ON THE EXPLICIT SOLUTIONS OF VARIOUS ISOTROPIC THIN CLOSED CIRCULAR CYLINDRICAL SHELL EQUATIONS APPLIED TO G.W.HOUSNER,S DYNAMIC FLUID PRESSURE

by Hirohumi Ogata

ABSTRACT

This paper deals with the derivation of general explicit solutions of various isotropic thin closed circular cylindrical shell equations.

These basic partial differential equations are called to be ‘uncoupled equations’ and their explicit solutions have never been derived yet.

As one of numerical examples, upright cylindrical tank under G.W.Housner,s dynamic fluid pressure is calculated.

BASIC PARTIAL DIFFERENTIAL EQUATIONS

A: FLÜGGE-BIEZENO-GRAMMEL EQUATION

$$
\begin{aligned}
&\nabla^8 w+2\mu(1+3k)\cdot w^{vi}+\left[6+3k(2-\mu+\mu^2)\right]\ddot{w}^{iv}+\left[2(4-\mu)+(7-5\mu)k+3(1-\mu)k^2\right]\ddddot{w}''\\
&+2(1+k)\overset{\cdot\cdot\cdot}{\dddot{w}}+\frac{1}{k}(1+3k)(1+k-\mu^2)w^{iv}+\left[2+\frac{1}{2}(1-\mu)(1+k)(4+3k)\right]\ddot{w}''+(1+k)\cdot\ddddot{w}\\
&+k\Big[(2-3k)w^{viii}+\frac{1}{2}\{(11-3\mu)+9(1-\mu)k\}\ddot{w}^{vi}+\{3(2-\mu)-\mu^2k\}\ddddot{w}^{iv}+\frac{1}{2}\{(7-3\mu)+3(1-\mu)k\}\overset{\cdot\cdot\cdot}{\dddot{w}}''\\
&+\overset{\cdot\cdot\cdot\cdot}{\ddddot{w}}\Big]-\frac{a^4}{D}\Big[\nabla^{*4}q_n+\dddot{q}_x'-\mu(1+3k)q_x'''+k\Big\{(1+3k)\cdot q_x^{v}-\ddddot{q}_x'-\frac{3}{2}(1-\mu)k\cdot\ddot{q}_x'''\Big\}\\
&-(2+\mu)\dot{q}_\theta''-(1+k)\dddot{q}_\theta+2k\dot{q}_\theta^{iv}+k\Big\{2+\frac{1}{2}(3-\mu)k\Big\}\overset{\cdot\cdot}{\dddot{q}}_\theta''=0
\end{aligned}
$$

$$
\begin{aligned}
\nabla^{*4}u=&(1+3k)\mu w'''-\ddot{w}'+k\left[\ddddot{w}'-(1+3k)w^{v}+\frac{3}{2}(1-\mu)k\ddot{w}'''\right]\\
&-\frac{a^4}{D}\cdot k\left[(1+3k)q_x''+\frac{2}{1-\mu}\ddot{q}_x-\frac{1+\mu}{1-\mu}\dot{q}_\theta'\right]
\end{aligned}
$$

$$
\begin{aligned}
\nabla^{*4}v=&(2+\mu)\dot{w}''+(1+k)\dddot{w}-2k\left[\dot{w}^{iv}+\left\{1+\frac{3-\mu}{4}\cdot k\right\}\dddot{w}''\right]\\
&+\frac{a^4}{D}\cdot k\left[\frac{1+\mu}{1-\mu}\dot{q}_x'-\frac{2}{1-\mu}q_\theta''-(1+k)\ddot{q}_\theta\right]
\end{aligned}
$$

where

$$
\nabla^{*4}=\nabla^4+3k\frac{\partial^4}{\partial y^4}+\frac{k}{2}(4+3k)(1-\mu)\frac{\partial^4}{\partial y^2\partial\theta^2}+k\frac{\partial^4}{\partial\theta^4}
$$

$$
\nabla^4=\frac{\partial^4}{\partial y^4}+2\frac{\partial^4}{\partial y^2\partial\theta^2}+\frac{\partial^4}{\partial\theta^4},\quad D=\frac{Eh^3}{12(1-\mu^2)},\quad k=\frac{1}{12}\left(\frac{h}{a}\right)^2
$$

$$
\frac{\partial^2}{\partial y^2}=(---)'',\quad \frac{\partial^2}{\partial\theta^2}=\ddot{(---)},\quad x=ay,\quad s=a\theta
$$

