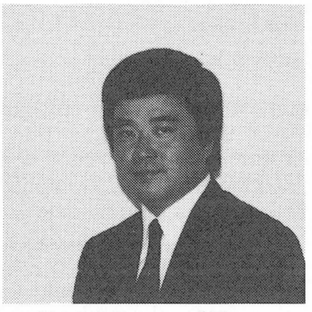
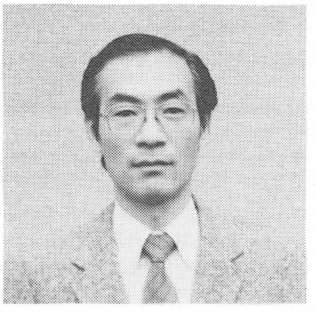
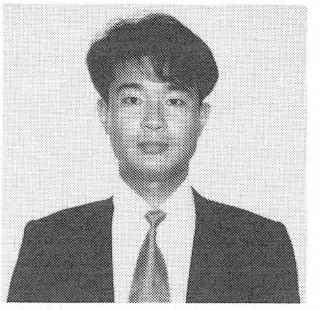
CONCRETE LIBRARY OF JSCE NO.18, DECEMBER 1991

ANALYTICAL STUDIES ON REINFORCED CONCRETE LINEAR MEMBERS SUBJECTED TO TORSION

(Translation from the Proceedings of JSCE, No.420/V-13, Aug. 1990)

Junichiro NIWA Takeshi HIGAI Norikazu MORIYA

SYNOPSIS

Analytical studies on the mechanical behavior of reinforced concrete linear members subjected to torsion have been carried out. The characteristics of the analytical method are to assume an original solid section of members to be equivalent to a hollow one and to convert the applied torsion into the uniform shear flow. Concerning stress-strain relationships of cracked concrete, softening, tension stiffening, and shear transfer along cracks have been incorporated into the analysis. The effect of local yielding of reinforcement is also considered in the analysis. From the verification with experimental results, it has been confirmed that the ultimate strength and the overall deformation behavior of reinforced concrete linear members subjected to torsion can be predicted reasonably by the proposed analytical method.

J. Niwa is associate professor of civil engineering at Nagoya University, Nagoya, Japan. He received his Doctor of Engineering Degree from the University of Tokyo, in 1983. He specializes in reinforced concrete design and analysis for shear and torsion, application of finite-element analysis, and durability design. He is a member of JSCE, JCI, IABSE, ACI, and also a corresponding member of CEB. He got the IABSE Prize in 1990.

T. Higai is a professor of Civil and Environmental Engineering at Yamanashi University, Kofu, Japan. He received his Doctor of Engineering Degree in 1972 from the University of Tokyo. His research interests include problems concerning shear failure of reinforced concrete structures and aseismic design of reinforced concrete structures. He is a member of several research committee on concrete and reinforced concrete at JSCE and JCI.

N. Moriya is a construction engineer at Shimizu Corporation, Tokyo, Japan. He received his Master of Engineering Degree in 1989 from Yamanashi University. His research interests include design and analysis of reinforced concrete structures subject to torsion. He is a member of JSCE and JCI.

---

1. INTRODUCTION

Torsion problems on reinforced concrete members have been the research subject for long years. Traditionally, the theory of elasticity or plasticity for continuum was the basis of torsional analysis for reinforced concrete. The analysis for shear stresses due to torsion by using Prandtl's membrane analogy or Nadai's sand heap analogy is very famous. However, because of the nonlinearity pertaining to reinforced concrete members, such as crack initiation and compression failure under multi-axial stresses, or yielding of reinforcement, the mechanical behavior of reinforced concrete members cannot be explained accurately by only the theory of elasticity or plasticity. According to elasto-plastic behavior of concrete subjected to tension, neither the theory of elasticity nor plasticity can predict the torsional cracking moment itself accurately [1].

In subsequent researches, therefore, the characteristics of reinforced concrete were considered. Main efforts of researches were laid on the macroscopic modeling of resisting mechanism. For example, the skew bending theory that predicts the ultimate torsional strength on the analogy of the bending failure and the space truss theory that was extended from the plane truss theory applicable for shear problems are very famous. The greatest advantage in these macroscopic models is that they can predict the ultimate torsional strength by relatively simple calculations. For the examination of the ultimate limit state, these macroscopic models are actually very useful. Particularly, the space truss theory has been adopted in CEB-FIP Model Code 1978 and JSCE Standard Specification for Design of Concrete Structures, because of its practical usefulness.

Although these macroscopic models are effective, such a modeling that torsional failure can be considered as a type of bending failures or resisting mechanism like as a space truss can be built up in a member is not the fact but the assumption. These models can predict the ultimate torsional strength easily; however, it is quite difficult for them to predict the overall mechanical behavior, that is, stress states and deformation of a member corresponding to arbitrary loading conditions. Practical usefulness of these models cannot be ignored; however, the establishment of more general and comprehensive analytical method is being desired for an elaborate design.

In such a trend, remarkable researches have been performed by Collins et al. They state that what exists in a member is not a truss mechanism, such as struts and ties, but a reinforced concrete element, and that the clear understanding for mechanical behavior of reinforced concrete elements with cracks is the fundamental of all analyses. Okamura et al. have also been carrying forward their researches from the same viewpoint, and have proposed the constitutive laws including the effect of cyclic loadings [2].

The common concept in these researches is that if the mechanical behavior of cracked reinforced concrete elements can be estimated accurately, problems on shear and torsion will be treated analytically based on obtained elements as like as flexural problems. From this viewpoint, Collins et al. have formulated both the softening behavior of cracked concrete and the tension stiffening by using experimental results obtained from the test of reinforced concrete plates subjected to in-plane forces [3], and have tried to apply the formulas into torsional analysis. Hsu et al. have also been performing torsional analysis considering the softening of cracked concrete [4]. In Japan, Okamoto et al. [5] and the authors [6] have proceeded the research on torsion in this way.

According to this kind of approach, not only the ultimate strength of a member but also the load-deformation relationship can be predicted continuously. Although calculation procedures required become more complicated than macroscopic models, except for this point such an approach mentioned in this paper surpasses macroscopic models in all other aspects. In this paper, the analytical method for reinforced concrete linear members subjected to torsion is described. The method has been formulated based on constitutive models for a reinforced concrete plate element. The applicability of the method is examined by comparing obtained analytical results with experimental data.

2. TORSIONAL ANALYSIS USING REINFORCED CONCRETE PLATE ELEMENT

2.1 Outline

The analytical method for torsion presented here comprises of the part peculiar to linear members subjected to torsion and the other part pertaining to reinforced concrete plates subjected to in-plane forces. In the beginning, the analytical procedure for reinforced concrete plates is outlined, and afterward the analytical procedure for torsion is summarized.

2.2 Analytical Procedure for Reinforced Concrete Plate Elements Subjected to In-Plane Forces

(a) Strain Compatibility Condition

Consider a rectangular and hollow cross section subjected to torsion, T. Even though an actual cross section concerned is solid, it may be assumed to be an imaginary hollow one. According to the theory of elasticity, shear stresses on solid cross section due to torsion become zero at the centroid of section. As the location where stresses are applied is approaching to the perimeter of cross section, the magnitude of shear stresses is becoming large. Resisting moment for torsion becomes large in proportion to the distance from the centroid of section. Therefore, neglecting the influence of center part of cross section is not an unrealistic assumption.

According to Bredt's torsional theory on thin-walled closed cross section, a uniform shear flow, q, is applied to every side wall forming the member (Fig. 1). Fig. 2 shows a free-body diagram of a reinforced concrete element taken out from an arbitrary side wall. The shear stress on a reinforced concrete plate, τ, can be obtained by dividing q by the thickness of a plate, t. Applied shear stress, τ, is resisted by normal and shear stresses of concrete and tensile stresses of torsional reinforcement in longitudinal (x) and transverse (y) directions.

