(31) WAVES IN NONLINEAR-ELASTIC "RATE-TYPE" VISCOUS MATERIALS

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SUMMARY. Provided the effects of nonlinearity and dissipation are small but definitely not negligible, a wide class of wave phenomena can be investigated by means of the quasilinear parabolic equation known as Burgers' equation. The analysis of Burgers' equation in its form corresponding to the one-dimensional propagation of a longitudinal plane wave in homogeneous isotropic solids constitutes the main content of this paper. The closed system of equations forming the basis for the derivation of Burgers' equation and representing nonlinear-elastic "rate-type" viscous media is considered. The importance of the geometric nonlinearity caused by the strain-displacement relation relative to that of the physical nonlinearity due to the stress-strain relation is discussed. Quantitative results illustrating how the parameters of the input (maximum amplitude and frequency) influence the distortion of an initially sinusoidal pulse are given for aluminium. A method for the evaluation of the parameter characterizing the physical (or material) nonlinearity is presented. It involves the approximation of a material's experimental stress-strain curve by the quadratic stress-strain relation.

The results show that: (1) Consideration of nonlinearity brings qualitatively new effects into the study of wave phenomena. (E.g., the distortion of the wave profile, which leads to the formation of a "weak shock".) (2) Various materials show a notably nonlinear behaviour, as expressed, in particular, in terms of the shapes of their experimental stress-strain curves. (3) The effect of both the physical and the geometric nonlinearities together must be taken into account in the study of nonlinear wave phenomena.

INTRODUCTION. Mathematical modelling of nonlinear wave phenomena has become an important topic in various fields of science and engineering. As far as civil (and earthquake) engineering is concerned, J.T.Oden (1972) has noted that the introduction of new materials whose response cannot be described by classical linear theories has encouraged the interest in nonlinear solid mechanics.

The propagation of deformation waves of moderate (small but finite) amplitude in nonlinear viscoelastic media is discussed in this paper. The basic model equations describing the propagation of waves of moderate amplitude (also known as "weak shocks") are Burgers' equation in the case of a dissipative medium and the Korteweg-de Vries equation in the case of a dispersive medium.

The main objectives of this work are 1) to provide a minimum theoretical background needed for the derivation and analysis of the one-dimensional Burgers' equation in the case of viscoelastic continua and 2) to illustrate some of the properties of Burgers' equation that are relevant in the context of the study of wave phenomena in solids.

BASIC EQUATIONS. In the case of solids, at the basis of the derivation of Burgers' equation lie the mathematical models (closed systems of equations) of nonlinear viscoelastic and thermoelastic continua. Here, the former model is analyzed with emphasis on the investigation of the separate and combined effects of the geometric and physical nonlinearities. The presentation follows that of Engelbrecht and Nigul (1981). The material (or Lagrangian) coordinates are used, thus facilitating the solution of boundary-value problems.

Consider the one-dimensional plane problem in a homogeneous isotropic "rate-type" viscoelastic medium. Its mathematical model consists of

(a) the strain-displacement relation (geometric nonlinearity)

$$E=U,+(1/2)U,^2 \qquad U=U(X;t) \tag{1}$$

and (b) the equation of motion

$$T,(1+U,)+TU,,=\rho_0 U.. \qquad T={}_ET+{}_DT \tag{2}$$

where the elastic part of the stress is given by the stress-strain relation (physical or material nonlinearity)

$${}_ET=(\lambda+2\mu)E+3(\nu_1+\nu_2+\nu_3)E^2 \tag{3}$$

and the dissipative part is defined in accordance with the Kelvin-Voigt model of viscosity:

$${}_DT=[\zeta+(4/3)\eta]U., \tag{4}$$

The above equations yield the wave equation

$$c_0^2[1+3(1+m)U,]U,,+nU.,,= U.. \tag{5}$$

$$c_0^2=(\lambda+2\mu)/\rho_0 \quad m=2(\nu_1+\nu_2+\nu_3)(\lambda+2\mu)^{-1} \quad n=[\zeta+(4/3)\eta]/\rho_0 \tag{6}$$

The following notation convention is adopted in this paper: the dot denotes differentiation with respect to time t (or a time-like variable) and the comma denotes differentiation with respect to material coordinate X (or a space-like variable).

