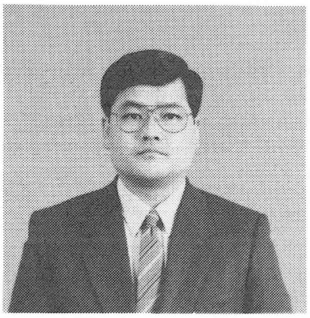
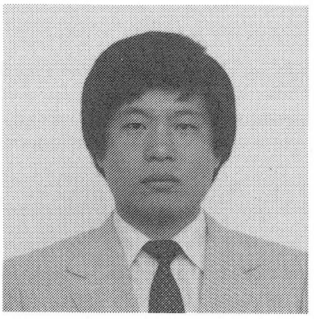
CONCRETE LIBRARY OF JSCE NO.18, DECEMBER 1991

ANALYTICAL STUDY ON THE SHEAR CAPACITY OF REINFORCED CONCRETE BEAMS WITH WEB REINFORCEMENT

(Reprint from Proceedings of JSCE, No.426, Feb.1991)


Motoyuki SUZUKI   Suck Hwa KANG   Yoshio OZAKA

SYNOPSIS

The purpose of this study is to theoretically clarify the ultimate shear strength and the deformation of reinforced concrete beams with web reinforcement, subjected to combined bending moment and shear. To achieve this, an ultimate equilibrium method based on the energy principle is employed. The proposed equations, which are derived from the equilibrium condition of force and moment for the failure surface at the ultimate state, are compared with test results and good agreement is noted. Compatibility condition of strains on the shear element is also considered to predict the failure mode of beams. In addition, an interaction between bending moment and shear is proposed.

M. Suzuki is an associate professor of civil engineering at Tohoku University, Sendai, JAPAN. He received his Dr. Eng. degree from Tohoku University in 1988. In addition to structural safety, his research interests are related to anti-seismic behavior of RC structures. He is a member of JSCE, JCI, IABSE and ACI.

S.H. Kang is a research associate of civil engineering at Tohoku University, Sendai, JAPAN. He received his BS and MS degree from Yonsei University in Seoul, KOREA, and Dr. Eng. degree from Tohoku University in 1991. His research interests are in the shear behavior of RC structures and the application of high strength concrete to RC structures. He is a member of JSCE and JCI.

Y. Ozaka, a professor of structural engineering, Tohoku University, Sendai, JAPAN since 1972, received his Dr. Eng. degree from that university. Previously, he was Senior Engineer, Structures Design Office, Japanese National Railways. The Japanese delegate to CEB and the Commission on "Seismic Design" of FIP, he is a member of ACI, IABSE, JSCE, JCI, and JPCEA. His interests are in anti-seismic behavior, cracking properties of concrete structures and structural safety problems.

1. INTRODUCTION

Shear failure, together with bending failure, is the representative failure mode of RC beams. However, the shear failure mechanism is still not clear despite many intensive studies because it is influenced by many factors other than the bending failure mechanism.

Theoretical approaches to explain the shear capacity of RC beams have been employed, and many experimental formulae, semi-experimental formulae and analytical models have been proposed since the truss analogy was proposed by Ritter and Mörsch in 1899.

Mechanical models may be divided into two groups: one group of models utilizes geometrical modeling such as the truss analogy [1] or the arch theory [2],[3], and the other employs theoretical approaches such as the ultimate strength theory [4] or the plastic theory. Until the 1960s, mechanical models based on experimental studies were predominant [5]. But, in the early 1970s, Nielsen's plastic theory [6] and Collins' compression field theory [7] were proposed resulting in rapid development of studies on the shear capacity of RC members. The former type of study estimates the shear capacity by applying upper and lower bound theorem to plastic material. In the latter type of study the shear capacity is estimated by considering the equilibrium condition of the stresses and the compatibility condition of the strains. Limit analysis based on the plastic theory is thought to be an advantageous method because of the rationality of its assumptions and the simplicity of mechanical modeling or analysis.

On the other hand, micro model approaches [8], such as the nonlinear finite element method, have been employed since the end of the 1960s. Although some useful mechanical models have been proposed, the micro model has not been applied to design formulae because of some unsolved problems such as the modeling of crack and bond properties between steel and concrete, and so on.

The purpose of this study is to theoretically clarify the ultimate shear strengths and the inclination of diagonal cracks of RC beams with web reinforcement, subjected to combined bending moment and shear force. In this study, the ultimate equilibrium method of the upper bound theorem of the plastic theory is used in addition to the equilibrium condition of forces and the compatibility condition of strains. Moreover, an equation representing the interaction relationship between shear and bending moment is proposed. To make comparisons between calculated values by the proposed equation and experimental data, we employ our test data, as well as that of Frantz and Regan.

2. QUESTIONS REGARDING STATE OF THE ART

There are two approaches to estimate the shear capacity and the deformation of RC beams subjected to combined bending moment and shear by the plastic theory. One is the static approach by which the inner stress state that equilibrates with the external load under the condition of the yield of material is determined. The other is the kinematic approach by which the deformation state is determined considering the internal and external energies, regardless of yield criteria for material. Furthermore, the kinematic approach can also be divided into two methods. The first method is a work method based on the minimum potential energy principle which states that the crack propagates in the direction in which the internal energy becomes minimum [9],[10]. The second method is the equilibrium method, which is based on the theorem that the failure of the member occurs in the direction in which the internal resistance becomes minimum.

It is very important to estimate the direction of the diagonal crack analytically in order to evaluate the number of web reinforcements crossing the crack. Past studies on the shear capacity and the inclination of the diagonal crack on the basis of the work method or the ultimate equilibrium method, are reviewed as follows.

Kupfer [11] postulated that the inclination (α) of concrete strut varies with the amount of web reinforcement(A_v) and that the ratio of the width of the compression zone of concrete to that of web (b/b_0) varies on the basis of the truss analogy. Furthermore, he hypothesized that the inclination of concrete strut could be found by differentiating the sum of strain energy along the length of the beam with respect to the inclination of the concrete strut. On the basis of this concept, the inclination of the diagonal crack is found from 35° to 40°. This theory is useful for designing web reinforcement against shear failure, but not for evaluating shear capacity. However, use of this theory involves problems, basically questions of the truss analogy: the shear transfer of concrete of the compression zone is disregarded and the increase of the shear capacity can not be expressed as the shear span depth ratio decreases.

Collins [7] assumed that shear failure occurs due to the failure of the inclined compressive strut. He used the tension field theory proposed for metal material by Wagner [12], and he developed a new theory based on the assumption that the inclination of the compressive strut is equal to that of the principal compressive strain. According to this theory, the inclination necessary to minimize the strain energy is obtained by differentiating the shear strain (γ) of the element per unit length with respect to the inclination of the diagonal crack (α).