With the increase in applied shear stress, cracks start initiating in reinforced concrete plates. In the case of pure torsion, the stress state of reinforced concrete plates is equivalent to pure shear. Therefore, reinforced concrete plates are subjected to biaxial principal tensile and compressive stresses ($\sigma_1 = -\sigma_2$). Traditionally, the assumption that cracks occur perpendicular to principal tensile stress was made for this kind of problem [7]. In this research, the same assumption has been utilized.

The directions parallel and perpendicular to cracks are designated as c- and t-axes, respectively. For the case of pure torsion, the angle β between x- and t-axes becomes *45* degrees. Generally, c-t axes do not coincide with the principal strain axes of reinforced concrete plates because shear stresses should be transferred along crack surfaces. The principal strain axes are designated as *1-2* axes in a reinforced concrete plate (Fig. 2). The angle between the principal tensile strain axis (*1*-

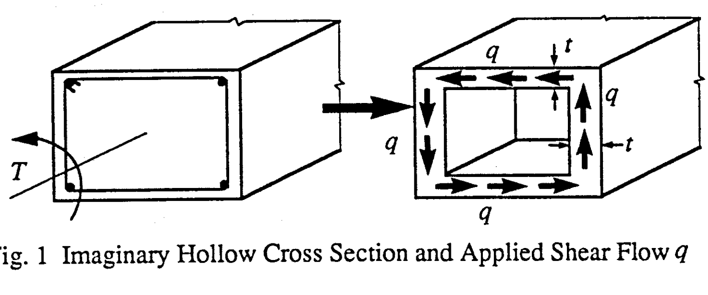


Fig. 1 Imaginary Hollow Cross Section and Applied Shear Flow q

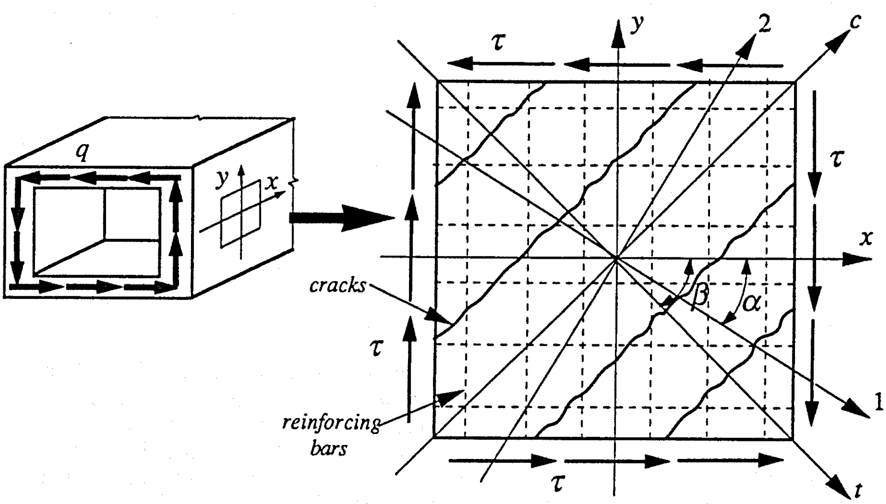


Fig. 2 Reinforced Concrete Plate Element Taken Out From Hollow Section

axis) and x-axis is defined as α. Before the initiation of cracks, the angle α is equal to the angle β. However, once cracks occur, α is not coincident with β because of the existence of shear stresses along crack surfaces.

The Mohr's strain circle of Fig. 3 exhibits the relationship of strains. The strains in Fig. 3 are averaged strain in the direction of plate thickness (z-axis). According to graphical relationships exhibited in Fig. 3, the following equations are obtained for these strains:

$$\varepsilon_x, \varepsilon_y = (\varepsilon_{ct} + \varepsilon_{cc})/2 \pm \frac{1}{2}(\varepsilon_{ct} - \varepsilon_{cc})\frac{cos2\alpha}{cos2(\beta\text{-}\alpha)} \quad (1)$$

$$\gamma_{xy} = (\varepsilon_{ct} - \varepsilon_{cc})\frac{sin2\alpha}{cos2(\beta\text{-}\alpha)} \quad (2)$$

$$\gamma_{cct} = (\varepsilon_{ct} - \varepsilon_{cc})\ tan2(\alpha\text{-}\beta) \quad (3)$$

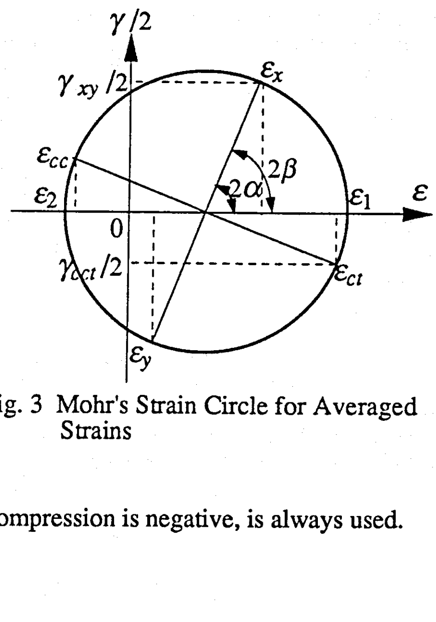


Fig. 3 Mohr's Strain Circle for Averaged Strains

In this paper, the notation that tension is positive, and compression is negative, is always used.

(b) Equilibrium Condition

When stresses of both concrete and reinforcement have been obtained from strains by using stress-strain relationships that will be described in Chapter 3, the average stresses of concrete in the c-t directions can be converted into the following average stresses in the x-y directions by coordinate transformation.

$$\sigma_{cx} = \sigma_{ct}\ cos^2\beta + \sigma_{cc}\ sin^2\beta - \tau_{cct}\ sin2\beta \quad (4)$$

2. TORSIONAL ANALYSIS USING REINFORCED CONCRETE PLATE ELEMENT

2.1 Outline

The analytical method for torsion presented here comprises of the part peculiar to linear members subjected to torsion and the other part pertaining to reinforced concrete plates subjected to in-plane forces. In the beginning, the analytical procedure for reinforced concrete plates is outlined, and afterward the analytical procedure for torsion is summarized.

2.2 Analytical Procedure for Reinforced Concrete Plate Elements Subjected to In-Plane Forces

(a) Strain Compatibility Condition

Consider a rectangular and hollow cross section subjected to torsion, T. Even though an actual cross section concerned is solid, it may be assumed to be an imaginary hollow one. According to the theory of elasticity, shear stresses on solid cross section due to torsion become zero at the centroid of section. As the location where stresses are applied is approaching to the perimeter of cross section, the magnitude of shear stresses is becoming large. Resisting moment for torsion becomes large in proportion to the distance from the centroid of section. Therefore, neglecting the influence of center part of cross section is not an unrealistic assumption.

According to Bredt's torsional theory on thin-walled closed cross section, a uniform shear flow, q, is applied to every side wall forming the member (Fig. 1). Fig. 2 shows a free-body diagram of a reinforced concrete element taken out from an arbitrary side wall. The shear stress on a reinforced concrete plate, τ, can be obtained by dividing q by the thickness of a plate, t. Applied shear stress, τ, is resisted by normal and shear stresses of concrete and tensile stresses of torsional reinforcement in longitudinal (x) and transverse (y) directions.

With the increase in applied shear stress, cracks start initiating in reinforced concrete plates. In the case of pure torsion, the stress state of reinforced concrete plates is equivalent to pure shear. Therefore, reinforced concrete plates are subjected to biaxial principal tensile and compressive stresses ($\sigma_1 = -\sigma_2$). Traditionally, the assumption that cracks occur perpendicular to principal tensile stress was made for this kind of problem [7]. In this research, the same assumption has been utilized.

