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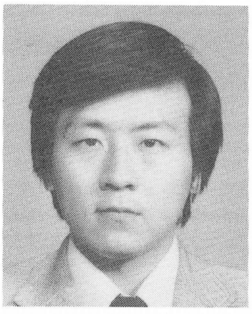
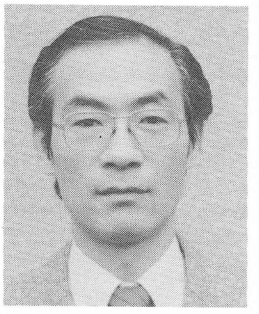
Numerical Problems in Non-Linear Finite Element Analysis of the Post-Failure Behavior of Structural System





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SYNOPSIS

Investigation was made of the sources of problems and peculiarities encountered during finite element methods of analysis of strain softening structures. Typical problems were demonstrated first by using reinforced concrete deep beam example structure. Second, a one-dimensional three element example structure was used to demonstrate the problems of solution divergence,oscillations,or non-uniqueness of solution. Effects were discussed of the materials models, such as hysteretic as contrasted to non-linear elasticity, load increment step size, element size, tangent stiffness and non-negative stiffness convention.

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1. INTRODUCTION

1.1 Problem

There exist the needs to evaluate reserve strength, deformation capacity, and deterioration process of reinforced concrete structures under repetitive or reversal loadings. Successful results seem to be scarce in obtaining reliable and convergent solution in the strain softening region of the load-displacement relationships.

Two possible sources of the difficulty are, the ambiguities in representation of the materials properties in large deformations, and the numerical problems such as the divergence of the solution process.

This paper is first to report such problems encountered in analyses using an existing finite element analysis program which takes into account of materials non-linearity [1][2][3] , and second to investigate the process of oscillation and divergence of the solution using a one-dimensional analysis model structure.

1.2 Existing Finite Element Methods of Analysis for Strain Softening Materials

Two aspects of analytical procedure govern the reality of the solution of the structural analysis in the strain softening regions of load-displacement relationships. The one is the treatment of the negative element stiffness when the stiffness at the integration point went into strain softening region, and the other is the convergence algorithm.

In some of the existing analysis programs [5] there has been accepted convention to let the argument of the element stiffness equal to zero (or small enough positive number) if that element is in the strain softening region, and then to assemble the structure stiffness matrix. Then the unbalanced nodal forces are relaxed until a convergence criterion is satisfied. During the process, the decrease in stresses may be considered, or the stresses may be assumed to remain constant. For the former procedure no report is known to the writers that demonstrated convergence. In the latter procedure, the unbalanced forces are relaxed in the next step, which means that the actual constitive relation has not been used.

A method referred to the hyper elliptic function method [6] has been effectively

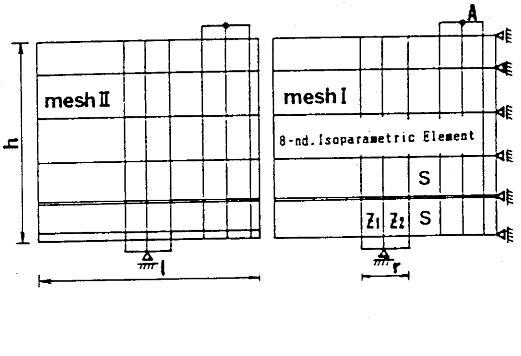

Fig.1 Mesh layout of example structure [10]

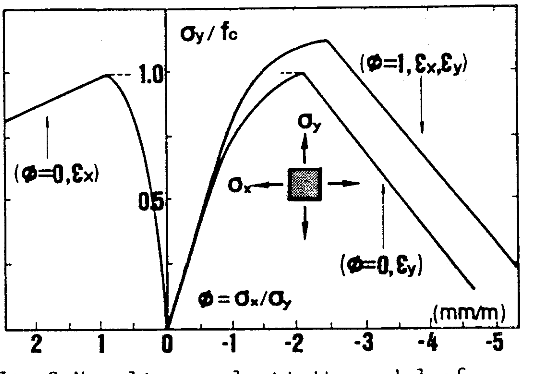

Fig.2 Non-linear elasticity model of stress-strain relationship

used for analyses of geometrically non-linear problem. No report is known to the writers of successful application of this method to the strain softening concrete structures.