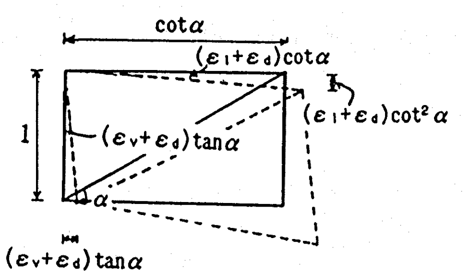


Fig.1 Shear strain on the element in the truss analogy

Namely, the shear strain on the element in the truss analogy is given as follows, as shown in Fig.1.

$$\gamma=(\varepsilon_l+\varepsilon_d)\cot\alpha+(\varepsilon_v+\varepsilon_d)\tan\alpha \tag{1}$$

where ϵ_1 denotes tensile strain in the longitudinal reinforcement, ϵ_v denotes tensile strain in the transverse reinforcement, and ϵ_d denotes the compressive strain in the compressive strut.

Differentiating the shear strain with respect to the inclination of the principal compressive strain (α), the following equation is obtained.

$$\frac{d\gamma}{d\alpha}=0 \tag{2}$$

The compatibility condition of the strains is written as follows:

$$\tan^2\alpha=\frac{\varepsilon_l+\varepsilon_d}{\varepsilon_v+\varepsilon_d} \quad (3)$$

In this theory, the inclination of principal compressive stress (α), which satisfies the equilibrium condition of forces and the compatibility condition of strains for any shear stress (τ), can be calculated. When the stress of the inclined compressive strut (f_d) reaches critical stress (f_{du}), the member fails. But this theory also has demerits in that the accuracy of estimation is low when the shear span depth ratio is small or the amount of web reinforcement is small.

Nielsen [6] proposed an exact solution of the shear capacity of the RC member, namely, that the upper bound solution is equal to the lower bound solution on the basis of the plastic theory. According to this theory, the upper bound solution is given by differentiating the internal dissipating energy with respect to the inclination of the diagonal crack at which the internal energy becomes minimum by applying the yield line theory based on the virtual work theory.

$$P\cdot u=0.5 f_{ck}b(1-\cot\alpha)h\cdot u/\sin\alpha+\rho_v\cdot f_{vy}bh\cot\alpha\cdot u \quad (4)$$

where ρ_v denotes the web reinforcement ratio, f_{vy} denotes the yield strength of the web reinforcement, and u denotes virtual displacement.

The first term of Eq.(4) is the dissipation energy in the concrete along the yield line, and the second term is the dissipation energy in the web reinforcement crossing the yield line. Namely,

$$\frac{d(P\cdot u)}{d\alpha}=0 \quad (5)$$

We expressed Eq.(5) by rearranging it as follows:

$$\tan\alpha=2\sqrt{\psi(1-\psi)}/1-2\psi \quad (6)$$

where, $\psi=\rho_v f_{vy}/f_{ck}$

The shear capacity is expressed as follows:

$$V_u=bhf_{ck}\sqrt{\psi(1-\psi)} \quad (7)$$

When a/h is smaller than $\tan\alpha$, namely,

$$\psi<\frac{(\sqrt{a^2+h^2}-a)}{2\sqrt{a^2+h^2}}$$

The shear capacity is written as follows:

$$V_u=bhf_{ck}\{(0.5\sqrt{1+(a/h)^2}-a/h)+\psi\cdot a/h\} \quad (8)$$

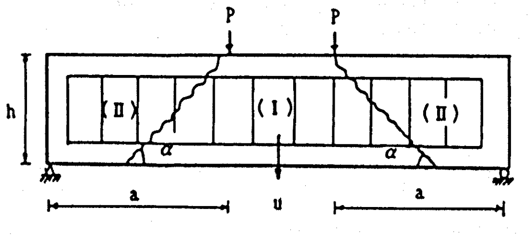


Fig.2 Shear failure mechanism in Nielsen's theory

In this theory, T-type beams and rectangular beams are dealt with in the same category. It is known that the estimation accuracy is comparatively good when a/d is larger than 3 for T-type beams, but the accuracy is not conservative for rectangular beams [13]. Furthermore, this theory has unsolved problems. For example, the determination of the so-called effectiveness factor (ν), and the underestimation of shear capacity when a/d is small in case of application of the theory to design.

Regan [14] also proposed the inclination of the diagonal crack at which the strain energy due to the web reinforcement and aggregate interlocking becomes minimum.

Studies to evaluate the capacity of members by the equilibrium method have been carried out for the members subjected to torsion moment rather than shear force. Although studies to estimate the shear capacity have been rarely carried out, a semi-theoretical equation based on experimental data has been proposed by the RC Central Institute of the USSR [15]. In this equation, the shear capacity (V_u) is given as the sum of the shear strength provided by concrete (V_c) and the shear strength provided by web reinforcement (V_s).

$$V_u = V_c + V_s = 2\,bd^2 f_t / c + A_v \cdot f_{vd} \cdot c/s \tag{9}$$

where f_t denotes tensile strength of concrete, f_{vd} is yield strength of web reinforcement, A_v represents area of web reinforcement, and s denotes spacing of web reinforcement.

According to this theory, the shear capacity is found by differentiating the shear resistance with respect to the length of the longitudinal projection of the diagonal crack (c), namely

$$\frac{dV_u}{dc} = 0 \tag{10}$$

$$c = \sqrt{2\,bd^2 f_t s / A_v f_{vd}} \tag{11}$$

The shear capacity is given by

$$V_u = 2\sqrt{2\,bd^2 A_v f_{vd} f_t / s} \tag{12}$$

As mentioned above, the approaches based on the energy method have been carried out by many investigators. However, each approach has merits and demerits, and the deformation mechanism of RC beams subjected to combined bending moment and shear does not seem to be sufficiently understood.

In this study, the shear capacity and the inclination of diagonal cracks in RC beams subjected to combined bending moment and shear are analytically examined on the basis of the ultimate equilibrium method. Furthermore, a method of estimating the failure mode is proposed based on considerations of the compatibility of strains in the failure surface.

3. ANALYSIS OF SHEAR CAPACITY BY THE ULTIMATE EQUILIBRIUM METHOD

3.1 Outline

The ultimate equilibrium method has been proposed by a research group in the USSR [15] for the RC member subjected to torsion. Deformation corresponding to the minimum resistance of the RC member subjected to combined flexure and torsion or torsion, flexure and shear, is determined by this theory, which is a kinematic approach based on the upper bound theorem. In this theory, an examination of the deformation mechanism and capacity of members in the ultimate state is attempted by taking into consideration the equilibrium conditions of forces and moments.