The directions parallel and perpendicular to cracks are designated as c- and t-axes, respectively. For the case of pure torsion, the angle β between x- and t-axes becomes 45 degrees. Generally, c-t axes do not coincide with the principal strain axes of reinforced concrete plates because shear stresses should be transferred along crack surfaces. The principal strain axes are designated as 1-2 axes in a reinforced concrete plate (Fig. 2). The angle between the principal tensile strain axis (1-

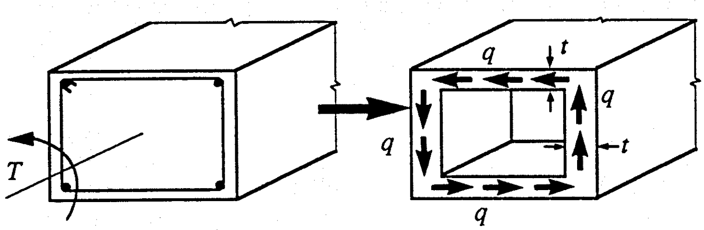


Fig. 1 Imaginary Hollow Cross Section and Applied Shear Flow q

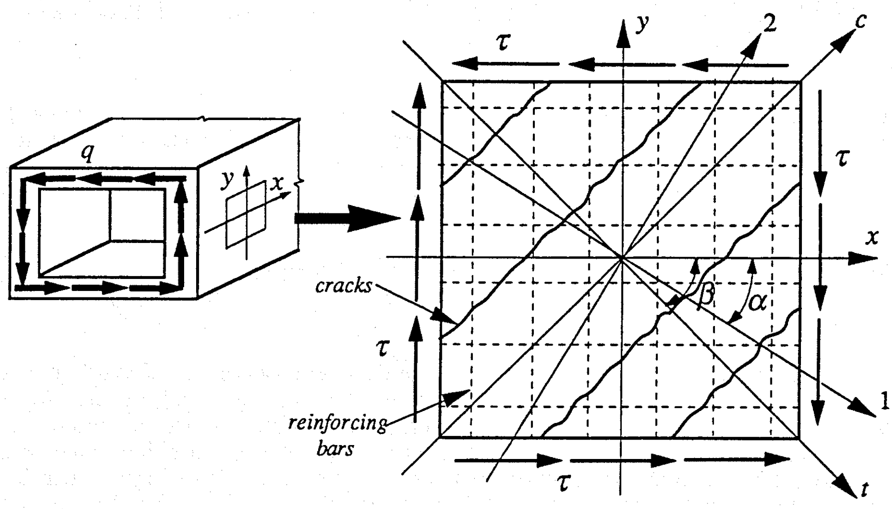


Fig. 2 Reinforced Concrete Plate Element Taken Out From Hollow Section

axis) and x-axis is defined as α. Before the initiation of cracks, the angle α is equal to the angle β. However, once cracks occur, α is not coincident with β because of the existence of shear stresses along crack surfaces.

The Mohr's strain circle of Fig. 3 exhibits the relationship of strains. The strains in Fig. 3 are averaged strain in the direction of plate thickness (z-axis). According to graphical relationships exhibited in Fig. 3, the following equations are obtained for these strains:

$$\varepsilon_x,\ \varepsilon_y = (\varepsilon_{ct} + \varepsilon_{cc})/2 \pm \frac{1}{2}(\varepsilon_{ct} - \varepsilon_{cc})\frac{cos2\alpha}{cos2(\beta\text{-}\alpha)} \quad (1)$$

$$\gamma_{xy} = (\varepsilon_{ct} - \varepsilon_{cc})\frac{sin2\alpha}{cos2(\beta\text{-}\alpha)} \quad (2)$$

$$\gamma_{cct} = (\varepsilon_{ct} - \varepsilon_{cc})\ tan2(\alpha\text{-}\beta) \quad (3)$$

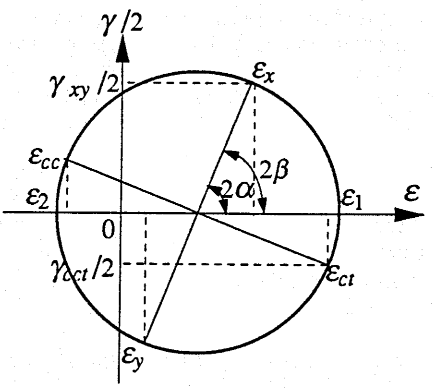


Fig. 3 Mohr's Strain Circle for Averaged Strains

In this paper, the notation that tension is positive, and compression is negative, is always used.

(b) Equilibrium Condition

When stresses of both concrete and reinforcement have been obtained from strains by using stress-strain relationships that will be described in Chapter 3, the average stresses of concrete in the c-t directions can be converted into the following average stresses in the x-y directions by coordinate transformation.

$$\sigma_{cx} = \sigma_{ct}\ cos^2\beta + \sigma_{cc}\ sin^2\beta - \tau_{cct}\ sin2\beta \quad (4)$$

$$\sigma_{cy} = \sigma_{ct} \sin^2\beta + \sigma_{cc} \cos^2\beta + \tau_{cct} \sin 2\beta \tag{5}$$

$$\tau_{cxy} = (\sigma_{ct} - \sigma_{cc}) \sin\beta \cos\beta + \tau_{cct} \cos 2\beta \tag{6}$$

The average stresses of reinforcement are σ_{sx} and σ_{sy}. For the case of pure torsion, applied normal stresses in x-y axes are zero. After all, the following equilibrium equations are obtained.

$$p_x \sigma_{sx} + \sigma_{cx} = 0 \tag{7}$$

$$p_y \sigma_{sy} + \sigma_{cy} = 0 \tag{8}$$

$$\tau_{cxy} = \tau \tag{9}$$

where, p_x and p_y are reinforcement ratio in the direction of x-y axes, respectively.

2.3 Outline of Torsional Analysis of Reinforced Concrete Linear Members

(a) Deformation State of Reinforced Concrete Plate Elements

The Mohr's strain circle of Fig. 3 exhibits the relationship of averaged strains in the direction of plate thickness. Actually, each side wall forming a hollow cross section of a member deforms with two reverse curvatures such as a saddle (Fig. 4). Therefore, the strain distribution in the direction of plate thickness is not uniform, and the average stress cannot be calculated directly from the average strain in the direction of plate thickness. This is a peculiar aspect of torsional problem.

If a displacement in the direction of plate thickness (z-direction) at an arbitrary point, $A(x,y)$, within a reinforced concrete plate element is designated as w, the displacement w can be represented by using the angle of twist per unit length, θ.

$$w = \theta \cdot x \cdot y \tag{10}$$

Since the directions parallel and perpendicular to cracks have been designated as c-t axes, respectively, the relationships of $x = c \sin\beta$ and $y = c \cos\beta$ are obtained as shown in Fig. 4. The curvature in c-direction, ϕ_c, can be determined as follows:

