is obtained for the case where the density increases exponentially regarding with depth. A peculiar type of wave velocity is also obtained.

Where the density gradient has a peak in the finite depth of water, it is theoretically shown that the stream function has an oscillatory form regarding with depth for any type of density distribution.

INTRODUCTION

Two-layered, or three-layered systems of internal waves are by all means too mathematical to deal with many types of problems in the real world, such as those in the ocean, lakes and rivers, though those multi-layered systems keep certain important features of the density current problems. Since any of the two-layered systems never take place in nature, except the cases of super-imposed two or three layered ones consisting of immiscible fluids, the continuous density gradient systems might be considered as the better models to deal with a series of real problems.

The basic equations for wave motions in a continuous density gradient field were perhaps first obtained by Love (6). Lamb already referred to that work of Love in his famous book. He showed that if the density gradient was expressed by an exponential function, the problems were to be solved analytically. There are a lot of research works after him, and they are well sammarized by Eckart (1,2), Phillips (9) and Yih (14,15).

It is also possible to reach an analytical solution by assuming the core of the stream function be exponential function regarding the vertical component. In this case the vertical density gradient may be determined later. Recently, the authors learned the work of Palm et al (8), which is one of the works along this line.

It is also interesting that, though many authors have worked on the cases of the waves in the infinite field, very few works worked on the cases of the finite depth of water (10,11). Since in this case, which is of the waves in the finite depth of water, the wave velocity takes a peculiar form, such as expressed by Eq. 20 or Eq. 21 in this paper, a detailed derivation is examined later. As a continuation, this paper deals with the general case of continuous density gradient systems with a peak, or a maximum value in the water.

THE BASIC EQUATIONS IN TWO DIMENSIONS

The basic equation of the motion is written as

$$(\rho_y\psi_{yt} + \rho\Delta\psi_t)_t - g\rho_y\psi_{xx} = 0 \tag{1}$$

where ψ = stream function; ρ = mean density; and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ (6,11). Note that ρ is a function of x, y and t in general. $u = -\psi_y$ and $v = \psi_x$ respectively. The suffices in Eq. 1 denote the differentiation regarding each component. Note also that the above equation may be valid when the mean motion is relatively small when the equation is written in a non-dimensional form. In other words, the small wave assumption is necessary to make use of Eq. 1.

If there is no diffusion or dissipation, ρ may be a function of the position (x,y) only. Thus, Eq. 1 may be reduced to

$$\Delta\psi_{tt} + (\psi_{ytt} - g\psi_{xx})\rho_y/\rho = 0 \tag{2}$$

Note that Eq. 2 is essentially the equation obtained by Love (6).

At the free surface, the governing equation is

$$\psi_{ytt} - g\psi_{xx} = 0 \qquad \text{at } y = 0 \tag{3}$$

If ψ is given by

$$\psi = F(y)\exp(ikx - i\omega t) \tag{4}$$

then $\psi_{ytt} = -\omega^2F'(y)\exp(ikx - i\omega t)$ and $\psi_{xx} = -k^2F\cdot\exp(ikx - i\omega t)$. Therefore, the surface condition is given by

$$\psi_{ytt} - g\psi_{xx} = (-\omega^2F' + gk^2F)\exp(ikx - i\omega t) = 0$$

Further, if $F=A\exp(ky)$, then $F'=Ak\exp(ky)$ and we obtain

$$-\omega^2F'(y) + gk^2F(y) = kA\exp(ky)(-\omega^2 + gk) \tag{5}$$

Eq. 5 may be equal to zero if $\omega^2 = gk$, which is the deep water condition. Therefore, where there is a deep water wave on the surface of the water, the motion is always irrotational whatever the vertical density distribution is. This was already pointed by Lamb(6).

Writing that $-\rho_y/\rho = \beta$, we obtain

$$\Delta\psi_{tt} - \beta\psi_{ytt} + g\beta\psi_{xx} = 0 \tag{6}$$

If we assume that

$$\psi = X(x)Y(y)T(t) \tag{7}$$

then we obtain the following relation.