3.2 Assumption

In this study, the failure modes in which stress redistribution does not develop, such as diagonal tension failure, are not taken into consideration. The subject of this study is RC beams with web reinforcement, and its analysis is based on the following assumptions.

a) As shown in Fig.3(a), the diagonal crack propagates linearly at an inclination of α_1 until $(h-d_e)$, namely, the distance from tension reinforcement to the neutral axis at the uncracked state. Thereafter, the crack propagates at an inclination of α_2 due to internal stress redistribution [16], and the member reaches the ultimate state. In this study the diagonal crack is modeled by a bilinear line.

b) This study assumes that the initial crack develops due to the extreme tension fiber, that tension reinforcement and web reinforcement subsequently yield, and that the concrete of the compression zone finally reaches the compressive strength of concrete resulting in beam failure.

c) On the basis of the concept that shear failure is basically equal to bending failure when web reinforcement is arranged sufficiently along the member [17], [18], the depth of the compression zone is assumed to be the same as that for pure bending.

d) The failure surface is composed of both the surface of the diagonal crack and that of the compression zone. The spacing of web reinforcement is constant.

e) The influences of the dowel action of reinforcement and the tensile strength of concrete are neglected.

f) Upper longitudinal bars of the compression zone are ignored.

g) The direction of the principal tensile strain is perpendicular to the direction of the diagonal crack .

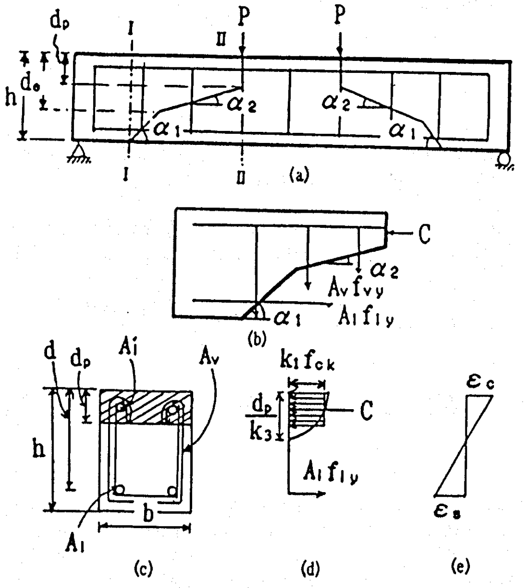


Fig.3 Crack propagation and equilibrium conditions of the forces at the ultimate state

Ultimate shear resistance is determined by the equilibrium condition between the internal and external forces at the failure surface on the basis of the above assumptions.

3.3 Derivation of the shear capacity equation

As shown in Fig.3(a), RC beam with vertical web reinforcements which is loaded with two equal concentrated loads is taken into consideration in this analysis. Section I-I, which is the starting point of the diagonal crack, and section II-II, which is located at the loading point, are considered.The stress of the tensile reinforcement is experimentally larger than that calculated by the traditional bending theory when the diagonal cracks exist in the member due to shear force [11],[19],[20]. Therefore, the moment (M_{ext}) due to the external force must be equilibrated in the summation of the moment (M_u) caused by bending force and moment due to shear force.

$$M_{ext}=M_u+V_u(h-d_e)\cot\alpha_1+V_u(d_e-d_p)\cot\alpha_2 \qquad (13)$$

The direction of the principal compressive stress at an arbitrary point of the member in the elastic state is expressed as follows by Mohr's theory.

$$\tan 2\alpha=-\frac{2\tau}{\sigma_x-\sigma_y} \qquad (14)$$

where, τ is shear stress, σ_x is normal stress in the longitudinal direction, and σ_y is normal stress in the transverse direction.

The direction of principal compressive stress, namely, the direction of the crack, is 45° on the neutral axis. It is known that the direction of the crack propagates linearly to the neutral axis near which the stress redistribution occurs internally [21] due to the brittleness of concrete. Applying this assumption to past experimental data [22], the results are as shown in Table 1.

Table 1 Angle of the diagonal crack in past experimental data

number of test specimens	average(°)	coefficient of variation(%)
54	44.5	12.2

In this analysis, the angle of the first crack (α_1) is assumed to be 45°. But when the distance from the loading point to the bearing point is shorter than d/2, the condition α_1 =45° can not be applied to this model. The distance from the extreme compression fiber to the neutral axis in the elastic state (d_e) is expressed as follows:

$$d_e=\frac{0.5\,bh^2+(n-1)A_l d+(n-1)A_l' d'}{bh+(n-1)A_l+(n-1)A_l'} \qquad (15)$$

where n is the ratio of Young's modulus of steel to concrete, A_l is the area of the lower longitudinal bars, A_l' is the area of the upper longitudinal bars, and d' is the concrete cover of the upper longitudinal bars.

The internal resistance moment (M_{int}) is expressed as a sum of the three components at the neutral axis on the failure surface: the resistance moment due to tension reinforcement (M_{ul}), the resistance moment due to web reinforcement (M_{uv}) and the resistance moment due the to concrete of the compression zone (M_{uc}).

$$M_{int}=M_{ul}+M_{uv}+M_{uc} \tag{16}$$

The terms on the right side of Eq.(16) may be written as follows.

$$M_{ul}=A_l f_{ly}(d-d_p) \tag{17}$$

$$M_{uv}=\frac{(d_e-d_p)^2}{2s}\cot^2\alpha_2\cdot A_v f_{vy}+A_v f_{vy}(h-d_e)\{(d_e-d_p)\cot\alpha_2+0.5(h-d_e)\}/s \tag{18}$$

$$M_{uc}=k_1\cdot f_{ck}\cdot b\cdot d_p^2/2 \tag{19}$$

where d_p is the depth of the neutral axis at the ultimate state, A_v is the area of the web reinforcement, f_{vy} is the yield strength of the web reinforcement, s is the spacing of the web reinforcement, k_1 is the coefficient related to the average stress of the concrete, and k_3 is the coefficient related to the depth of the compression zone.