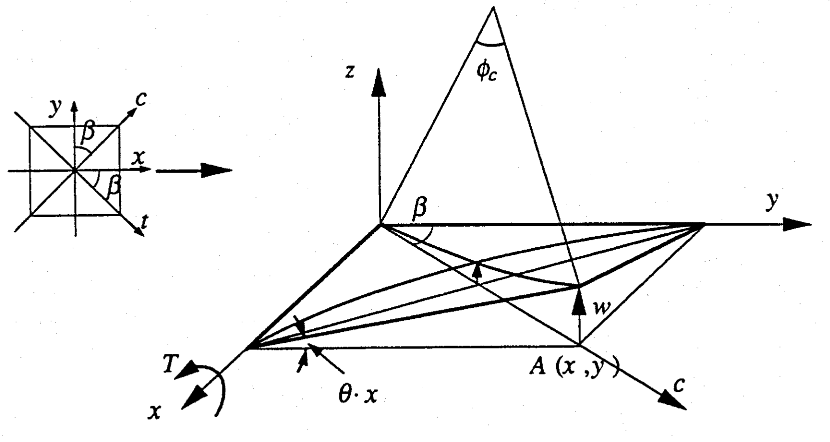


Fig. 4 Deformation of Reinforced Concrete Plate Subjected to Torsion

$$dx/dc = \sin\beta,\quad dy/dc = \cos\beta$$
$$\frac{dw}{dc} = \frac{\partial w}{\partial x}\frac{dx}{dc} + \frac{\partial w}{\partial y}\frac{dy}{dc} = \theta\, y \sin\beta + \theta\, x \cos\beta$$
$$\therefore\ \phi_c = -\frac{d^2w}{dc^2} = -\frac{\partial}{\partial x}\left(\frac{dw}{dc}\right)\frac{dx}{dc} - \frac{\partial}{\partial y}\left(\frac{dw}{dc}\right)\frac{dy}{dc} = -\,\theta \sin 2\beta \qquad (11)$$

The curvature in the t-direction, ϕ_t, can be determined in the same way. ϕ_t becomes $-\phi_c$.

The strain ε_{cc} is not a principal strain. However, the curvature ϕ_c becomes the maximum value according to Eq. (11), because of the assumption that β is *45* degrees for pure torsion. This means that the directions of principal curvatures and principal strains do not coincide. The reason why these directions are not coincident with each other is that β has been assumed to be constant throughout the plate thickness. As a result of calculations, the influence of this discrepancy was admitted to be slight. Therefore, it was decided to use Eq. (11) without any modification in the following analysis.

The normal strain distribution on each cross section perpendicular to c- and t- directions was assumed as Fig. 5. The average normal strains on each cross section are ε_{ct} and ε_{cc}, respectively. The strains ε_{ct} and ε_{cc} are shown in Fig. 3. For the elastic stage before cracking, the surface strains of a plate are considered to have the relationship of $\varepsilon_{cs} = -\varepsilon_{ts}$ as shown in Fig. 5(a). However, for both the inelastic stage before cracking and the stage after cracking, the tensile strain increases very rapidly with the decrease in the tension stiffening of concrete, and thus the strain distribution in the c- and t- directions are no more symmetric each other. The strain distributions are considered to become such a state as shown in Fig. 5(b).

The effective depth of a plate for torsional analysis, t_d, was assumed to be equal to the distance between the surface of a plate and the location where compressive strain on a cross section becomes zero. According to this assumption, the surface strain in a compression side, ε_{cs}, becomes $2\varepsilon_{cc}$, and the strain in a tension side changes linearly from $\varepsilon_{ct}-\varepsilon_{cc}$ to $\varepsilon_{ct}+\varepsilon_{cc}$. The surface strain ε_{cs} can be obtained from Eq. (12).

$$\varepsilon_{cs} = 2\,\varepsilon_{cc} = \phi_c\, t_d = -\,t_d\,\theta \sin 2\beta \qquad (12)$$

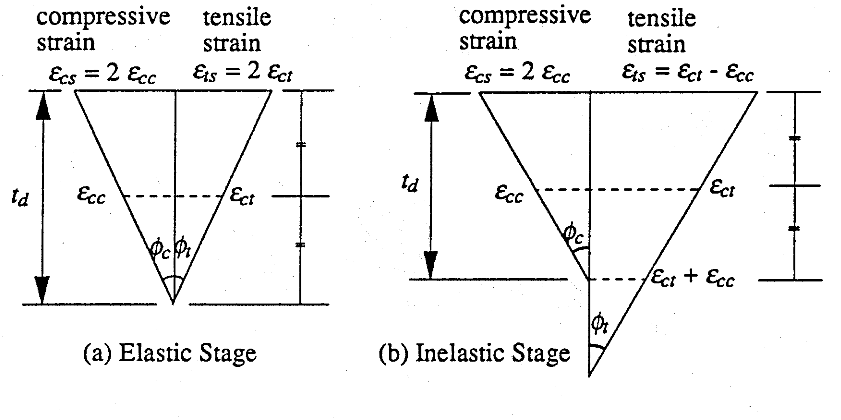
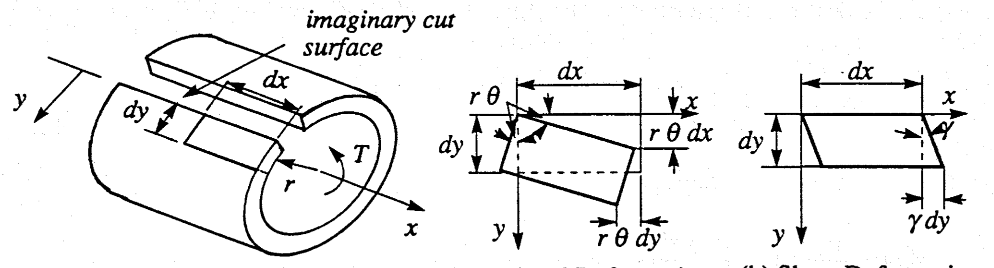


Fig. 5 Strain Distribution of Reinforced Concrete Plate in the Direction of Plate Thickness



(a) Torsional Deformation (b) Shear Deformation

Fig. 6 Longitudinal Displacement of Closed Cross Section Subjected to Torsion

(b) Relationship between Shear Strain and Angle of Twist

The relationship between the shear strain of a reinforced concrete plate, γ_{xy}, and the angle of twist per unit length, θ, can be obtained by utilizing the assumption that the difference of longitudinal displacement should not occur for a closed cross section. In a hollow cross section as shown in Fig. 6, we consider an infinitesimal rectangular element along an imaginary cut surface. The longitudinal displacement due to the angle of twist, du_1, can be obtained as follows:

$$du_1 = -r\,\theta\,dy$$

The longitudinal displacement due to the shear strain, du_2, can be obtained as follows:

$$du_2 = \gamma_{xy}\,dy$$

The integrated value for infinitesimal displacements with respect of the perimeter of a whole cross section should be zero because the difference of longitudinal displacement cannot exist for a closed cross section. According to this condition, the relationship between the shear strain and the angle of twist per unit length can be derived as Eq. (13).

$$\oint du = -\theta \oint r\,dy + \oint \gamma_{xy}\,dy = -2A_o\,\theta + \gamma_{xy}\,p_o = 0$$

$$\therefore\ \theta = \frac{\gamma_{xy}\,p_o}{2A_o} \tag{13}$$

where, A_o is the area enclosed by the center line of effective thickness, and p_o is the length of perimeter of the center line. In the case of a rectangular cross section having the width b and the height h, A_o and p_o can be calculated by Eqs. (14) and (15).

$$A_o = (b - t_d)(h - t_d) \tag{14}$$

$$p_o = 2(b + h - 2t_d) \tag{15}$$

(c) Compatibility Condition of Deformation

By arranging the above relationships, the compatibility equation of deformation for torsional analysis can be derived. By substituting the shear strain of Eq. (2) into Eq. (13), Eq. (16) can be obtained.

$$\theta = \frac{p_o}{2\,A_o}(\,\varepsilon_{ct} - \varepsilon_{cc}\,)\frac{\sin 2\alpha}{\cos 2(\beta\text{-}\alpha)} \tag{16}$$

Next, θ of Eq. (16) is substituted into Eq. (11).

$$\phi_c = -\frac{p_o}{2\,A_o}(\,\varepsilon_{ct} - \varepsilon_{cc}\,)\frac{\sin 2\alpha\,\sin 2\beta}{\cos 2(\beta\text{-}\alpha)} \tag{17}$$

By substituting ϕ_c of Eq. (17) into Eq. (12) and arranging it, finally the following equation can be obtained.

$$\varepsilon_{cc} = \frac{p_o\,t_d \sin 2\alpha \sin 2\beta}{p_o\,t_d \sin 2\alpha \sin 2\beta - 4\,A_o \cos 2(\beta\text{-}\alpha)}\,\varepsilon_{ct} \tag{18}$$

Eq. (18) is the compatibility equation of deformation for torsional analysis. When the average strain of a reinforced concrete plate, ε_{ct}, perpendicular to cracks is given, the average strain, ε_{cc}, parallel to cracks can be determined uniquely based on the assumed α and t_d values.