Those are the reasons which urge the investigation of divergence and oscillation of the solution process.

2. EXAMPLE OF DIVERGENCE IN ANALYSES OF DEEP BEAM STRUCTURE (CASE STUDY)

The structure to be analysed was shown in Fig.1 with finite element mesh layout [10]. Prescribed displacements were imposed on the node A, and the reactions at that node were computed. Convergence algorithm was Newton's iteration. For simplification the non-linear materials property was as shown in Fig.2. For this model there is one-to-one relationship between stress and strain. The tangent stiffness was used to form the structure stiffness matrix.

Fig.3 shows the load-displacement relationship that satisfied equilibrium. The convergence process is also shown of the reaction for the prescribed displacements (case A,mesh I). The eqilibrium equation is given by Eq.1. The convergence index was defined by Eq.2 and denoted by ϕ. The convergence criterion was prescribed by letting ϕ equal or less than 1.0%.

$$F_i = F_{eq,i}, \quad F_{eq,j} = \iiint_v B_{ji}\sigma_i dv \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

$$\phi = \sqrt{\Sigma(\Delta F_i)^2}/R_A, \quad \Delta F_i = F_i - F_{eq,i} \quad \cdots\cdots\cdots\cdots (2)$$

where, F_i is external nodal load, $F_{eq,i}$ is equivalent nodal force, ΔF_i is unbalanced nodal force, R_A is reaction at node A, B_{ij} is strain matrix, and i is degree of freedom.

At step number 7, where the stiffness of the elements Z_1 and Z_2 in Fig.1 entered into the strain softening region, the solution diverged. Fig.3 exhibits that the computed reaction diverged with increasing number of iteration with some oscillation.

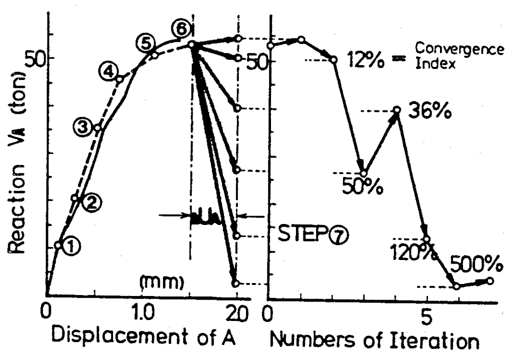

Fig.3 Displacement vs. reaction and convergence process (case A)

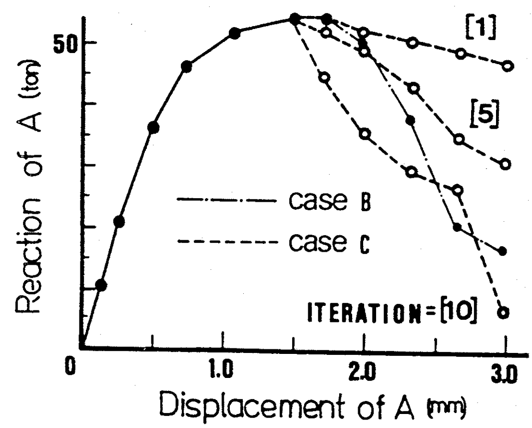

Fig.4 Effect of ieration cycle number 1,5, and 10

The next analysis was performed as follows. The iteration was terminated after one execution, and the unbalanced nodal forces were carried over to the next step (case B, refer to Fig.4). This process resembles the incremental integration[7]. In appearance the results seem as if reasonable behavior, but in reality error accumulated as the load increment advanced, and the convergence index ϕ exceeded 100%. This can hardly be accepted as correct solution.

Next, the stiffness of the element which was strained into the strain-softening region was let equal to zero, and convergence procedure was executed with iteration cycle number 1,5, and 10. The load-displacement relationships for respective case were given in Fig.4 (case C). Convergence was slow. The reaction kept decreasing as the cycles of iteration was increased. Deviation from equilibrium also became more pronounced.