From the equilibrium condition between the external and internal moments, the following equation can be written.

$$M_{ext}=M_{int} \tag{20}$$

From Eqs.(13) and (16), the following equation results.

$$M_u+V_u(h-d_e)+V_u(d_e-d_p)\cot\alpha_2=k_1 f_{ck} b d_p^2/2+A_l f_{ly}(d-d_p)+A_v f_{vy}\frac{(d_e-d_p)^2\cdot\cot^2\alpha_2}{2s}$$
$$+A_v f_{vy}\frac{(h-d_e)}{s}\left\{(d_e-d_p)\cot\alpha_2+\frac{(h-d_e)}{2}\right\} \tag{21}$$

The stress-strain relationship of the concrete of the compression zone is described in Fig.3(d),(e).The stress-strain relationship is expressed as follows by Hognestad's theory.

$$\sigma_c=f_{ck}\left\{2\left(\frac{\varepsilon_c}{\varepsilon_0}\right)-\left(\frac{\varepsilon_c}{\varepsilon_0}\right)^2\right\} \tag{22}$$

Assuming the stress distribution of concrete is rectangular, it is expressed as follows:

$$k_1 f_{ck}=\int_0^{d_p/k_3}\sigma_c\cdot dx/(d_p/k_3) \tag{23}$$

Horizontal force equilibrium requires the following equation.

$$k_1 f_{ck} b d_p-A_l f_{ly}=0 \tag{24}$$

The depth of neutral axis is expressed as follows:

$$d_p=\frac{A_l f_{ly}}{k_1 f_{ck} b} \tag{25}$$

Substituting Eq.(25) into the right side of Eq.(21), the following equation is derived.

$$M_u+V_u(h-d_e)+V_u(d_e-d_p)\cot\alpha_2=A_l f_{ly}(d-d_p/2)+A_v f_{vy}\frac{(d_e-d_p)}{s}\cot\alpha_2\cdot\frac{(d_e-d_p)}{2}\cot\alpha_2$$
$$+A_v f_{vy}\frac{(h-d_e)}{s}\left\{(d_e-d_p)\cot\alpha_2+\frac{(h-d_e)}{2}\right\} \tag{26}$$

It is clear that the relationship between the applied moment and the shear force at the ultimate state can be described as follows:

$$M_u = aV_u \tag{27}$$

where a is the length of shear span.

Then, the following equation is derived.

$$V_u = \frac{1}{\{a+h-d_e+(d_e-d_p)\cot\alpha_2\}}\Big[A_l f_{ly}(d-d_p/2) + A_v f_{vy}\cdot\frac{(d_e-d_p)}{s}\cot\alpha_2\cdot\frac{(d_e-d_p)}{2}\cot\alpha_2$$
$$+A_v f_{vy}\cdot\frac{(h-d_e)}{s}\Big\{(d_e-d_p)\cot\alpha_2+\frac{(h-d_e)}{2}\Big\}\Big] \tag{28}$$

Moreover, in order to evaluate the inclination of the diagonal crack at which the internal shear resistance becomes minimum, Eq.(28) is differentiated with respect to α_2 (inclination of second diagonal crack) and rewritten as follows.

$$\frac{dV_u}{d\alpha_2}=0 \tag{29}$$

$$\cot\alpha_2=\frac{-a-h+d_e}{d_e-d_p}+\sqrt{\left(\frac{a+h-d_e}{d_e-d_p}\right)^2+\frac{2s(d-d_p/2)A_l f_{ly}}{(d_e-d_p)^2A_v f_{vy}}-\frac{(h-d_e)}{(d_e-d_p)^2}\left(a+\frac{h-d_e}{2}\right)} \tag{30}$$

Eq.(30) is an equation to estimate the inclination of the diagonal crack at which the shear resistance becomes minimum, and thus the shear capacity can be expressed as follows.

$$V_u=\frac{1}{\sqrt{(a+h-d_e)^2+\frac{2s(d-d_p/2)A_l f_{ly}}{A_v f_{vy}}-(h-d_e)\left(a+\frac{h-d_e}{2}\right)}}\Big[A_l f_{ly}\left(d-\frac{d_p}{2}\right)$$
$$+\frac{A_v f_{vy}}{2s}\Big\{-a-h+d_e+\sqrt{(a+h-d_e)^2+\frac{2s(d-d_p/2)A_l f_{ly}}{A_v f_{vy}}-(h-d_e)\left(a+\frac{h-d_e}{2}\right)}\Big\}^2$$
$$+\frac{A_v f_{vy}(h-d_e)}{s}\Big\{-a-h+d_e+\sqrt{(a+h-d_e)^2+\frac{2s(d-d_p/2)A_l f_{ly}}{A_v f_{vy}}-(h-d_e)\left(a+\frac{h-d_e}{2}\right)}+\frac{(h-d_e)}{2}\Big\}\Big]$$
$$=\frac{A_v f_{vy}}{s}\Big[\sqrt{(a+h-d_e)^2+\frac{2s(d-d_p/2)A_l f_{ly}}{A_v f_{vy}}-(h-d_e)\left(a+\frac{h-d_e}{2}\right)}-a\Big] \tag{31}$$

Also in this study, the modeling of only one diagonal crack is examined for simplicity of calculation. Using the same method by which Eqs.(30) and (31) were derived, the diagonal crack can be described by the following equation.

$$\cot\alpha=-\frac{a}{(h-d_p)}+\sqrt{\left(\frac{a}{h-d_p}\right)^2+\frac{2s(d-d_p/2)}{(h-d_p)^2}\cdot\frac{A_l f_{ly}}{A_v f_{vy}}} \tag{32}$$

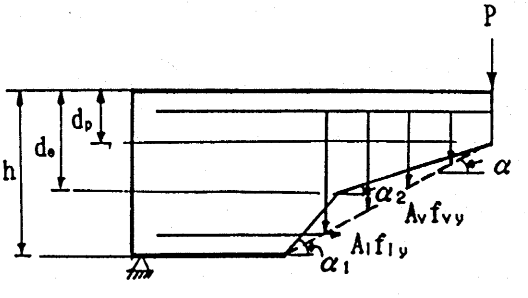


Fig.4 Modeling of the diagonal crack

And the shear capacity can be expressed as follows.

$$V_u=\frac{1}{\sqrt{a^2+2s\left(d-\frac{d_p}{2}\right)\frac{A_l f_{ly}}{A_v f_{vy}}}}\left[A_l f_{ly}\left(d-\frac{d_p}{2}\right)+\frac{A_v f_{vy}}{2s}\left\{2a^2-2a\sqrt{a^2+2s\left(d-\frac{d_p}{2}\right)\frac{A_l f_{ly}}{A_v f_{vy}}}\right.\right.$$

$$\left.\left.+2s\left(d-\frac{d_p}{2}\right)\frac{A_l f_{ly}}{A_v f_{vy}}\right\}\right]$$

$$=\frac{A_v f_{vy}}{s}\left[\sqrt{a^2+\frac{2s(d-d_p/2)A_l f_{ly}}{A_v f_{vy}}}-a\right] \qquad (33)$$

Shear capacities (V_{u2}) based on two diagonal cracks as shown in Fig.3 are compared with those (V_{u1}) based on only one diagonal crack as shown Fig.4 on the basis of past experimental data (see Fig.5).