For the elastic stage of concrete, the compatibility equation becomes simple. As exhibited in Fig. 5(a), ε_{cc} is equal to $-\varepsilon_{ct}$, and thus γ_{xy} becomes $-2\varepsilon_{cc}$. Considering the assumption that β is 45 degrees, t_d becomes $-2\varepsilon_{cc}/\theta$ from Eq. (12). By using the relationship of Eq. (13) for θ, t_d becomes $2A_o/p_o$. For a rectangular cross section, t_d is equal to $(b+h-\sqrt{b^2-bh+h^2})\,/\,3$ because A_o and p_o are represented by Eqs. (14) and (15).

3. THE CALCULATION OF AVERAGE STRESSES AND THE LOCATION OF STRESS RESULTANTS

3.1 Average Tensile Stress of Concrete

Concrete stresses of a reinforced concrete plate are considered in the direction parallel and perpendicular to cracks, and then they are converted into stresses in x-y directions by coordinate transformation. Since stresses are not uniform within effective thickness, the average value has to be determined. Most analytical methods previously proposed assumed that the location of stress resultants coincides with the half of effective thickness. Strictly speaking, it is obviously wrong. To determine the location of stress resultants rigorously for both tension and compression stresses is one of the characteristics of this analysis.

As for the estimation of tensile stress of concrete in the direction perpendicular to cracks, the stress-strain relationship considering the tension stiffening, that was proposed by Okamura et al. [8], has been adopted.

$$0 \le \varepsilon_t < \varepsilon_{cr} \;\Rightarrow\; \sigma_t = E_c\,\varepsilon_t \le f_t$$
$$\varepsilon_{cr} \le \varepsilon_t \;\Rightarrow\; \sigma_t = f_t\,(\varepsilon_{cr}/\varepsilon_t)^c$$

where, $\varepsilon_{cr} = 0.002$, and $c = 0.4$. Young's modulus of concrete has been determined by the following equation.

$$E_c = 40000\, f_c'^{1/3} (kgf/cm^2)$$

As for the tensile strength of concrete, experimental values reported were generally used. When experimental data was not available, it has been predicted by using the following equation.

$$f_t = 0.583\, f_c'^{2/3} (kgf/cm^2)$$

When the surface strain in tensile side does not exceed f_t / E_c, concrete is judged to be in the elastic stage. Whenever the surface strain surpasses f_t / E_c, the compatibility condition of Eq. (18) should be applied.

The average tensile stress of concrete within the plate thickness can be calculated from Eq. (19).

$$\sigma_{ct} = \frac{1}{t_d} \int_0^{t_d} \sigma_t \, dz \tag{19}$$

The location of stress resultant, z_t, can be obtained by Eq. (20).

$$z_t = \int_0^{t_d} \sigma_t \, z \, dz \, / \, (\sigma_{ct}\, t_d) \tag{20}$$

where, the origin of z-axis is the location of depth t_d from the surface.

3.2 Average Compressive Stress of Concrete

As for the estimation of compressive stress of concrete in the direction parallel to cracks, the stress-strain relationship considering the softening of concrete, that was proposed by Collins et al. [9], has been applied.

$$\sigma_c = -\,\eta\, f_c' \left[2\left(\frac{\varepsilon_c}{\varepsilon_o}\right) - \left(\frac{\varepsilon_c}{\varepsilon_o}\right)^2 \right] \tag{21}$$

where, $\eta = 1/[0.8 - 0.34\, (\varepsilon_t / \varepsilon_o)] \leq 1$, and $\varepsilon_o = -0.002$. The case that ε_c reaches -0.0035 was defined as the ultimate state.

The ultimate state, in other words, the failure of reinforced concrete is considered to be governed by the failure of concrete unless reinforcement is broken. This is a general principle, and is not restricted to only torsion. In this research, we have assumed that when ε_c reaches -0.0035 is the ultimate state. Although it is quite clear that the magnitude of ultimate strength is greatly dependent on the amount of reinforcement provided, we do not consider that the yield of reinforcement directly means the failure of reinforced concrete.

If the stress distribution within the effective thickness is obtained, the average compressive stress σ_{cc} and the location of stress resultant z_c can be calculated in the same way as Eqs. (19) and (20).

3.3 Shear Stress of Concrete

For the estimation of the average shear stiffness after cracking, the method proposed Izumo et al. [7] has been adopted. In this method, the average shear stiffness is estimated as a series of shear stiffness on crack surfaces and elastic shear stiffness between cracks. As for the estimation of the shear stiffness on crack surfaces, Aoyagi-Yamada's equation has been applied. Therefore, the average shear stiffness after cracking, G_{av}, can be calculated by Eq. (22).

$$\frac{1}{G_{av}} = \frac{1}{G_c} + \frac{1}{G_{cr}} \tag{22}$$

where, G_c is the elastic shear modulus, and it can be obtained from Young's modulus and Poisson's ratio. G_{cr} is the shear stiffness on crack surfaces that is given by Aoyagi-Yamada's equation [Eq. (23)].

$$G_{cr} = 36 / \varepsilon_{cr} \tag{23}$$

Eventually, the average shear stress after cracking, τ_{cct}, can be obtained as follows:

$$\tau_{cct} = \gamma_{cct}\, G_{av}$$

The location of stress resultants was assumed to be $z_s = t_d /2$.

The shear stress of concrete in the x-y direction, τ_{cxy}, is obtained from Eq. (6). Considering the equilibrium of moments with respect to x axis, the location of stress resultant for τ_{cxy} can be obtained as follows:

$$z_q = \left[\left(\sigma_{ct}\, z_t - \sigma_{cc}\, z_c\right) sin\beta\, cos\beta + \tau_{cct}\, z_s\, cos2\beta\right] / \tau_{cxy} \tag{24}$$

For a rectangular cross section, the area enclosed by the location of stress resultant, A_1, becomes $(b\text{-}2z_q)(h\text{-}2z_q)$. The torsional moment, T, is calculated by using A_1.

$$T = 2\, A_1\, \tau_{cxy}\, t_d \tag{25}$$

3.4 Tensile Stress of Reinforcement

In principle, the tensile stresses of reinforcement, σ_{sx} and σ_{sy}, are obtained by multiplying strains in the x- and y-directions by Young's modulus of reinforcement. However, once reinforcing bars yield locally at the location of cracks, it is known that the average stiffness of reinforcement decreases less than the stiffness of a reinforcing bar itself. To deal with this phenomenon, the judgment on local yielding of reinforcement has been performed by using local stresses of reinforcement at the location of cracks. The local stresses are calculated from Eq. (26).

$$\begin{aligned} \sigma_{sxcr} &= (\,\tau + \tau_{ctcr}\,)\, tan\beta \;/\, p_x \\ \sigma_{sycr} &= (\,\tau - \tau_{ctcr}\,)\, cot\beta \;/\, p_y \end{aligned} \tag{26}$$

where, τ is the applied shear stress, and τ_{ctcr} is the shear stress carried by concrete on crack surfaces, that can be obtained as $\tau_{ctcr} = G_{cr}\, \gamma_{cct}$. Whenever the local yielding is recognized, the average stress-average strain relationship of reinforcement proposed by Okamura et al. [11] has been utilized.