Next, a path-dependent elasto-plastic-fracture model[8] was used for the material property. This model recognized negative stiffness. The problem was the same as before in reference to Fig.1. Newton's iteration was again used. The results are shown in Fig.5 (case D). In this case, convergence was attained even in the strain-softening region up to step 9. However, divergence suddenly occurred in the next increment, when the unbalanced nodal forces were relaxed due to cracking occurred in the element surrounding the strain softening element.

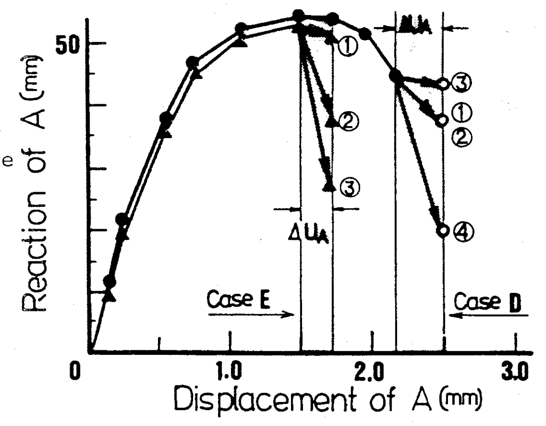

Fig.5 Effect of hysteretic material (case D) and Mesh subdivision (case E)

The mesh layout was changed in such a manner that the strain softening element was subdivided as was shown by the mesh II in Fig.1. Divergence occurred in step 7, two steps earlier, despite the identical analysis procedure as the last case D (case E). Particularly, oscillation occurred for the nodes connected to the subdivided elements and surrounding elements. This dependence of the solution on the mesh layout cast doubt about the validity of the solution obtained in the previous case D.

3. EXAMPLE ANALYSES OF ONE-DIMENSIONAL MODEL STRUCTURE

In the preceding example analyses it was demonstrated that the behavior in the strain softening region of the load-displacemnt relationships depend on the nonlinear solution procedure, the mesh layout, and the constitutive law of the materials.

The next series of analyses were made of the model structure composed of three of the one-dimensional elements for more detailed pursuit of convergence behavior. From the analyses of the preceding section it was found that the nodes where the solution oscillated or diverged were confined within the limited area of the structure. Those nodes were indicated by solid dots in Fig.6, and the nodes where solution converged were indicated by open circles for the case A at step 7. The objective of this section was to simulate the behavior of those problematic nodes.

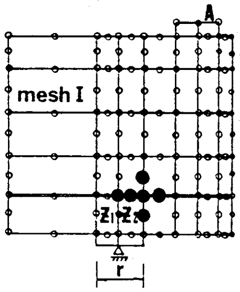

Fig.6 Location of divergent and oscillating nodes (solid dots)

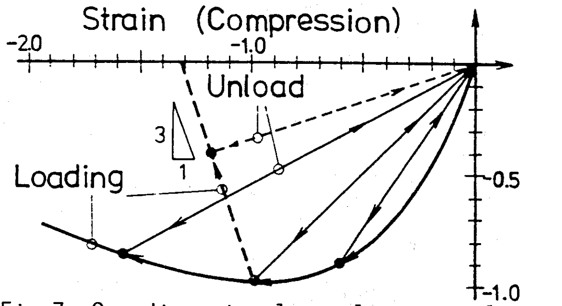

Fig.7 One-dimensional non-linear inelastic stress-strain relationship (Solid line:model 1, Broken line:model 2)

Three elements were connected in a row as shown in Fig.8. The stress-strain relationship was given in Fig.7. The solid lines show the first model, and the broken lines show the second model. (The second model is to be used in the discussion of section 3.3.) Prescribed displacement was imposed on the node A, and the displacement of the internal node S was to be obtained. The cross sectional area of the element A was made equal to 98% of that of element B. The graduated variation in cross sectional area of the three elements was to simulate the stress pattern in the critical region of the preceding example deep beam structure, where the stress flow was funneled into a narrow channel.

3.1 Strain Softening Materials Model (Element length equal)

Comparison was made of the three different step size of the load increment after the last stable step 9 which was just before attainment of the maximum resisting strength of the structure. Fig.9 shows the load-displacement relationships for the three cases. Fig.10 shows the variation in the strains in the elements with the progress of iteration. The structural stiffness matrix to be used while relaxing the unbalanced nodal forces were tangential stiffness even for the strain softening region. (The data with solid dots in Fig.10 are for the case of non-negative stiffness, which is to be discussed in section 3.2.)