B: FLÜGGE-LUR’E-BYRNE EQUATION

$$
\nabla^4(\nabla^2+1)^2w+\frac{1-\mu^2}{k}w^{iv}+2(1-\mu)\left[\ddddot{w}''+\ddot{w}''-\overset{\cdot\cdot\cdot}{\dddot{w}}\right]=-\frac{a^4}{D}\nabla^4q_n
$$

$$
\nabla^4u=-\mu w^{iii}+\ddot{w}'-k\left[\ddddot{w}'-w^{v}+\frac{3}{2}(1-\mu)k\ddot{w}^{iii}\right]
$$

$$
\nabla^4v=-(2+\mu)\dot{w}''-\dddot{w}+k\;2\dot{w}^{iv}+2\dddot{w}^{ii}
$$

C: L.S.D MORLEY EQUATION

$$
\nabla^4(\nabla^2+1)^2w+\frac{1-\mu^2}{k}w^{iv}=-\frac{a^4}{D}q_n
$$

$$
\nabla^4u=-\mu w^{iii}+\ddot{w}^{i}
$$

$$
\nabla^4v=-(2+\mu)\dot{w}^{ii}-\dddot{w}
$$

where u,v,w denote the longitudinal, circumferential and normal displacement respectively. x,s denote the longitudinal, circumferential coordinate. $y$, $\theta$ are non-dimensional quantities.

D: APPROXIMATE L.H.DONNELL EQUATION (MUSHTARI-VLASOV-DONNELL EQUATION)

$$\nabla^8 w + \frac{1-\mu^2}{k} \cdot w^{iv} = -\frac{a^4}{D} \cdot \nabla^4 q - \frac{a^4}{D}\left[ \dddot{q}_\theta + \mu\, q_x^{iii} + (2+\mu)\, \dot{q}_\theta^{ii} - \ddot{q}_x' \right]$$

$$\nabla^4 u = \ddot{w}' - \mu\, w^{iii} - \frac{a^4}{D} k \left[ q_x^{ii} + \frac{2}{1-\mu} \cdot \ddot{q}_x - \frac{1+\mu}{1-\mu} \cdot \dot{q}_\theta^{i} \right]$$

$$\nabla^4 v = -(2+\mu)\, \dot{w}^{ii} - \dddot{w} - \frac{a^4}{D} \cdot k \left[ \ddot{q}_\theta + \frac{2}{1-\mu} \cdot q_\theta^{ii} - \frac{1+\mu}{1-\mu} \cdot \dot{q}_x^{i} \right]$$

E: KOOKI-MIZOGUCHI EQUATION

$$w^{viii} + \frac{4B}{A} \cdot \ddot{w}^{vi} + \frac{6C}{A} \cdot \ddddot{w}^{iv} + \frac{8-2\mu^2}{A} \cdot \ddot{w}^{iv} + \frac{6L}{A} w^{iv} + \frac{4}{A}\left[ \overset{\bullet\bullet\bullet\bullet\bullet\bullet}{w}{}^{ii} + 2\ddddot{w}^{ii} + \ddot{w}^{ii} \right]$$

$$+ \frac{1}{A}\left[ \overset{\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet}{w} + 2\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{w} + \ddddot{w} \right] = \frac{a^4}{AD}\left\{ A \cdot q_n^{iv} + 2U \cdot \ddot{q}_n^{ii} + B \cdot \ddddot{q}_n \right\} + \left\{ \mu A \cdot q_x^{iii} + \frac{(1+\mu)(2-\mu)}{1-\mu} k \cdot \ddot{q}_x^{iii} - V \cdot \ddot{q}_x^{i} \right.$$