4. ANALYTICAL PROCEDURES AND CALCULATED EXAMPLES

4.1 Analytical Procedures and Flow Chart

Analytical procedures are as follows:

(1) Input the properties of materials, and the dimensions and details of cross section.

(2) Consider a reinforced concrete plate element taken out from an imaginary hollow cross section. Assign the average tensile strain, ε_{ct}, of the plate in the direction perpendicular to cracks.

(3) Assume the angle α between the directions of the principal tensile strain and the x axis.

(4) Assume the effective thickness t_d of the plate.

(5) Calculate the area enclosed by the center line of effective thickness, A_o, the length of perimeter of the center line, p_o, and the reinforcement ratios in the x- and y- directions, p_x and p_y.

(6) By using the equation of compatibility condition [Eq. (18)], calculate the average compressive strain, ε_{cc}, of the plate in the direction parallel to cracks.

(7) Calculate the strains, ε_x, ε_y, γ_{xy}, and γ_{cct} from Eqs. (1) ~ (3).

(8) Calculate the average tensile, compressive, and shear stresses of concrete (σ_{ct}, σ_{cc}, and τ_{cct}), the location of stress resultants (z_t, z_c, and z_s), and the tensile stresses of reinforcement (σ_{sx} and σ_{sy}).

(9) By using Eqs. (4) ~ (6), convert the average concrete stresses into the stresses in the x- and y- directions.

(10) Verify the equilibrium condition in x direction [Eq. (7)]. If it is not satisfied, return to the step (4). Assume t_d again and repeat the procedure. If t_d converges to a certain value and the equilibrium condition is satisfied, go to the step (11).

(11) Verify the equilibrium condition in y direction [Eq. (8)]. If it is not satisfied, return to the step (3). Assume α again and repeat the

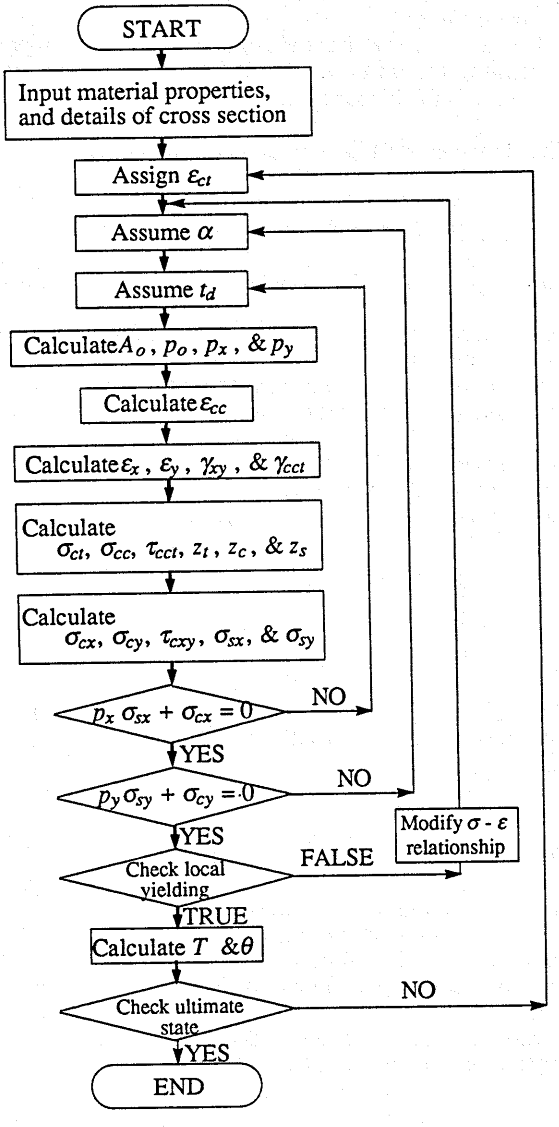


Fig. 7 The Flow Chart for Analytical Procedures

procedure. If α converges to a certain value and the equilibrium condition is satisfied, go to the step (12).

(12) Judge as to whether or not the reinforcement yields locally. If obtained status of reinforcement is different from previously assumed one, return to the step (3). Modify the stress-strain relationship of reinforcement, and repeat the procedure. If obtained status coincides with the assumed one, go to the step (13).

(13) Calculate the torsional moment, T, from Eq. (25) and the angle of twist per unit length, θ, from Eq. (13).

(14) Unless the surface strain of concrete in the compression side of the plate reaches the ultimate strain, return to the step (2). Assign ε_{ct} again and repeat the procedure. If the surface strain exceeds the ultimate strain, stop the calculation.

The flow chart of Fig. 7 exhibits this procedure.

4.2 Calculated Examples for Model Specimens

(a) Comparison with the solution of the theory of elasticity

Before cracks occur due to torsion, reinforced concrete exhibits the behavior as an elastic body. Elastic analysis for torsion, after all, comes to find a stress function, Ψ, that satisfies Poisson's differential equation and also the boundary condition that the function has to be zero on the perimeter of cross section.

When the axis of a member is designated as z axis, and x-y axes are assigned on the cross section, the stress function for torsion, Ψ, has to satisfy the following Poisson's equation [Eq. (27)] and the boundary condition [Eq. (28)].

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -2\,G\,\theta \qquad (27)$$

$$\Psi = \Psi(x, y) = 0 \quad \text{(on the perimeter of the cross section)} \qquad (28)$$

If the stress function, Ψ, can be obtained, the torsional moment, T, will be calculated from Eq. (29).

$$T = 2\iint \Psi\, dx\, dy \qquad (29)$$

The stress function for circular or elliptic cross section can be presented easily by using a simple quadratic function. However, a theoretical stress function cannot be obtained for rectangular cross section because it is difficult to represent an analytical boundary condition. Generally, Fourier's series has been utilized to give the stress function for rectangular cross section, and in practice, such an approximate equation as Eq. (30) is often used.

$$T = \beta\, b^3\, h\, G\, \theta \qquad (30)$$

where, b and h are the length of the short and long sides of a rectangular, respectively. β is the

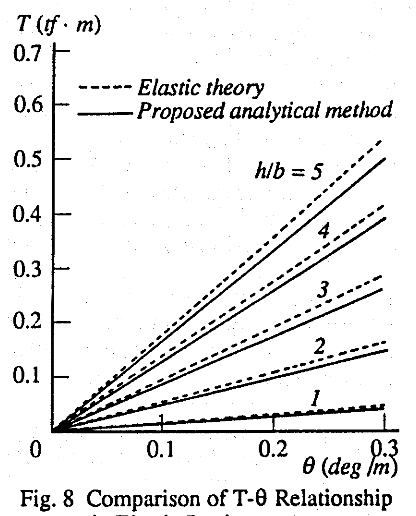


Fig. 8 Comparison of T-θ Relationship in Elastic Region

coefficient depending on the ratio of h/b. The torsional stiffness of rectangular cross section changes remarkably with the ratio of the length of sides.

In Fig. 8, the T-θ relationships obtained from Eq. (30) are shown by broken lines. In this example, b was fixed to *10 cm* and the ratio of h/b was changed from *1* to *5*. Obtained torsional stiffness (T/θ) is designated as K_a. As for the shear modulus of concrete, G_c = *70000 kgf/cm²* was used.

The T-θ relationships obtained from the analytical method proposed are also shown in Fig. 8 by solid lines. The relationships exhibit the elastic behavior of reinforced concrete before cracking. In this region, the effect of reinforcement seems to be negligible. The torsional stiffness obtained from the analysis is called as K_b. The compressive strength of concrete used was *300 kgf/cm²*.