As a result, the average ratios of V_{u1} to V_{u2} using Frantz's [24] and Regan's experimental data [26], were 1.001 and 1.045, respectively. Thus, it can be seen that it is practical to evaluate the shear capacity on the basis of a model of only one diagonal crack.

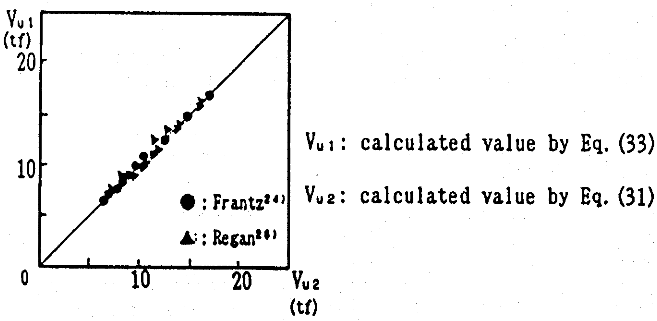


Fig.5 Comparison of shear capacity

3.4 Compatibility condition of the strains

As described in Fig.6(a), each component of the strain at a certain shear element A in the area of development of the diagonal crack is given as follows:

$$\varepsilon_l=\varepsilon_{cr}\sin^2\alpha+\varepsilon_d\cos^2\alpha \qquad (34)$$

$$\varepsilon_v=\varepsilon_{cr}\cos^2\alpha+\varepsilon_d\sin^2\alpha \qquad (35)$$

$$\gamma=2(\varepsilon_{cr}+\varepsilon_d)\sin\alpha\cos\alpha \qquad (36)$$

where ϵ_l is the strain in the of longitudinal direction, ϵ_v is the strain in the transverse direction, ϵ_d is the strain in the direction perpendicular to the crack, and γ is the shear strain.

Eqs.(34),(35) and (36) are rewritten as follows because ϵ_d is small enough to be ignored.

$$\varepsilon_l=\varepsilon_{cr}\sin^2\alpha \qquad (37)$$

$$\varepsilon_v = \varepsilon_{cr} \cos^2 \alpha \tag{38}$$

$$\gamma = \varepsilon_v \tan \alpha + \varepsilon_l \cot \alpha \tag{39}$$

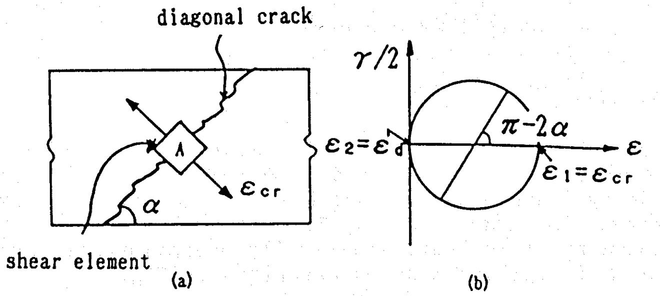


Fig.6 Strain component at the diagonal crack surface

Based on the minimum potential energy theory, the angle of crack α, at which the shear strain at element A becomes minimum, may be found by differentiating the expression for the shear strain with respect to α.

$$\frac{d\gamma}{d\alpha} = 0 \tag{40}$$

The compatibility condition required to determine the internal deformation mechanism at the diagonal crack surface is expressed by the following equation.

$$\tan^2 \alpha = \frac{\varepsilon_l}{\varepsilon_v} \tag{41}$$

3.5 Failure mode

The equilibrium condition for the yielding of reinforcements which is needed to determine whether web reinforcement or tension reinforcement yields first must be considered. First, when both reinforcements yield at the same time, that is to say when ϵ_l reaches ϵ_{ly} and ϵ_v reaches ϵ_{vy}, the compatibility condition of the strains at that state may be written as follows.

$$\tan^2 \alpha = \frac{\varepsilon_{ly}}{\varepsilon_{vy}} \tag{42}$$

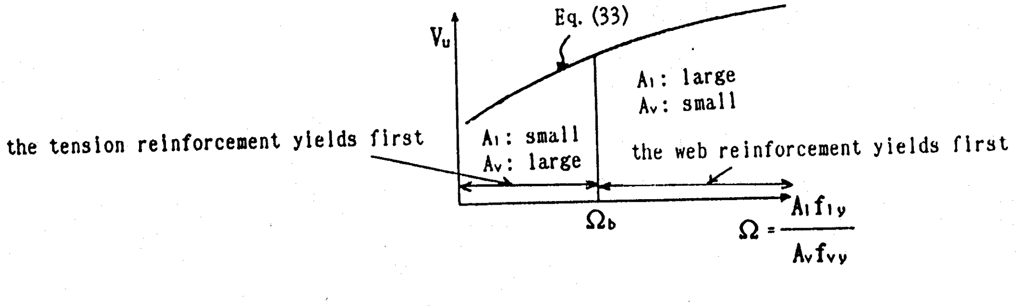


Fig.7 Failure mode

Applying Eq.(32) and Eq.(42), the following equation results.

$$\Omega=\frac{A_l f_{ly}}{A_v f_{vy}}=\frac{(h-d_p)^2}{2\,s(d-d_p/2)}\left\{\frac{\varepsilon_{vy}}{\varepsilon_{ly}}+\frac{2\,a}{(h-d_p)}\sqrt{\frac{\varepsilon_{vy}}{\varepsilon_{ly}}}\right\} \quad (43)$$

If we assume $\epsilon_{1y}/\epsilon_{vy}=1$, Eq.(43) is expressed as follows.

$$\Omega_b=\frac{(h-d_p)^2}{2\,s(d-d_p/2)}\left\{1+\frac{2\,a}{(h-d_p)}\right\} \quad (44)$$

When the ratio of force applied to tension reinforcement to that applied to web reinforcement is smaller than that based on Eq.(44), tension reinforcement yields first, followed by web reinforcement, and finally by compression zone failure. When the ratio is larger than that based on Eq.(44), web reinforcement yields first, followed by tension reinforcement, and finally by failure of the concrete in the compression zone. This failure mode is described in Fig.7.