It is useful to write the wave equation (5) in matrix form:

$$V.+AV,+BV.,=0 \qquad V=\begin{bmatrix} U. \\ U, \end{bmatrix} \tag{7}$$

To make use of the ray method, the following relations are assumed

$$V=V_0+\varepsilon V_1+O(\varepsilon^2) \qquad A=A_0+A_1 \qquad A_1=O(\varepsilon) \qquad B=O(\varepsilon)$$

$$A_0=\begin{bmatrix} 0 & -c_0^2 \\ -1 & 0 \end{bmatrix} \qquad A_1=\begin{bmatrix} 0 & -c_0^2[3(1+m)U,] \\ 0 & 0 \end{bmatrix} \qquad B=\begin{bmatrix} 0 & -n \\ 0 & 0 \end{bmatrix} \tag{8}$$

where ε is a small parameter to be specified later.

Introducing (8) in (7), one gets for the terms of order O(1)

$$V_0.+A_0V_0,=0 \tag{9}$$

For equation (9) the ray method yields the eiconal equation $t=X/c_0$, which determines the wavefront.

We now focus our attention on the region just behind the wavefront of a wave travelling in the X-direction by defining the new variables

$$\xi=c_0t-X \qquad \tau=\varepsilon X \tag{10}$$

By (8) and (10), equation (7) yields

$$(Ic_0-A_0)V_0,=0 \tag{11}$$

$$(Ic_0-A_0)V_1,+ A_0V_0.-A_1V_0,=c_0BV_0,, \tag{12}$$

The solution of equation (11) is given by

$$V_0=aR \qquad a=a(\xi;\tau) \qquad R=\begin{bmatrix} 1 \\ -1/c_0 \end{bmatrix} \tag{13}$$

where R is the right eigen-vector of matrix A_0 and a is an amplitude factor having a unit of velocity.

By (10) and (13), equation (12) yields Burgers' equation

$$a_{.} + \frac{3}{2}\,\frac{1+m}{\varepsilon c_0}\, a a_{,} = \frac{1}{2}\,\frac{n}{\varepsilon c_0}\, a_{,,} \tag{14}$$

whose non-dimensional form reads

$$b_{.} + \operatorname{sign}(1+m)\, b b_{,} = \Gamma^{-1} b_{,,} \qquad \Gamma = 3|1+m|\tau_c a_0/n \tag{15}$$

where the following dimensionless variables have been introduced:

$$b(\zeta;\sigma) = a/a_0 \qquad \zeta = \xi/\tau_c \qquad \sigma = \frac{3}{2}|1+m|(\varepsilon c_0)^{-1} a_0 \tau \tau_c^{-1} \tag{16}$$

In (15) and (16), a_0 is the maximum amplitude of the wave and τ_c is the characteristic wavelength. Γ is the fundamental parameter of the process, parameter known as the acoustic Reynolds number. For a given input $(a_0;\tau_c)$, it expresses the importance of nonlinearity $(1+m)$ relative to that of dissipation (n).

The above presentation reveals the relative contribution to the overall nonlinearity from the geometric and physical nonlinearities determined by equations (1) and (3), respectively. In Burgers' equation, the effect of the former is represented by unity (1), whereas that of the latter is determined by a parameter m that is a function of the third- and the second-order elastic coefficients (see equation (6)).

Finally, the small parameter ε in equation (8) is given by

$$\varepsilon = \frac{3}{2}|1+m|M \qquad M = a_0/c_0 \qquad \sigma = \varepsilon \tau_c^{-1} X \tag{17}$$

and the significance of Mach's number M is thus revealed.

<u>COMPUTATIONAL EXAMPLES</u>. Burgers' equation (15) was solved numerically for a sinusoidal initial condition (curve 0 in Figures 1-3).

Figure 1 shows, for the case of $m=-3$ and $\Gamma=50$, the distortion due to the physical and geometric nonlinearities considered separately (curves 2 and 3, respectively) and together (curve 1). The linear viscoelastic case is given for comparison (curve 4).