Since calculated torsional stiffness in the elastic region is influenced greatly by the magnitude of shear modulus employed, broken and solid lines in Fig. 8 are not necessarily coincident with each other as shown in the figure. Table 1 shows the magnitude of torsional stiffness, K_a and K_b, and the change of torsional stiffness with the increase in the ratio of h/b. According to Table 1, although K_a and K_b are not necessarily coincident, it can be admitted that the change of torsional stiffness is very similar in both methods. Proposed analytical method cannot provide the same torsional stiffness as the elastic theory; however, it can predict the change of torsional stiffness with the increase in the ratio of h/b reasonably.

(b) Overall behavior

Fig. 9 shows the T-θ relationship for the same rectangular cross section having different amount of reinforcement. The details on this example are represented in

Table 1 Comparison of Torsional Stiffness in Elastic Region

No.	b *cm*	h *cm*	A_{sx} *cm²*	A_{sy} *cm²*	s *cm*	K_a *tf·m²/deg*	K_{ai}/K_{a1}	K_b *tf·m²/deg*	K_{bi}/K_{b1}
1	10	10	1.5	0.38	10	0.172	1.00	0.159	1.00
2	10	20	3.0	0.50	10	0.560	3.26	0.531	3.34
3	10	30	4.5	0.56	10	0.968	5.63	0.933	5.87
4	10	40	6.0	0.60	10	1.373	7.98	1.300	8.18
5	10	50	7.5	0.63	10	1.778	10.3	1.658	10.4

*K_a is the torsional stiffness according to the elastic theory ($= T/\theta$)

** K_b is the torsional stiffness from the proposed analysis

*** $f_c' = 300$ *kgf/cm²*, $G_c = 70000$ *kgf/cm²*

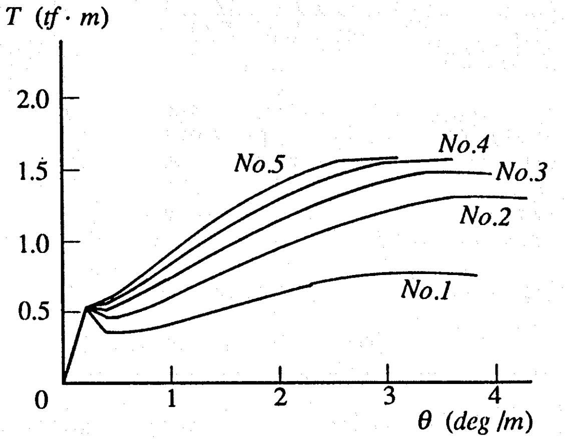


Fig. 9 T-θ relationship for rectangular cross section

Table 2 Details of Section and Material Properties for Calculated Examples

No.	b *cm*	h *cm*	A_{sx} *cm²*	A_{sy} *cm²*	s *cm*	f_{xy} *kgf/cm²*	f_{yy} *kgf/cm²*	f_c' *kgf/cm²*
1	20	20	2.56	0.4	10	3500	3500	300
2	20	20	5.12	0.8	10	3500	3500	300
3	20	20	7.68	1.2	10	3500	3500	300
4	20	20	10.2	1.6	10	3500	3500	300
5	20	20	12.8	2.0	10	3500	3500	300

Table 2. In Table 2, A_{sx} and A_{sy} mean the total cross-sectional area of longitudinal reinforcement and the cross-sectional area of one transverse reinforcement, respectively, and s means the spacing of transverse reinforcement. According to Fig. 9, the following aspects are observed.

(1) When reinforcement ratio is low, the resisting moment for torsion decreases rapidly after cracking and then it recovers gradually with the increase in torsional deformation.
(2) With the increase in reinforcement ratio, the tendency that the resisting moment decreases rapidly after cracking has disappeared.
(3) Once the resisting moment reaches the maximum value, the moment is being kept nearly constant.

It is considered that the rapid decrease in the resisting moment after cracking is due to the fact that the analysis has been performed by displacement control. The characteristics of the tension stiffening adopted also affects this tendency.

5. COMPARISON WITH PREVIOUS EXPERIMENTAL RESULTS

To verify the accuracy of the analytical method, previous experimental data for pure torsion [12]~[17] have been collected. The summary of experimental data are shown in Table 3.

Fig. 10 shows the comparison of

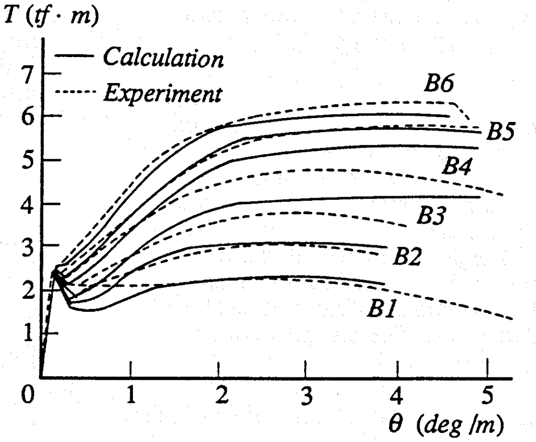


Fig. 10 Comparison of Experimental and Analytical T-θ Relationships (Hsu *B* Series [12])

Table 3 Summary of Experimental Data and Calculated Results

Researcher	Number of data	b *cm*	h/b	p_x* %	p_y* %	f_{xy} kgf/cm^2	f_{yy} kgf/cm^2	f_c' kgf/cm^2	$T_{cr}/T_{cr.cal}$		$T_u/T_{u.cal}$	
										Av. & C.V.		Av. & C.V.
Hsu [12]	51	15.24 ~25.4	1 ~3.25	0.40 ~3.16	0.45 ~3.76	3160 ~3590	3250 ~3680	146 ~467	0.64 ~1.31	0.97 14.5%	0.83 ~1.20	1.02 7.87%
McMullen [13]	13	12.7 ~25.4	1~3	0.44 ~1.79	0.64 ~1.98	2360 ~3870	2360 ~3870	281 ~407	0.62 ~1.37	0.85 32.2%	0.81 ~1.22	1.04 11.7%
Ernst [14]	7	15.25	2	0.61 ~1.72	0.35 ~1.24	2880 ~3760	3900	275	0.69 ~0.79	0.73 5.80%	0.74 ~0.90	0.83 5.84%
Victor [15]	5	8.03 ~8.18	1.97 ~2.01	1.32 ~2.01	0.47 ~1.79	2230 ~2351	2615 ~3390	221 ~235	—	—	0.87 ~1.12	0.99 8.07%
Okamoto [16]	10	14.0	1.59	1.63 ~2.56	0.55 ~2.96	3560 ~3790	3360 ~4020	185 ~345	—	—	0.81 ~1.21	0.94 12.2%
Authors [17]	5	15.0 ~20.0	1 ~1.33	0.95 ~2.65	1.14 ~3.33	3620 ~3800	3290 ~3860	178 ~368	1.03 ~1.32	1.23 8.91%	0.73 ~1.09	0.91 15.4%
									Data	*76*	*Data*	*91*
									Ave.	*0.95*	*Ave.*	*0.99*
									C.V.	*20.8%*	*C.V.*	*11.1%*

* $p_x = 100\, A_{sx} / (b \cdot h)$, $p_y = 100 \dfrac{2(b+h)A_{sy}}{b \cdot h \cdot s}$