$$\left. + \frac{1+\mu}{1-\mu} k \cdot \ddddot{q}_x^{i} \right\} - \left\{ \frac{2(1-\mu^2)}{1-\mu} k \cdot \dot{q}_\theta^{iv} - (2+\mu)\, \dot{q}_\theta^{ii} + \frac{4-3\mu+\mu^2}{1-\mu} k \cdot \dddot{q}_\theta^{ii} + k \cdot \overset{\bullet\bullet\bullet\bullet\bullet}{q}_\theta - \dddot{q}_\theta \right\}$$

$$u^{iv} + \frac{2U}{A} \cdot \ddot{u}^{ii} + \frac{B}{A} \ddddot{u} + \frac{(1+\mu)(2-\mu)}{(1-\mu) \cdot A} k \cdot \ddot{w}^{iii} + \mu w^{iii} + \frac{(1+\mu)\, k}{(1-\mu)\, A} \cdot \ddddot{w}^{i} - \frac{V}{A} \cdot \ddot{w}^{i}$$

$$+ \frac{a^4}{D} \cdot k \left\{ q_x^{ii} + \frac{2B}{(1-\mu) A} \cdot \ddot{q}_x - \frac{(1+\mu) k}{(1-\mu) A} \cdot \dot{q}_\theta^{i} \right\} = 0$$

$$v^{iv} + \frac{2U}{A} \cdot \ddot{v}^{ii} + \frac{B}{A} \cdot \ddddot{v} - \frac{2(2-\mu) k}{(1-\mu) A} \cdot \dot{w}^{iv} - \frac{4-3\mu+\mu^2}{(1-\mu) A} \cdot k \cdot \dddot{w}^{ii} + \frac{2+\mu}{A} \dot{w}^{ii} - \frac{k}{A} \cdot \overset{\bullet\bullet\bullet\bullet\bullet}{w} + \frac{1}{A} \dddot{w}$$

$$+ \frac{a^4}{D} \cdot k \left\{ \frac{2}{(1-\mu) A} \cdot q_\theta^{ii} + \frac{1}{A} \cdot \ddot{q}_\theta - \frac{1+\mu}{(1-\mu) A} \cdot \dot{q}_x^{i} \right\} = 0$$

where

$$A = 1 + \frac{1}{3}\left(\frac{h}{a}\right)^2, \quad B = 1 + \frac{1}{12}\left(\frac{h}{a}\right)^2, \quad C = 1 + \frac{1-\mu^2}{72}\left(\frac{h}{a}\right)^2, \quad L = 2(1-\mu^2)\left(\frac{a}{h}\right)^2 A$$

$$U = 1 + \frac{1+(1-\mu)^2}{12(1-\mu)} \cdot \left(\frac{h}{a}\right)^2, \quad V = 1 - \frac{\mu}{6(1-\mu)}\left(\frac{h}{a}\right)^2, \quad k = \frac{1}{12}\left(\frac{h}{a}\right)^2$$

## DERIVATION OF GENERAL EXPLICIT SOLUTIONS

As we are discussing a shell complete in the circumferential direction, then we require that all quantities be periodic in the $S = a\theta$ coordinate. We can assume that

$$w = W(y) \cos n\theta, \quad v = V(y) \cdot \sin n\theta, \quad u = U(y) \cdot \cos n\theta$$

We assume the normal surface load per unit area in the next form.

$$q_n = (\alpha_n \cdot y^2 + \beta_n y + \gamma_n + S e^{\nu y} + T e^{-\nu y}) \cos n\theta$$

$n$ is the circumferential wave number. Putting $q_x$, $q_\theta$ equal to zero, we substitute above expressions into the basic partial differential equations.