3.6 Interaction between shear force and bending moment

Defining $2s(d-0.5d_p)A_1f_{1y}/A_vf_{vy}=\kappa$, Eq.(33) is rewritten as follows.

$$V_u=\frac{1}{\sqrt{a^2+\kappa}}\left\{A_l f_{ly}\left(d-\frac{d_p}{2}\right)+\frac{A_v f_{vy}}{2\,s}(2\,a^2-2\,a\sqrt{a^2+\kappa}+\kappa)\right\} \quad (45)$$

Bending moment (M_0) subjected to pure bending is given as follows.

$$M_0=A_l f_{ly}(d-d_p/2) \quad (46)$$

Therefore,

$$V_u=2\,M_0\cdot\frac{1}{\kappa}(\sqrt{a^2+\kappa}-a) \quad (47)$$

Substituting $a=M_u/V_u$ into the above equation and adjusting accordingly,

$$\left(\frac{V_u}{M_0}\right)^2+\frac{4\,M_u}{M_0}\cdot\frac{1}{\kappa}=\frac{4}{\kappa} \quad (48)$$

The shear force (V_0) at the pure shear is derived from the condition $M_u=0$. Inserting this into Eq.(48) and adjusting,

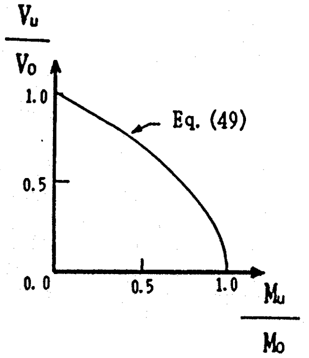


Fig.8 Interaction relationship between shear force and bending moment

$$\left(\frac{V_u}{V_0}\right)^2+\left(\frac{M_u}{M_0}\right)=1 \tag{49}$$

The interaction relationship between shear force and bending moment at the ultimate state is plotted in Fig.8.

4. CONSIDERATIONS ON THE PROPOSED EQUATION

4.1 Effect of factors on the shear capacity

It is well known that there are numerous factors influencing the shear failure. In this section, the shear span-depth ratio (a/d), tension reinforcement ratio (ρ_1) and web reinforcement ratio (ρ_v), which are thought to be the main factors, are examined on the basis of the proposed equation. The following dimensionless factors are used.

$$\omega=\tau_u/f_{ck},\quad \phi=\rho_l f_{ly}/f_{ck},\quad \psi=\rho_v f_{vy}/f_{ck},\quad \eta=a/d \tag{50}$$

Application ranges of dimensionless factors are defined as follows, on the basis of the JSCE model code.

$$0.03\leqq\phi\leqq0.7,\ 0.01\leqq\psi\leqq0.3,\ 1\leqq a/d\leqq4 \tag{51}$$

The relationships between the shear capacity and dimensionless factors are plotted in Fig.9.

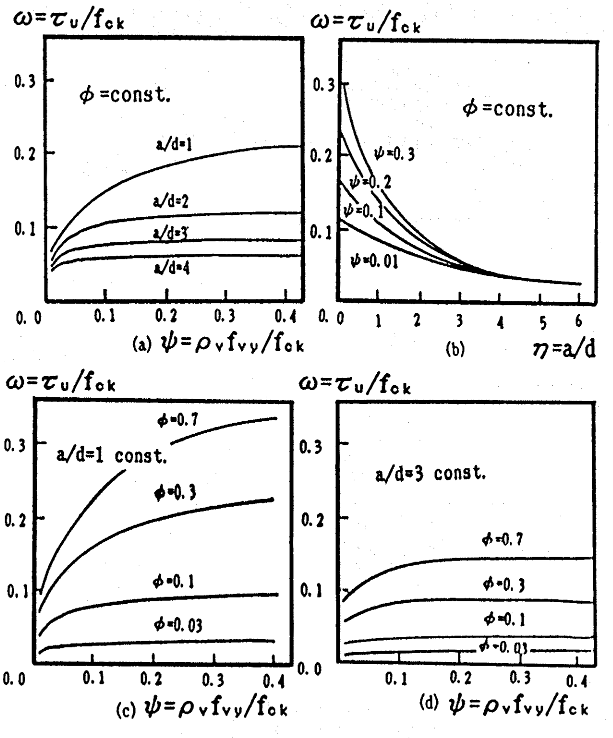


Fig.9 Relationship between the shear capacity and a/d, ϕ, ψ

According to the results, the larger the shear capacity, the smaller the ratio of a/d, and the larger the area of web reinforcement, the larger the shear capacity of the member. However, the tendency for the shear capacity to increase may not be linear, because the shear capacity has an upper limit for the area of web reinforcement. A comparison of shear capacity according to the proposed equation, various published equations and several design codes are given in Fig.10.

As the ACI formula, the JSCE formula, and the CEB-FIP Model code formula are derived on the basis of the truss analogy, the shear capacity has a tendency to increase in proportion to the web reinforcement. The calculated value by the proposed equation has the same tendency as that of Arakawa's equation [23] and Thürlimann's equation [19].

Moreover, the relationship between the inclination of the diagonal the crack (α) and the web reinforcement ratio(ρ_v) under the conditions of ρ_1=2% and f_{ck}=300kgf/cm^2, is described in Fig.11.

According to results, the greater the inclination of the diagonal crack, the larger the ratio of a/d or the web reinforcement ratio. As it is thought that ρ_v is less than 1 to 1.5% in practical design, the inclination of the diagonal crack seems to be less than 60°.

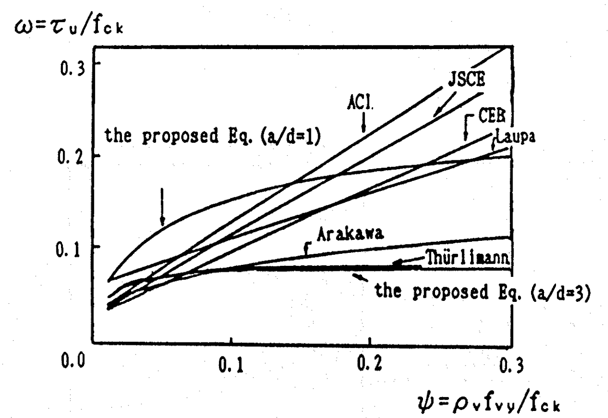


Fig.10 Comparison of the proposed equation and the published experimental data

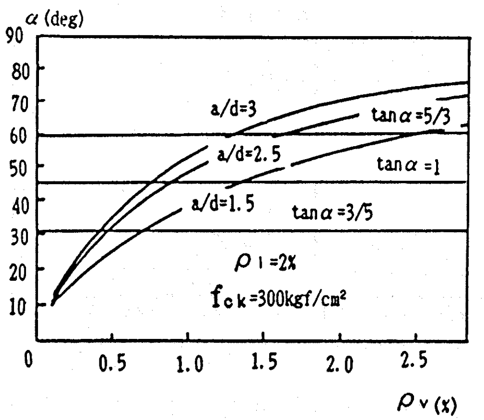


Fig.11 Angle of the diagonal crack

In order to control cracking and to ensure the yielding of both tension reinforcement and web reinforcement, the CEB-FIP Model Code sets up the range of the angle as follows.

$$3/5 \leqq \tan\alpha \leqq 5/3 \tag{52}$$

4.2 Comparison with the published experimental data

In order to evaluate the rationality of the proposed equation, comparisons of analytical results with experimental data are carried out. The number of experimental data to be used for comparison on purposes is 98, an outline of those data being given in Table 2. The experimental data used for comparison are restricted to RC beams with web reinforcement. Frantz carried out a test to study the influence of compressive strength of concrete and the web reinforcement ratio, Smith examined RC beams in which a/d is small, and Regan carried out a test to study the influence of shear span depth ratio for RC members with a larger web reinforcement ratio and a larger tension reinforcement ratio.