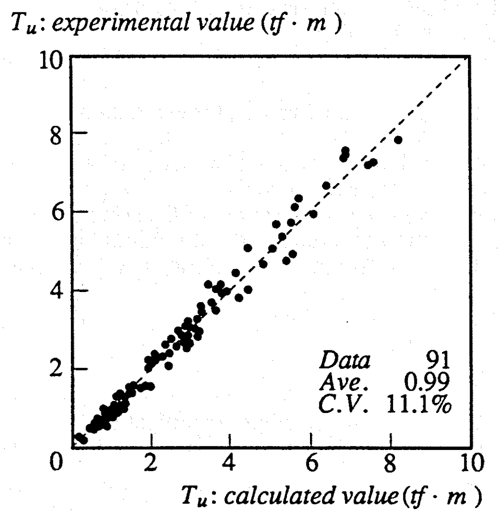


Fig. 11 Comparison of Experimental and Analytical Results (Torsional Strength: T_u)

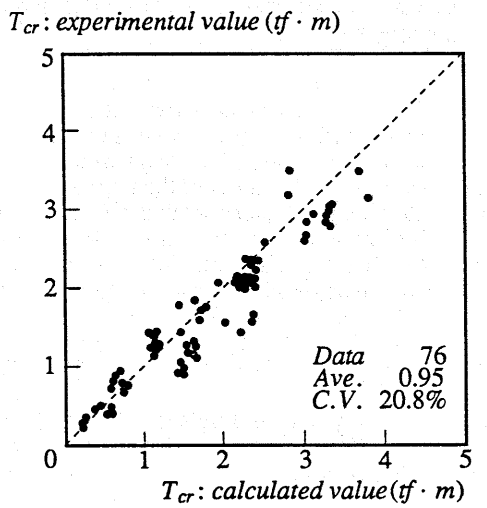


Fig. 12 Comparison of Experimental and Analytical Results (Cracking Moment: T_{cr})

experimental T-θ relationships with analytical results. These data are obtained from Hsu's *B* series [12]. As for the estimation of torsional cracking moment, the prediction can is valid. Overall prediction for T-θ relationship after the cracking up to the ultimate state is also reasonably accurate.

Fig. 11 shows the comparison of experimental ultimate strength with analytical result. For previous ninety-one experimental data, the mean and coefficient of variance for the ratio of the experimental value to the analytical one are *0.99* and *11.1%*, respectively. The accuracy of the estimation for the ultimate strength is quite well. As for the torsional cracking moment, however, the mean and coefficient of variance are *0.95* and *20.8%* (FIG. 12), and thus the accuracy of the estimation is not satisfactory.

The effect of slight decrease in the tensile strength of concrete under pure shear, that is, biaxial tension and compression stress state, has not been incorporated into the analysis. The uniaxial tensile strength has been used as the tensile strength of concrete in this analysis. If the effect of this slight decrease had been considered in the analysis, the accuracy of the estimation for cracking moment would have been improved somewhat, and the mean would have approached to *1.0* slightly.

6. CONCLUDING REMARKS

The purpose of this paper was to present the analytical method for the prediction of the T-θ relationship of reinforced concrete linear members subjected to torsion. In the analysis of reinforced concrete plates that is the basis of torsional analysis, the results of recent researches on tension stiffening and softening of concrete, shear transfer on crack surfaces, and local yielding of reinforcement have been utilized. As for the torsional analysis, based on the assumption that effective thickness can be determined by the location where compressive strain becomes zero, the equation of strain compatibility condition has been derived. Conclusions obtained from this research are as follows:

(1) The torsional stiffness in the elastic region predicted by the analysis is not necessarily coincident with the stiffness calculated from the elastic theory. However, the change of torsional stiffness with the ratio of dimensions of a cross section can be predicted reasonably by the analysis.
(2) Considering the large scatter between experimental and analytical values, the estimation of torsional cracking moment is not sufficient.
(3) In case the estimation of the magnitude of torsional cracking moment has been accurate, the following T-θ behavior until the ultimate state can be predicted accurately.
(4) The ultimate torsional strength itself can be estimated accurately regardless of the accuracy of the prediction for torsional cracking moment. Although the magnitude of torsional cracking moment has the substantial influence on the following deformation behavior, it hardly affects the magnitude of the ultimate strength of the member.

REFERENCES

[1] Japan Society of Civil Engineers, "Data for the Revision of the Standard Specification for Design and Construction of Concrete Structures — 1986 —", Concrete Library of JSCE, No. 61, 1986 (in Japanese)

[2] Shin,H., Izumo,J., Maekawa,K., and Okamura,H., "Finite Element Analysis of Reinforced Concrete Walls Subjected to Reversed Cyclic Loadings", Proceedings of JCI, Vol. 10-1, 1988, pp. 95~104 (in Japanese)

[3] Vecchio,F.J. and Collins,M.P., "The Modified Compression Field Theory for Reinforced Concrete Elements Subjected to Shear", ACI Journal, Vol. 83, No. 2, 1986, pp.219~231

[4] Hsu,T.C.C. and Mo,Y.L., "Softening of Concrete in Torsional Members — Theory and Tests", ACI Journal, Vol. 82, No. 3, 1985, pp. 290~303

[5] Nagataki,S., Li,S., and Okamoto,T., "A Consideration on the Torsional Resisting Mechanism of Reinforced Concrete Members", Proceedings of JSCE, No. 390/V-8, 1988, pp. 179~188 (in Japanese)

[6] Moriya,N., Niwa,J., and Higai,T., "Analysis of Reinforced Concrete Members Subjected to Torsion Based on In-Plane Shear Theory", Transactions of JCI, Vol. 11, 1989, pp. 503~510

[7] Izumo,J., Shima,H., and Okamura,H., "Analytical Model for RC Panel Elements Subjected to In-Plane Forces", Journal of JCI, Vol. 25, No. 9, 1987, pp. 107~120 (in Japanese)

[8] Okamura,H. and Maekawa,K., "Nonlinear Finite Element Analysis for Reinforced Concrete", Proceedings of JSCE, No. 360/V-3, 1985, pp. 1~10 (in Japanese)

[9] Vecchio,F.J. and Collins,M.P., "Predicting the Response of Reinforced Concrete Beams Subjected to Shear Using Modified Compression Field Theory", ACI Journal, Vol. 85, No. 3, 1988, pp.258~268

[10] Aoyagi,Y. and Yamada,K., "Strength and Deformation Characteristics of Reinforced Concrete Shell Elements Subjected to In-Plane Forces", Proceedings of JSCE, No. 331, 1983, pp. 167~180 (in Japanese)

[11] Okamura,H., Maekawa,K., and Izumo,J., "Reinforced Concrete Plate Element Subjected to Cyclic Loading", Proceedings of IABSE Colloquium, Delft, 1987, pp. 575~589

[12] Hsu,T.T.C., "Torsion of Structural Concrete — Behavior of Reinforced Concrete Rectangular Members", ACI SP-18, 1968, pp. 261~306

[13] McMullen,A.E. and Rangan,B.V., "Pure Torsion in Rectangular Sections — A Re-Examination", ACI Journal, Vol. 75, No. 10, 1978, pp. 511~519

[14] Ernst,G.C., "Ultimate Torsional Properties of Rectangular Reinforced Concrete Beams", ACI Journal, Vol. 54, No. 4, 1957, pp. 341~356

[15] Victor,D.J. and Muthukrishnan,R., "Effect of Stirrups on Ultimate Torque of Reinforced Concrete Beams", ACI Journal, Vol. 70, No. 4, 1973, pp. 300~306

[16] Okamoto,T. and Nagataki,S., "Strength and Deformation of Reinforced Concrete Members Subjected to Flexure, Shear, and Torsion", Proceedings of JSCE, No. 360/V-3, 1985, pp. 81~89 (in Japanese)

[17] Niwa.J, Higai,T., and Moriya,N., "Analysis on Mechanical Behavior of Reinforced Concrete Members Subjected to Torsion", Research Reports of the Faculty of Engineering, Yamanashi University, No. 39, 1988, pp. 81~86 (in Japanese)