Then, we can get a set of ordinary differential equations as follows.

$$\boxed{\begin{aligned}
&\frac{d^8 W}{dy^8} + W_1 \frac{d^6 W}{dy^6} + W_2 \frac{d^4 W}{dy^4} + W_3 \frac{d^2 W}{dy^2} + W_4 \cdot W \\
&= W_5 \cdot y^2 + W_6 \cdot y + W_7 + W_8 \cdot e^{\nu y} + W_9 \cdot e^{-\nu y} \\
&\frac{d^4 U}{dy^4} + U_1 \frac{d^2 U}{dy^2} + U_2 \cdot U = U_3 \cdot \frac{d^5 W}{dy^5} + U_4 \frac{d^3 W}{dy^3} + U_5 \cdot \frac{dW}{dy} \\
&\qquad + U_6 y^2 + U_7 \cdot y + U_8 \\
&\frac{d^4 V}{dy^4} + V_1 \frac{d^2 V}{dy^2} + V_2 \cdot V = V_3 \cdot \frac{d^4 W}{dy^4} + V_4 \cdot \frac{d^2 W}{dy^2} + V_5 \cdot W \\
&\qquad + V_6 \cdot y^2 + V_7 \cdot y + V_8 \qquad\qquad (a)
\end{aligned}}$$

In the case of FLÜGGE-LUR'E-BYRNE equations, $W_1 \sim W_9, U_1 \sim U_9, V_1 \sim V_9$ become as follows.

$$W_1 = 2(-2n^2+\mu),\ W_2 = 6n^4-6n^2+1+\frac{1-\mu^2}{k},\ W_3 = -4n^6+(8-2\mu)n^4+(-4+2\mu)n^2$$

$$W_4 = n^4(n^2-1)^2,\ W_5 = -\frac{a^4}{D}\cdot n^4 d_n,\ W_6 = -\frac{a^4}{D}\cdot n^4\beta_n,\ W_7 = -\frac{a^4}{D}(n^4\gamma_n - 4n^2 d_n)$$

$$W_8 = -\frac{a^4}{D}(\nu^2-n^2)^2 S,\ W_9 = -\frac{a^4}{D}(\nu^2-n^2)^2 T,\ U_1 = -2n^2,\ U_2 = n^4,\ U_3 = k$$

$$U_4 = -\mu+\frac{3}{2}(1-\mu)\cdot k^2 n^2,\ U_5 = -n^2-kn^4,\ U_6 = U_7 = U_8 = 0,\ V_1 = -2n^2,\ V_2 = n^4$$

$$V_3 = -2kn,\ V_4 = (2+\mu)n+2kn^3,\ V_5 = -n^3,\ V_6 = V_7 = V_8 = 0$$

In the equations of ( $a$ ),after normal displacement $w(y,\theta)$ is known, the other displacement $u(y,\theta)$,$v(y,\theta)$ can be sought. This procedure is the characteristic one of 'UNCOUPLED EQUATIONS'.

(1) WHEN THE WAVE NUMBER $n$ IS EQUAL TO ONE ; As are known from above relations, $W_3$ and $W_4$ become zero, when $n=1$, except approximate Donnell equation.

Then, the ordinary differential equation for $W(y)$ becomes as below.

$$\frac{d^8W}{dy^8}+W_1\frac{d^6W}{dy^6}+W_2\frac{d^4W}{dy^4} = W_5 y^2+W_6\cdot y+W_7+W_8\cdot e^{\nu y}+We^{-\nu y}$$

The explicit general solution for $w(y,\theta)$ can easily be obtained.

$$w(y,\theta) = \Big(C_1\cdot e^{\alpha y}\cos\beta y+C_2\cdot e^{\alpha y}\sin\beta y+C_3\cdot e^{-\alpha y}\cos\beta y+C_4\cdot e^{-\alpha y}\sin\beta y$$
$$+C_5+C_6\cdot y+C_7\cdot y^2+C_8\cdot y^3+W_{VI}\cdot y^6+W_V\cdot y^5+W_{IV}\cdot y^4+W_P\cdot e^{\nu y}+W_M\cdot e^{-\nu y}\Big)\cdot\cos\theta$$

where, $\alpha$ and $\beta$ are the real and imaginary parts of next quartic equation.