$$\{X''/X + Y''/Y - \beta Y'/Y\} = -(g\beta X''/X)(T/T'') \tag{8}$$

Apparently T has a form that

$$T'' = c_1T \tag{9}$$

and we obtain

$$(c_1 + g\beta)X''/X + c_1(Y''/Y - \beta Y'/Y) = 0 \tag{10}$$

If β is a function of y only, Eq. 10 becomes

$$X'' = c_2X \tag{11}$$

and finally we obtain

$$Y''/Y - \beta Y'/Y + c_2g\beta/c_1 + c_2 = 0 \tag{12}$$

It is natural for us to assume the form of the stream function as

$$X \sim \exp(ikx);\ T \sim \exp(i\omega t) \tag{13}$$

where $\omega^2=-c_1$ and $k^2=-c_2$; the governing equation for Y is

$$Y'' - \beta Y' + (\beta gk^2/\omega^2 - k^2)Y = 0 \tag{14}$$

Note that $\beta=\beta(y)$, which is not constant in general. $g\beta$ is often written as $N^2(y)$ (Brunt-Väisälä frequency). Using N, we obtain

$$Y'' - N^2Y'/g + k^2(N^2/\omega^2 - 1)Y = 0 \tag{15}$$

When Boussinesq approximation is applied to Eq. 15, the second term of Eq. 15 may be omitted and the equation would be that of Eckart(1).

$$Y'' + (N^2 - \omega^2)Y/c^2 = 0\ ;\ c \equiv \omega/k \tag{16}$$

Eq. 16 is equivalent to the equation for a pendulum with a variable spring coefficient, while Eq. 15 does include a damping term. When $N^2/\omega^2>1$, Y is oscillatory regarding y. Therefore, though Eq. 16 may be generally employed by several authors (9,14), Eq. 15 should not be forgotten.

The interesting case is that for $N^2>\omega^2$, since if $\omega^2>N^2$, Eq. 15 or Eq. 16 is not far different from the ordinary wave equation for the finite depth of water. From this reason, the case where $N^2>\omega^2$ is examined in this paper.

SOLUTION FOR CONSTANT N

Where N is constant, which is by all means unrealistic from the theoretical point of views, but reasonably practical, the solution for Eq. 16 is obtained (11) as,

$$Y = -c\cdot\sin\xi(h+y)/\sin\xi h \tag{17}$$

under the boundary conditions such as

$$v = 0 \text{ at } y = -h \text{ (at the bottom)};\ v = \partial\eta/\partial t \text{ at } y = 0 \text{ (at the surface)}$$

The stream function corresponding to Eq. 17 is

$$\psi = -ac\cdot\sin\xi(h+y)\cos(kx-\omega t)/\sin\xi h \tag{18}$$

where ξ is given by

$$\xi^2 = (N^2-\omega^2)/c^2 \tag{19}$$

The wave velocity c is obtained by using another surface condition which is dynamic that

$$\partial\psi/\partial y - gk^2\psi/\omega^2 = 0 \tag{20}$$

$$c^2 = gh\cdot\tan\xi h/\xi h \tag{21}$$

From the above two equations, we obtain the tendency of ξh regarding the change of $(N^2-\omega^2)h/g$.

$$\xi h \cdot \tan\xi h = (N^2 - \omega^2)h/g \quad (22)$$

which is graphically shown on Fig. 1.

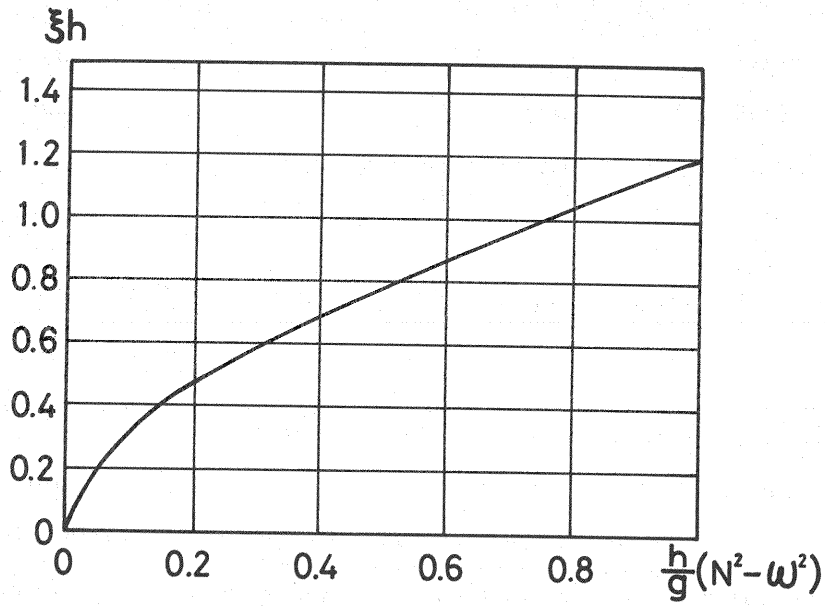

Fig. 1 Relation of the modified wave number ξ and N.
(Non-dimensional Plot)

However, under the normal situation, Eq. 21 gives the relation that $c^2 \simeq gh$, since ξh is small when $(N^2-\omega^2)h/g$ is small. The wave velocity c in this case shows the surface wave velocity affected by the density gradient in the water.