Figures 2 and 3 show, for aluminium ($c_0 = 6300$ m/s, $m=-9.2$, $n=0.7$ m^2/s [1]), the effects of the parameters of the input maximum amplitude a_0 and characteristic wavelength $\tau_c = L = c_0/f$, where L is the wavelength and f the frequency of the input (curve 0). Thus, curves 1, 2, 3 and 4 correspond to $a_0 = 5.625$, 56.25, 562.5 and 5625 cm/s, respectively, with f=0.1 MHz in Figure 2 and f=0.01 MHz in Figure 3.

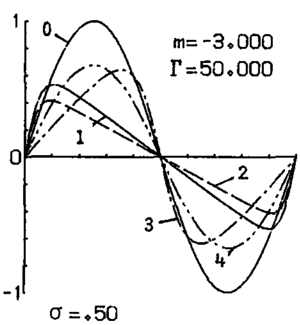


Figure 1

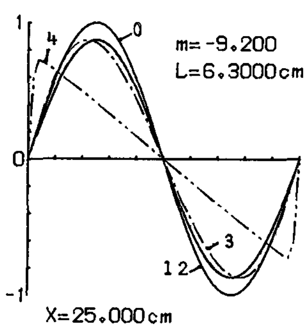


Figure 2

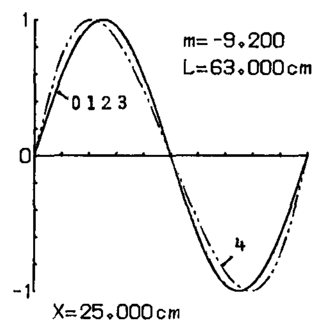


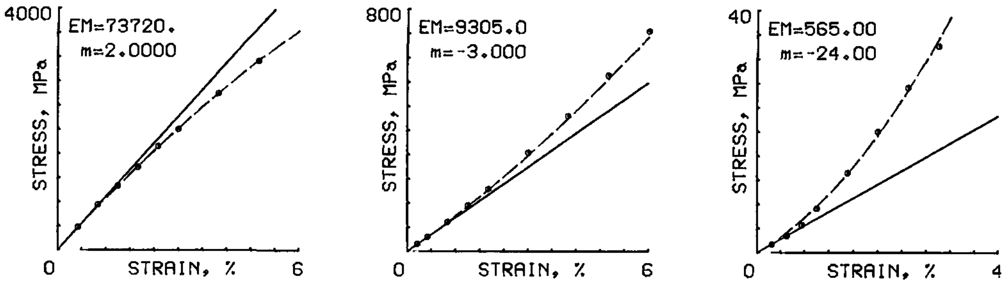
Figure 3

<u>CALCULATION OF THE PARAMETER OF PHYSICAL NONLINEARITY</u>. The physical nonlinearity is defined here by equation (3), which can be recasted as

$${}_{E}T = (\lambda + 2\mu)\left(1 + \frac{3}{2} m E\right) E \tag{18}$$

to display the characteristic parameter m, which is defined in (6).

The value of the parameter m, as well as that of the elastic modulus $EM=\lambda+2\mu$, was calculated for various materials by approximating their experimental stress-strain skeleton curves by means of equation (18). The experimental data (the dots in the figures below) were taken from J.W.Nunziato et al.(1974) (Fused Silica and Polymethyl Methacrylate) and from J.G.Jackson et al.(1980) (20-40 Ottawa Sand).



Fused Silica Polymethyl Methacrylate 20-40 Ottawa Sand

<u>DISCUSSION AND CONCLUSIONS</u>. The nonlinear theory discussed here is a straightforward generalization of the corresponding linear viscoelastic theory. The linear theory is obtained from the nonlinear one as a limiting case when dissipative effects are large compared with those due to nonlinearity. When wave phenomena are dealt with, however, these effects depend strongly on the parameters of the wave itself. Indeed, the parameter governing the propagation of nonlinear waves in viscoelastic media is given by (see equation(15))

$$\Gamma=GF \qquad G=3|1+m|/n \qquad F=\tau_c a_0$$

and it may change, for a given medium (G=const.), within very wide limits depending upon the input F. There is, however, a limit imposed by the smallness of the parameter ε in Eq.(8). By (17), this means that Mach's number must satisfy the condition $|M|<<1$. Since Mach's number is related through equations (13), (7) and (1) to the maximum uniaxial strain, equation (17) also implies the smallness of the product between the maximum strain and the parameter $|1+m|$ characterizing the overall nonlinearity.

The conclusions are given in the SUMMARY.

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