$\lambda^4+W_1\cdot\lambda^2+W_2 = 0$ ; The other relations are

$$W_{VI} = \frac{W_5}{360W_2},\ W_V = \frac{W_6}{120W_2},\ W_{IV} = \frac{W_7-720W_1\cdot W_{VI}}{24W_2},\ W_P = \frac{W_8}{\nu^2(\nu^4+W_1\cdot\nu^2+W_2)},\ W_M = \frac{W_9}{''}$$

(2) WHEN THE WAVE NUMBER $n$ IS GREATER THAN 1 ; The general solution becomes

$$w(y,\theta) = \Big[e^{\alpha_1 y}(C_1\cdot\cos\beta_1 y+C_2\cdot\sin\beta_1 y)+e^{-\alpha_1 y}(C_3\cdot\cos\beta_1 y+C_4\cdot\sin\beta_1 y)$$
$$+e^{\alpha_2 y}(C_5\cdot\cos\beta_2 y+C_6\cdot\sin\beta_2 y)+e^{-\alpha_2 y}(C_7\cdot\cos\beta_2 y+C_8\cdot\sin\beta_2 y)$$
$$+W_{II}\cdot y^2+W_I\cdot y+W_0+W_P\cdot e^{\nu y}+W_M\cdot e^{-\nu y}\Big]\cdot\cos n\theta$$

Where, $\alpha_1,\alpha_2,\beta_1,\beta_2$ are the real and imaginary parts of next eighth-degree equation.

$\lambda^8+W_1\cdot\lambda^6+W_2\cdot\lambda^4+W_3\cdot\lambda^2+W_4 = 0$ ; The eight roots can be expressed as

$\lambda_1 = \alpha_1+i\beta_1,\ \lambda_2 = \alpha_1-i\beta_1,\ \lambda_3 = -\alpha_1+i\beta_1,\ \lambda_4 = -\alpha_1-i\beta_1,\ \lambda_5 = \alpha_2+i\beta_2,\ \lambda_6 = \alpha_2-i\beta_2$

$\lambda_7 = -\alpha_2+i\beta_2,\ \lambda_8 = -\alpha_2-i\beta_2$ ; $\alpha_1,\alpha_2,\beta_1,\beta_2$ are real number. $i$ is imaginary symbol.

$$W_{II} = \frac{W_5}{W_4},\ W_I = \frac{W_6}{W_4},\ W_0 = \frac{W_7-2W_3\cdot W_{II}}{W_4},\ W_P = \frac{W_8}{\nu^8+W_1\cdot\nu^6+W_2\nu^4+W_3\cdot\nu^2+W_4},\ W_M = \frac{W_9}{''}$$

To obtain the explicit solution for $U(y)$ and $V(y)$, next ordinary differential equ-ation must be solved and solution becomes as follows.

$$\frac{d^4P}{dy^4}+Q_1\frac{d^2P}{dy^2}+Q_2\cdot P = e^{\alpha y}(A_1\cdot\cos\beta y+A_2\cdot\sin\beta y)\ ;$$ $Q_1, Q_2, A_1, A_2$ are some consta-nts.

$$P(y) = \frac{e^{\alpha y}}{\{(\alpha^4-6\alpha^2\beta^2+\beta^4)+Q_1(\alpha^2-\beta^2)+Q_2\}^2+\alpha^2\beta^2\{4(\alpha^2-\beta^2)+2Q_1\}^2}$$
$$\times\Big\langle\big[\{(\alpha^4-6\alpha^2\beta^2+\beta^4)+Q_1\cdot(\alpha^2-\beta^2)+Q_2\}A_1-\alpha\beta\{4(\alpha^2-\beta^2)+2Q_1\}\cdot A_2\big]\cdot\cos\beta y$$
$$+\big[\{(\alpha^4-6\alpha^2\beta^2+\beta^4)+Q_1\cdot(\alpha^2-\beta^2)+Q_2\}A_2+\alpha\beta\{4(\alpha^2-\beta^2)+2Q_1\}\cdot A_1\big]\cdot\sin\beta y\Big\rangle$$

By utilization of above formula, remaining explicit solutions for $u(y,\theta)$ and $v(y,\theta)$ can be easily obtained. As mentioned above, these solutions have different forms for wave number $n=1$ and $n\geq 2$. But, approximate Donnell equation has same solutions for $n=1$ and $n\geq 2$.