Table 2 Outline of the experimental data and comparison of the experimental data and the calculated values

outline of the experimental data	Frantz[24]		Smith[25]		Regan[26]		Kani[27]		Tohoku Uni.	
number of test specimens	12		47		18		15		6	
b(cm)	15.2		10.16		15.24		15.47 ～ 15.70		20.00	
d(cm)	29.8		27.9		25.4		27.2 ～ 28.3		36.0	
a/d	3.6		1.09 ～ 2.27		3.36 ～ 5.05		2.16 ～ 5.98		3.50	
ρ_l(%)	3.36		2.15 ～ 2.67		0.98 ～ 4.16		2.54 ～ 2.64		0.83 ～ 1.94	
ρ_v(%)	0.12 ～ 0.38		0.88 ～ 1.25		1.30 ～ 5.20		0.60 ～ 1.00		0.36 ～ 0.53	
f_{ck}(kgf/cm²)	229 ～ 846		164 ～ 231		130 ～ 438		267 ～ 366		263 ～ 341	
design formula	average	c.o.v.	average	c.o.v.	average	c.o.v.	average	c.o.v.	average	c.o.v.
proposed Eq. (33)	1.05	11.49	1.18	11.90	1.04	17.31	0.98	21.43	1.26	4.80
JSCE(1)	1.29	9.16	1.89	32.10	0.26	21.22	0.53	40.56	0.52	4.65
ACI(2)	1.33	8.59	1.64	31.60	0.23	21.96	0.50	39.71	0.45	5.41
AIJ(3)	3.22	13.72	3.75	12.84	1.41	17.46	1.71	28.03	1.55	8.46
AIJ(4)	1.11	11.57	1.32	7.95	0.30	18.49	0.54	30.71	0.46	7.73
Thürlimann(5)	1.15	14.15	1.22	17.83	0.85	31.60	0.96	22.59	1.36	6.72
Arakawa(6)	1.47	15.61	1.66	7.96	0.85	12.15	0.89	26.57	0.76	10.03
CEB1978(7)	1.16	14.14	1.90	32.67	0.41	22.74	0.51	40.59	0.44	6.12
USSR(8)	0.76	10.56	1.06	25.31	0.22	11.46	0.29	38.80	0.24	8.01

c.o.v.:coefficient of variance(%)

(note)

(1) $V_u=0.9\sqrt[4]{\frac{100}{d}}\sqrt[3]{100\rho_l}\sqrt[3]{f_{ck}}+\frac{A_v f_{vy}}{s}\cdot z$

(2) $V_u=\left(0.504\sqrt{f_{ck}}+\frac{176\rho_l Vd}{M}\right)bd+\frac{A_v f_{vy} d}{s}$

(3) $V_u=\left[\alpha f_s+\frac{1}{2}f_t(\rho_v-0.002)\right]bjd$

(4) $V_u=\frac{1}{2}\left\{\sqrt{\left(\frac{a}{h}\right)^2+1}-\frac{a}{h}\right\}\left\{1-\frac{(1+\cot^2\phi)}{\nu f_{ck}}\rho_{vy}\right\}bh\nu f_{ck}+\rho_v f_{vy}\cot\phi b\cdot j$

(5) $V_u=-\frac{A_v f_{vy}}{s}d+\sqrt{\left(\frac{A_v f_{vy} d}{s}\right)^2+\frac{2jdA_l f_{ly} A_v f_{vy}}{s}}$

(6) $V_u=\left[\frac{0.092k_p k_u(f_{ck}+180)}{\frac{M}{V\cdot d}+0.115}+2.7\sqrt{\rho_v f_{vy}}\right]b\cdot j\cdot d$

(7) $V_u=0.6f_t bd+0.9\rho_v f_{vy} bd$

(8) $V_u=2\sqrt{2bd^2\frac{A_v f_{vd}}{s}\cdot f_t}$

The average value and coefficients of variation of the ratio of the experimental data to the calculated value based on the proposed equation and other design formula are given in Table 2.

The ratios of the experimental values to the calculated values seem to be more than 1 for the RC member with a/d of 3 or less. But the ratio is less than 1 for Regan's and Kani's data. Generally, a weakness of several design equations and published equations is that they do not have a wide range of applications.

The average and the coefficient of variation of the ratios of experimental data to calculated values based on the proposed equation vary from 0.98 to 1.26 and from 4.8 to 21.4%, respectively. Moreover, the average and the coefficient of variation of the ratio for all experimental data in Table 2 are 1.10 and 13.4%, respectively.

The relationships between the value of V_{exp}/V_{cal} and the shear span depth ratio, and the dimensionless web reinforcement ratio (ψ) are described in Fig.12. As mentioned above,it is clear that the proposed equation has a wide range of applications and estimates the shear capacity with acceptable accuracy.

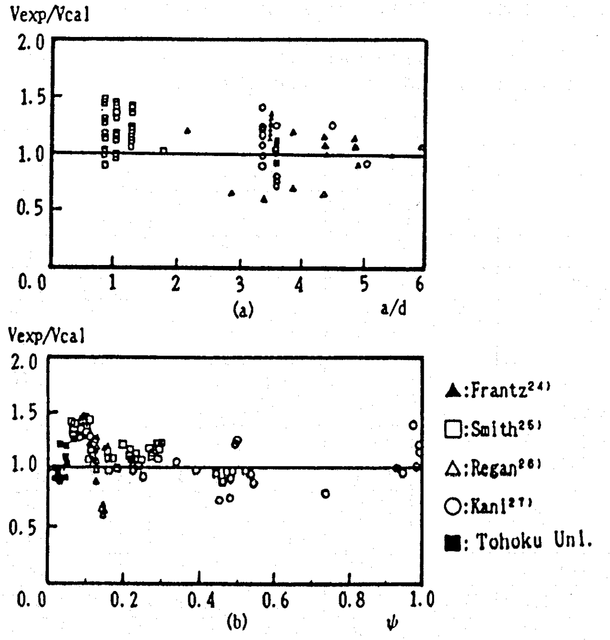


Fig.12 Relationships between V_{exp}/V_{cal} and a/d, ψ

5. CONCLUSION

The purpose of this study was to theoretically evaluate the ultimate shear strengths and the deformation mechanism, especially the inclination of the diagonal crack, of RC beams with web reinforcement subjected to combined bending moment and shear. Obtained results are as follows.