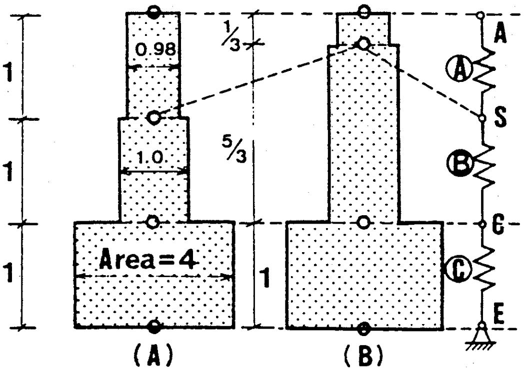

Fig.8 One-dimensional model structure

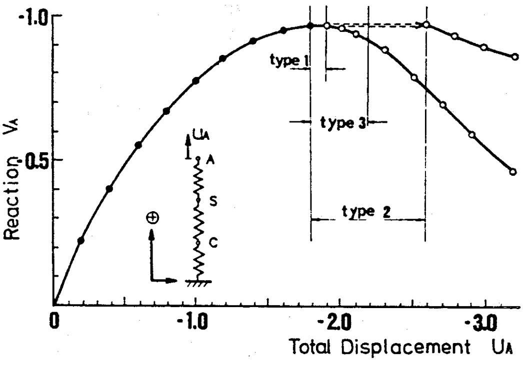

Fig.9 Effect of load increment step size (type 1,2,3)

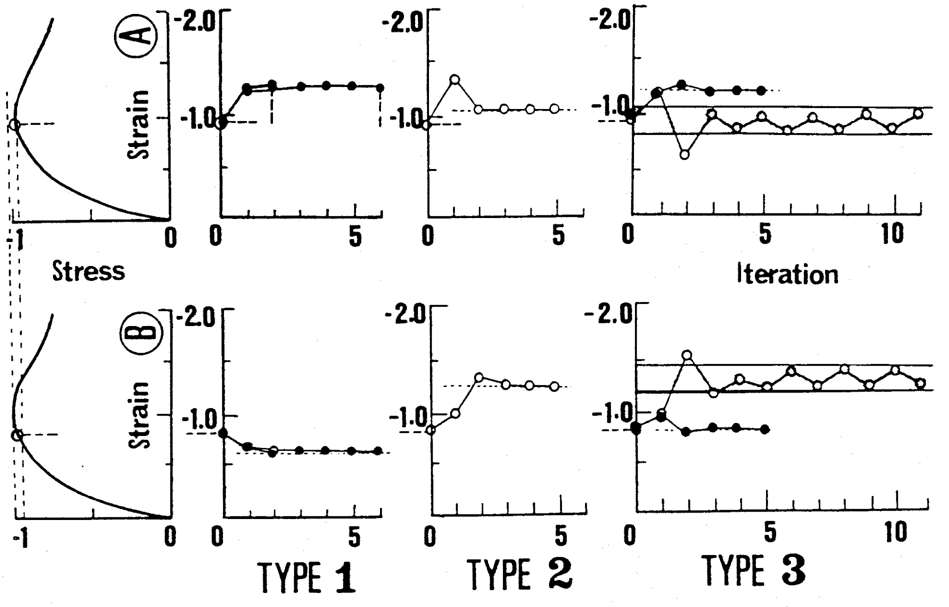

Fig.10 Variation of strains during iteration
o:tangent stiffness, •:non-negative stiffness convention

Type 1 was for the smaller step size of load increment. Convergence was attained after two cycles monotonically.
Type 2 was for the larger step size of the load increment. Convergence was attained but with larger number of iteration. For the successive load increments also solutions were obtained by two or three cycles of iteration for both types. However, the load-displacement relationships grossly differed. (Fig.9).
Type 3 was for the intermediate step size of load increment. The solution oscillated as shown in Fig.10 and convergence was not attained. This dependence of the solution on the step size of the load increment does not occur in the cases of positive tangent stiffness.