The general solution derived above has eight arbitrary constants which must be determined from four boundary conditions at each of the two edges of constant $y$.

Representative combination of boundary conditions are given as follows.

a) FREE EDGE ; $N_x=0,\ M_x=0,\ N^*_{x\theta}=0,\ Q^*_x=0.$

b) HINGED EDGE WITH FIXED SUPPORT ; $M_x=0,\ u=0,\ v=0,\ w=0.$

c) HINGED EDGE WITH SUPPORT FREE TO MOVE IN THE NORMAL DIRECTION ; $M_x=0,\ Q^*_x=0,\ u=0,\ v=0.$

d) CLAMPED EDGE ; $u=0,\ v=0,\ w=0,\ \frac{\partial w}{\partial x}=0$. Where, $N_x$ is longitudinal axial force and $M_x$ is bending moment in the longitudinal direction. $N^*_{x\theta}$ and $Q^*_x$ denote the Kirchhoff,s effective shearing stress resultants of In-plane and Out-of-plane (in the long.dir.) respe -ctively. In Flügge,s stress-displacement relations, $N_x, M_x, N^*_{x\theta}, Q^*_x$ can be expressed as ;

$$N_x=\frac{K}{a}\left(\frac{\partial u}{\partial y}+\mu\frac{\partial v}{\partial \theta}+\mu w\right)-\frac{D}{a^3}\cdot\frac{\partial^2 w}{\partial y^2}\ ,\quad M_x=\frac{D}{a^2}\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial \theta^2}-\frac{\partial u}{\partial y}-\mu\frac{\partial v}{\partial \theta}\right)\ ,\quad K=\frac{Eh}{1-\mu^2}$$

$$Q^*_x=\frac{D}{a^3}\left\{\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial y\,\partial \theta^2}-\frac{3-\mu}{2}\cdot\frac{\partial^2 v}{\partial y\,\partial \theta}-\frac{\partial^2 u}{\partial y^2}+\frac{1-\mu}{2}\cdot\frac{\partial^2 u}{\partial \theta^2}\right\},\ N^*_{x\theta}=\frac{1-\mu}{2}\cdot\frac{K}{a}\left\{\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial y}+3D\left(\frac{\partial v}{\partial y}-\frac{\partial^2 w}{\partial y\,\partial \theta}\right)\right\}$$

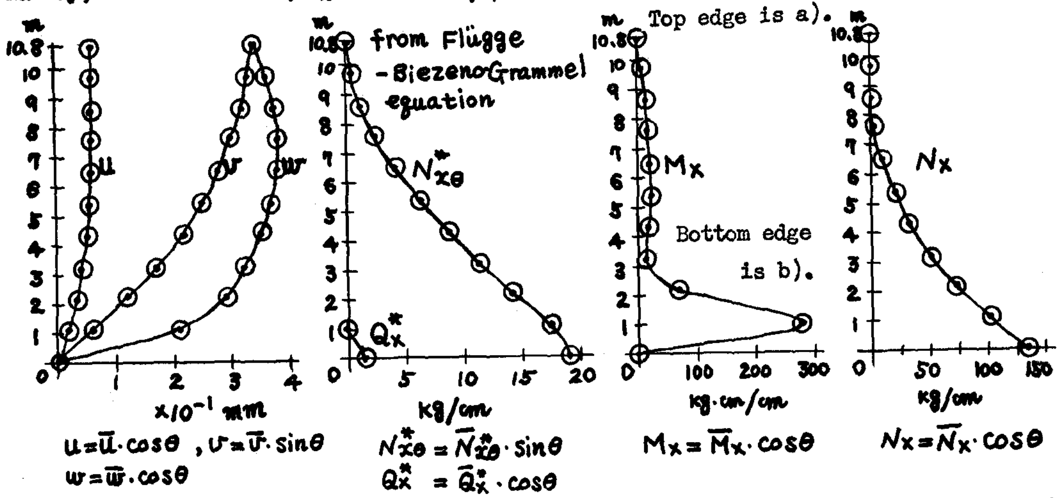
(NUMERICAL EXAMPLE) Here,let apply derived solutions to upright tank under G.W.Housner,s impulsive pressure which is proportional to horizontal acceleration. The pressure is