(1) An analytical model based on the equilibrium method of the upper bound theorem is proposed. That is to say, on the assumption that the diagonal crack propagates in the the direction necessary to minimize the internal resistance, the theoretical equation of the shear capacity and inclination of the diagonal crack in the ultimate state are proposed.

(2) The diagonal crack is idealized as only one straight line.

(3) According to the results of the analysis, it is theoretically clear that the smaller the a/d, the greater the shear capacity.

(4) It is theoretically clear that the larger the a/d and the larger the web reinforcement ratio, the larger the inclination of the diagonal crack.

(5) The compatibility between the calculated value and experimental data is very good. According to comparisons with results of other formulae, it is clear that the proposed equation has a wide range of applications and estimates the shear capacity with acceptable accuracy.

The experimental data used for comparison with the proposed equation are still insufficient. This is attributed to the fact that there have been few studies on the inclination of diagonal cracks. Therefore, comprehensive analysis is not possible. In the near future, the authors plan to undertake further experimental studies. This study ignores the influences of the tensile stress of concrete and dowel action, and these factors must also be further examined.

References

[1] Leonhardt, F., "Various problems about the shear in the concrete structure(1) (in Japanese)," Bridge and Foundation, Vol.77, No.4, pp.1-8, Apr., 1977

[2] Kani, G.N.J., "The riddle of shear failure and its solution," Journal of ACI, Vol.61, No.4, pp.441-466, Apr., 1964

[3] Kamiyama, H., "Shear mechanism of reinforced concrete (in Japanese)," Concrete Journal, Vol.6, No.8, pp.24-31, Aug., 1968

[4] Walther, R., "Über die Berechnung der Schubtragfahigkeit von Stahl-und Spannbetonbalken-Schübbruchttheorie-," BETON-UND STAHLBETONBAU, Heft11, pp.261-271, 1962

[5] Minami, K., "Limit analysis of reinforced members subjected to shear (in Japanese)," Proceeding of JCI colloquium on shear analysis of RC structure, JCI, pp.1-16, Jun., 1982

[6] Nielsen, M.P., Braestrup, M.W., Jensen, B.C., and Bach, F., "Concrete plasticity - shear in beam," CEB Bulletin D'Information, No.126, pp.285- 357, 1978

[7] Collins, M.P., "Towards a rational theory for RC members in shear," Journal of Structural Division, ASCE, Vol.104, No.ST4, pp.649-666, Apr., 1978

[8] Ngo, D. and Scordelis, A.C., "Finite element analysis of reinforced concrete beams," Journal of ACI, Vol.64, No.3, pp.152-163, Mar., 1967

[9] Tauchert, T.R., Energy Principles in Structural Mechanics, McGraw-Hill Book Company, 1974

[10] ACI Committee 439, "Effect of steel strength and of reinforcement ratio on the mode of failure and strain energy capacity of reinforced concrete beams," Journal of ACI, Vol.66, No.3, pp.165-173, Mar., 1969

[11] Kupfer, H., "Erweitung der Mörschschen Fachwerkanalogie mit Hilfe des Prinzips vom Minimum der Foränderungsarbeit," CEB Bulletin D'Information, No.40, pp.44-51, 1964

[12] Wagner, H., "Ebene Blechwandtrager mit Sehr dunnen Stegblech," Zeitschrift für Flugtechnic und Motorluftschiffahrt, Vol.20, Nov., pp.8-12, 1929

[13] Isikawa, Y., Yasunaka, K., and Niwa, J., "Methods of predicting shear strength of RC beams with web reinforcement (in Japanese)," Proceedings of the Japanese Concrete Institute, Vol.6, pp. 509-512, 1984

[14] Hamadi, Y.D. and Regan, P.E., "Behaviour of normal and lightweight aggregate beams with shear cracks," The Structural Engineer, Vol.58B, No.4, pp.71-79, Apr., 1980

[15] Cowan, H.J., Reinforced and Prestressed Concrete in Torsion, London, 1965

[16] Krefeld, W.J. and Thurson, C.W., "Studies of the shear and diagonal tension strength of simply supported reinforced concrete beams," Journal of ACI, Vol.63,

No.4, pp.451-476, Apr., 1966
[17] Hedmann, O. and Losberg, A., "Design of concrete structures with regard to shear forces," CEB Bulletin D'Information, No.126, pp.183-209, 1978
[18] Collins, M.P., Walsh, P.F., Archer, F.E. and Hall, A.S., "Ultimate strength of reinforced concrete beams subjected to combined torsion and bending," Torsion of Structural Concrete, SP-18, ACI, pp.379-402, 1968
[19] Thürlimann, B., "Shear strength of reinforced and prestressed concrete beams," CEB Bulletin D'Information, No.126, pp.17-38, 1978
[20] Leonhardt, F., "Shear in concrete structure,"CEB Bulletin D'Information, No.126, pp.67-124,1978
[21] Rensaa, E.M., "Shear, diagonal tension, and anchorage in beams," Journal of ACI, Vol.55, No.6, pp.695-715, 1958
[22] The Public Works Research Institute, Ministry of Construction, "Experimental data of shear test of RC members (in Japanese)," The Public Works Research Institute Material, No.2045, 1984
[23] Arakawa, S., "Study on the shear capacity of RC beams (in Japanese)," Proceeding of Japanese Architecture Engineering, No.66, pp.437-440, 1960
[24] Mphone, A.G. and Frantz, G.C., "Shear tests of high-and low strength concrete beams with stirrups," SP-87, ACI, pp.179-196, 1985
[25] Smith, K.N. and Vansitis, A.S., "Shear strength of deep beams," Journal of ACI, Vol.79, No.3, pp.207-213, Mar., 1982
[26] Placas, A., Regan, P.E. and Baker, A.L.L., "Shear failure of reinforced concrete beams," Journal of ACI, Vol.68, No.10, pp.763-773, Oct., 1971
[27] Kani, G.N.J., "A rational theory for the function of web reinforcement," Journal of ACI, Vol.66, No.3, pp.185-197, Mar., 1969