For type 1 solution the element A (weaker element) was in the strain softening region, whereas the element B was on the unloading path (from strain hardening region). This situation agrees with theoretically correct solution. (See Fig.10)
For type 2 solution both element A and element B were in the strain softening region, and the equilibrium was satisfied.
For type 3 solution it was assumed that the element A (weaker element) was on (the unloading path from the) strain hardening region, and element B was in strain softening region.

In the following discussion "mode 1,2,3" refers to the state of the strains in the two elements. The meaning of "type 1,2,3" is same as above.

Next, the equilibrium equation, Eq.1, was solved analytically and the results were shown in Fig.11 on the next page. The relationship between force and change in length were plotted of the stronger element B, the weaker element A, and the entire structure for the three modes.

The incremental changes of nodal forces due to small changes in nodal displacements were computed by Eq.3.

$$\{ d F_{eq,i} \} = [K]\{d u_i\} , \qquad [K] = \iiint_V B^T DB \, dv \qquad \text{........ (3)}$$

where, u_i is nodal displacement and D is tangent stiffness.

Mode 1 solution in Fig.11 is the case where the weaker element is in the strain softening region and the stronger element is in the unloading region, which is correct answer.
Mode 2 solution is the case where both elements are in the strain softening region.
Mode 3 solution is the case where the stronger element is in the strain softening region and the weaker elemnt is in the strain hardening region.

It can be seen from Fig.11 that there are regions where the displacement solution does not exist (jump region) for the mode 2 and mode 3.

For the type 2 analysis where the larger load increment was used, the first trial displacement value given was such that causes strain softening in the two elements. The convergence procedure was started by using the tangent stiffness at the point corresponding to the assumed displacement. Thus, convergence was completed at the equilibrium point for the mode 2 which was the location nearest to the trial starting point.

For the type 3 analysis where the intermediate step size for the load increment was used, the stronger element was assumed to be in the strain softening region, and hence, the procedure was the one to search the mode 3 solution. However, as indicated in Fig.11, the assumed trial displacement fell in the region where the correct solution did not exist. Hence, oscillation took place. For the case such as this one, where the assumed mode was incorrect, conventional remedies of numerical procedure [7] have no effect.

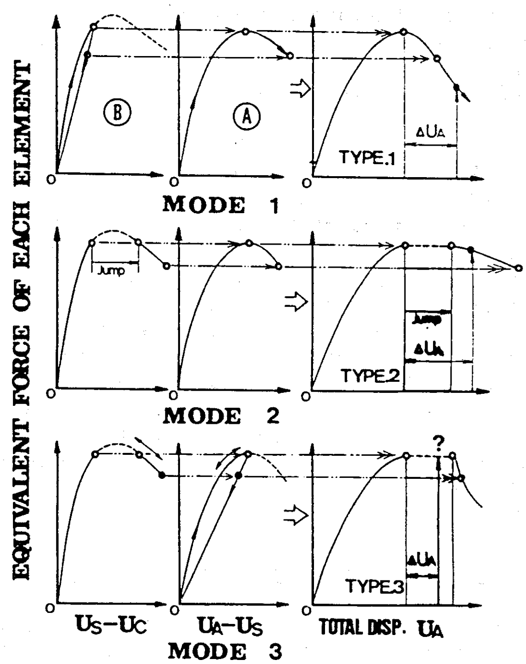

Fig.11 Mode, or state of strains in element A and B

Next, a description will be made of the process of divergence for the type 3 solution. The equilibrium of the node S is given by Eq.4, where the structure tangent stiffness is K_s,the unbalanced nodal forces are ΔF_s, the correction increment of the nodal displacement is Δu_s, and the tangent stiffness for element A and B are respectively K_a and K_b.

$$\Delta u_s = K_s \Delta F_s , \qquad K_s = K_a + K_b \qquad \text{......... (4)}$$

The relationship of Eq.4, or the relationship between the nodal displacement $\bar{u}_s$ and the unbalanced nodal force ΔF_s for the internal node S can be obtained analytically, and was plotted in Fig.12. The displacement $\bar{u}_s$ is the movement from the equilibrium position for the preceding step. The solution for $\bar{u}_s$ which satisfies equilibrium is the point on the curve of Fig.12, where the ΔF_s coordinate is zero. The tangent stiffness K_s corresponds to the slope of the $\bar{u}_s$ - ΔF_s curve. On the figure the modes that tell the state of the two elements that are dictated by the value of $\bar{u}_s$ were also indicated. The positive and negative sign of the tangent stiffness K_s is also indicated.