$$q_n=-\frac{1}{2}\rho\dot{u}_0 H\left\{1-\left(\frac{a}{H}\right)^2\cdot y^2\right\}\sqrt{3}\cdot\tanh\left(\sqrt{3}\,\frac{a}{H}\right)\cdot\cos\theta$$

Where, $\rho$ is the density of fluid. $\dot{u}_0$ is horizontal acceleraion, $a$ is radius of curvature, $H$ is depth of the fluid. Let be

$$\rho=\frac{0.001}{g},\ \dot{u}=0.40g,\ H=1080\text{ cm},\ a=610\text{ cm};\ q_n=(0.28220-0.09003\,y^2)\cdot\cos\theta$$

Namely, $\alpha_n=-0.09003$, $\gamma_n=0.28220$, $\beta_n=S=T=0.$ $E=3.2\times10^5\ \text{kg/cm}^2$, $\mu=\frac{1}{6}$




The summation of integrated value of $Q^*_x$ and $N^*_{x\theta}$,s horizontal component must coincide with total horizontal force induced by pressure.The turning moment Mo must be balanced by Mx and Nx. Namely,

$$Q_0=-\rho\dot{u}_0\pi a^2 H\,\frac{\tanh(\sqrt{3}\,a/H)}{(\sqrt{3}\,a/H)}=\int_0^{2\pi}(Q^*_x\cdot\cos\theta+N^*_{x\theta}\cdot\sin\theta)\,a\,d\theta=(\bar{Q}^*_x+\bar{N}^*_{x\theta})\pi a$$

$$M_0=Q_0\times\left(\frac{3}{8}H\right)=\int_0^{2\pi}(M_x\cdot\cos\theta+N_x\cdot a\cos\theta)\cdot a\,d\theta$$ at the bottom edge.

| HEIGHT | membrane | A: Flügge-Biezeno-Gra | | | B: Flügge-Lür'e-Byrne | | | C: L.S.D.Morley | | | D: Appr. L.H.Donnell | | | E: Kooki-Mizoguchi | | | kg/cm |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $N_{x\theta}$ | $N^*_{x\theta}$ | $Q^*_x$ | Total | $N^*_{x\theta}$ | $Q^*_x$ | Total | $N^*_{x\theta}$ | $Q^*_x$ | Total | $N^*_{x\theta}$ | $Q^*_x$ | Total | $N^*_{x\theta}$ | $Q^*_x$ | Total | |
| 0.00 | 20.27 | 19.08 | 1.23 | 20.31 | 19.06 | 1.21 | 20.27 | 18.91 | 1.37 | 20.28 | 18.95 | 1.12 | 20.07 | 19.05 | 1.21 | 20.26 | |

| $N_x$ (kg/cm) ($y=0.00$) | membrane | A: | B: | C: | D: | E: |
|---|---|---|---|---|---|---|
| | 134.55 | 134.40 | 134.53 | 134.60 | 127.63 | 134.51 |

Approximate Donnell equation reveals less precision than the other one.

(CONCLUSIONS) These solutions obtained here do not contain the rigid-body-displacement. But,numerical example reveals very good coincidence between external forces and induced stress resultants. Upright tank subject to dynamic pressure which derived from Potential theory(solution of Laplace equation) can be easily analyzed by the same methods above.