For the internal waves, we must seek the solution in the form (1) that

$$Y = A \cdot \sin(n\pi y/h); \; n = 0,1,2,3,\cdots \cdots \quad (23)$$

The stream function for this case is given by

$$\psi = a \cdot \sin(n\pi y/h)\cos(kx-\omega t) \quad (24)$$

The corresponding internal wave velocity is given by

$$c^2 = (N^2 - \omega^2)h^2/n^2\pi^2 \quad (25)$$

Note that the surface condition, Eq. 20, is not available in this case. This motion is rotational (6, 10, 16).

THE CASE FOR ARBITRARY N

Where N is a variable of y, Eq. 16 is written as

$$Y'' + f(y)Y = 0 \quad (26)$$

Normally, f(y) has a maximum value M and a minimum value m (m=0 when $N^2=\omega^2$). Consider another equation associated with Y (12) such that

$$Z'' + MZ = 0 \; ; \; Z = Z(y) \quad (27)$$

A short mathematical procedure gives us

$$(ZY' - YZ')' = (M - f(y))YZ \quad (28)$$

and integration of Eq. 28 from α to β gives

$$|ZY' - YZ'|_{\alpha}^{\beta} = \int_{\alpha}^{\beta}(M - f)YZdy \quad (29)$$

where α and β are the roots of Y=0 which have not yet been obtained.

Apparently,

$$Y(\alpha) = 0 \ ; \ Y(\beta) = 0 \quad (-h<\alpha<\beta<0) \quad (30)$$

and also

$$Y'(\alpha) > 0 \ ; \ Y'(\beta) < 0 \quad (31)$$

Using the above conditions, Eq. 29 may be written as

$$Z(\beta)Y'(\beta) - Z(\alpha)Y'(\alpha) = \int_{\alpha}^{\beta}(M - f)ZYdy \quad (32)$$

Since both $Z(\alpha)$ and $Z(\beta)$ must have the same sign throughout the region, we assume the sign be positive. Note that the negative sign assumption does not cause any changes in the result. Then, the left-hand side of Eq. 32 is positive except the case that $Z(\alpha) = Z(\beta) = 0$.

The variables of the right-hand side of Eq. 32 are $M-f>0$, $Y<0$ and $dy<0$. Therefore, the multiple except Z is negative. Since Z is an oscillatory function with respect to y, there is a possibility for Z to have a root between α and β. In conclusion, the right-hand side of Eq. 32 is generally negative, if Z has no root between α and β, which is a contradiction. Thus, there is at least a root of Z between two roots of Y.

One of the solution for Eq. 29 is

$$Z = \sin\{(y - \alpha)\sqrt{M}\}$$

which has no root between α and $\alpha+\pi\sqrt{M}$. Therefore, except the case that $f(y)=M$, the following condition must be satisfied.

$$\beta - \alpha > \pi\sqrt{M} \quad (33)$$

From those reasons if the value h/n is under the condition that

$$h/n < \pi\sqrt{M} < \beta - \alpha \quad (34)$$

Y(y) has only one root, or no root in the region (-h/n,0). In other words, the above condition gives us the possible mode of Y which is the solution for the arbitrary N(y). Note that only M is important in this analysis.

The maximum value of M, M_{max}, may be written as

$$M_{max} = (N^2 - \omega^2)/c_o^2 \quad (35)$$

where N_o is the maximum value of N and c_o is the corresponding wave velocity. The above relation may be written as

$$h^2/n^2 < c_o^2\pi^2/(N_o^2 - \omega^2) \quad (36)$$

CONCLUSION

The internal waves associated with the surface waves in a finite depth of water with density gradient was discussed. Generally speaking, waves are rotational ones and the stream function is periodical with respect to depth, and the wave velocity shows a peculiar form, where the density distribution follows a specific function form.

This tendency is also applied to certain types of density distribution and the condition where the stream function shows a periodicity was theoretically obtained.

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APPENDIX - NOTATION

The following symbols are used in this paper:

a	= wave amplitude;
c_o, c	= wave velocity;
c_1, c_2	= constants;
g	= gravitational acceleration;
i	= imaginary unit;
k	= wave number;
n	= mode constant;
N	= square root of Brunt-Väisälä frequency;
h	= water depth;
M	= constant;
M_{max}	= maximum value of M;
t	= time;
T	= time dependent part of stream function;
x, y	= coordinates of the space;
X, Y	= space dependent parts of stream function;
Z	= a function associated with Y;
α	= one root of $Y=0$;
β	= another root of $Y=0$;
ξ	= modified wave number $(\xi^2=(N^2-\omega^2)/c^2)$;
π	= 3.14159....;
ρ	= density of the fluid;
ψ	= stream function; and
ω	= wave frequency.