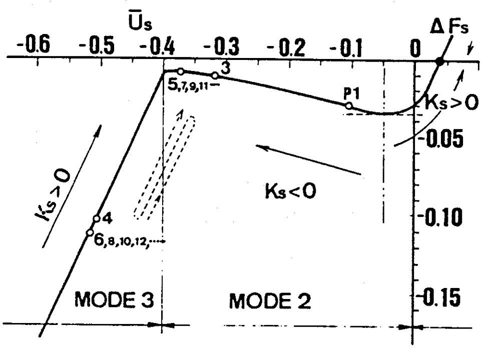

Fig.12 Unbalanced nodal force vs. nodal displacement (node S)

In the convergence algorithm which use the tangent stiffness, correction for the nodal displacement is always directed to the direction that decreases the value of ΔF_s. If the sign of K_s adopted was opposite to the one which is directing to the correct solution, the result is that the computed Δu_s is moved away from the correct solution. In the type 3 analysis the trial point for the first step was indicated by P_1 in Fig.12. This point was projected from the preceding converged solution using tangent stiffness and load increment at that stage. The point resides in mode 2 region. Locally the modification for the displacement is to move into the direction to reduce the unbalanced force. However, as is evident from Fig.12, the true solution is located in the direction opposite to the direction of the movement. Hence, the second trial value is moved to mode 3, giving rise to the unbalanced force. Then, if redistribution is executed using K_s at this point, the movement is then toward the true solution. Nevertheless, the displacement again falls inside mode 2 region, and the procedure oscillates.

3.2 Convergence Process When Using Non-negative Convention for Negative stiffness

The type 3 analysis was performed using the convention to let the element stiffness zero in forming the global stiffness matrix even if actual stiffness was negative. Then, for the global stiffness, which is to be denoted by $K_{s,NZ}$, each element of the stiffness matrix is positive. In the neighborhood of the trial displacement the direction of correction was to increase ΔF_s, but convergence was attained eventually(See Fig.10,12). In order for successful use of the negative stiffness, the starting (trial) value of displacement has to be positioned in the region of the correct mode (mode 1 for this case).

A question now would be whether or not there is generality in expecting convergence by using non-negative convention for the element stiffness as was just mentioned above. For the type 2 analysis it is possible that two elements enter into strain softening region simultaneously. (It is frequently experienced that large unbalanced force is carried over from other elements even when the load increment is small, because of stress released from cracked elements.) Then, if the global equilibrium equation, Eq.4, is considered, the stiffness $K_{s,NZ}$ for node S is zero or very small number, and hence by Eq.4, Δu_s should diverge. This is the mechanism that cracking is prescribed during solution process for the elements connected to the softening elements.

In the analysis of type 1 and convergent in mode 1 case, the number of repetiion increased when negative stiffness was ignored. This is because the process approaches the true solution in either way of treating the stiffness, since K_s and $K_{s,NZ}$ have the same sign owing to the fact that the first trial value for strain is set as mode 1 in Fig.11. In this case convergence is better with tangent stiffness.

Next, an analysis was performed for another material model which is to lose resistance very drastically after the peak stress as shown by broken lines in Fig.7. The relationship between the nodal displacements $\bar{u}_s$ and the unbalanced nodal forces ΔF_s was shown in Fig.13 for the type 2 case. When the tangent stiffness was used, immediate convergence was attained from point P_1 in Fig.13 to the solution of mode 2. Whereas, when positive stiffness was used, divergence occurred, since correction of the trial value moved away from the solution owing to the fact that K_s and $K_{s,NZ}$ are of the same sign.

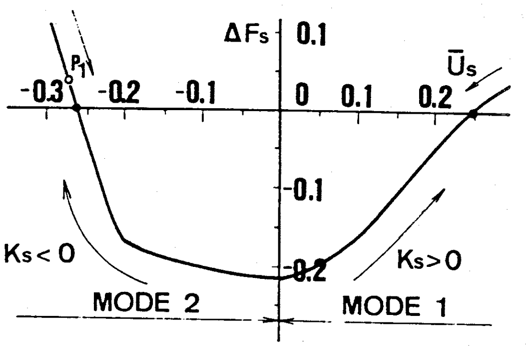

Fig.13 Unbalanced nodal force vs. nodal displacement with non-negative stiffness convention

Convergence is fast when unbalanced nodal force is relaxed using tangent stiffness, when the direction of displacement to reduce the unbalanced force and the direction toward the true solution coincide on the trial displacement position. There is no problem if load increment is small. There occurs cases where the direction toward true solution and the direction of correction by Eq.4 disagree. Then the stiffness to project the process toward the true solution is undecided.

3.3 Effect of Shape and Size of Element (Element Length Changed) and Effect of Hysteretic Unloading Path as Compared to Non-linear Elasticity

In section 2 the solution changed when some elements were subdivided. This motivated the analysis of this section for the one-dimensional model structure where the element A was made shorter while element B was made longer as shown on the right side of Fig.8. The results are shown on Fig.14. The load increments were made small and negative stiffness was recognized.

The load-displacement relationship until the maximum resisting strength was almost identical to Fig.9. In the strain softening region convergence was attained in a similar manner as the mode 1 of Fig.11. However, the load -displacement relationship in this region differed markedly from that of type 1. The recover of strain in element B is greater than the increase in strain in element A during the unloading of element B which is caused by softening of element A. (Release energy is large)

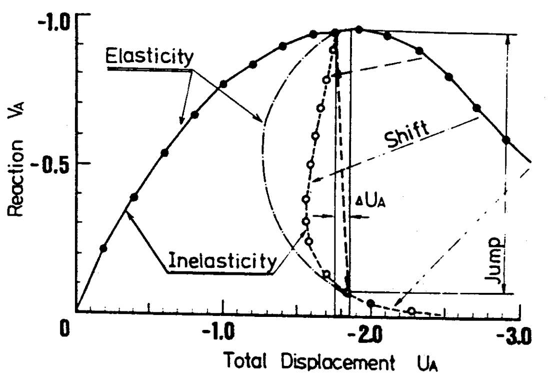

Fig.14 Effect of element size(Element B was made longer)

The true solution is also given in Fig.14. From this it can be seen that there exist "jump region". In the same figure the true solution for the materials model for which the unloading path traces back along the loading path (Non-linear elastic model) was shown. Such are the reasons for the variation of the solution due to the element subdivision and materials model as was experienced in Section 2.

The fact that the true solution changes with the element size (refer to case E, section 2) raised a problem in adopting materials model for strain softening behavior. In tests of materials the measured stress-strain relationships in softening region vary with gage length. The numerical analyses in softening regions could be unreal unless coherence is considered for shape and size of the finite element model and the test specimens.

4. CONCLUSION

The following characteristics of the behavior of the numerical solution process for the strain softening region were found from the investigation of example structures. The general solution process was not established.

(1) For the non-linear solution process by using Newton's iteration there could be cases where the direction of correction of trial solution projected by the tangent stiffness differs from the direction where the true solution exists.

Then, it is possible that divergence or oscillation occurs and that the true solution is not obtained. Further, it does not guarntee that there is the true solution in the direction opposite to the direction of modification that had led to oscillation.

(2) In the analyses of strain softening materials it was demonstrated that the true solution changed with the shape and size of the elements. This was attributed to the fact that the amount of energy released from the element depends on the shape and size of the element. Hence, when defining the materials models, there should be consistency between the shape and size of the finite element models and those of the physical test specimens used for experiments to derive the materials properties.

Acknowledgment

This research was supported by the Grant-in-Aid for Scientific Research, Ministry of education, No.58850100, under the supervision of Professor Hajime Okamura, University of Tokyo as the Principal Investigator.
Contributions made to the execution of this investigation by Messrs. Masashi Odagawa and Li Baolu, graduate students of Nagaoka Technological University, and Mr.Kazuto Ozawa, graduate student of University of Tokyo, are thankfully acknowledged.

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